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On Compare of Two Types of Battles in Lanchester's Classic Aimed-fire Model

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Abstract

Consider two types of battles in Lanchester's classic aimed-fired model of warfare. In one battle, the initial combatants of one force are divided into n parts. We analyze the conditions for victory and compare the battles' end times.

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1 Introduction

In this paper, we consider Lanchester's classic aimed-fire model of warfare :

$$\begin{cases} x'(t) = -\alpha y(t) \\ y'(t) = -\beta x(t) \end{cases} \tag{1.1}$$

which describes combat between X and Y forces, where x and y represent the number of engaged X and Y combatants, respectively. The parameters α and β are the individual fighting values of Y and X forces, respectively, and are positive constants. The ratio $E = \alpha/\beta$ is called the exchange ratio (see [2] and [5]).

Throughout this paper we consider only non-negative solutions. The initial value problem (1.1) for $t \geq t_0$ with initial conditions $x(t_0) = x_0 > 0$ and $y(t_0) = y_0 > 0$ satisfies the well-known Lanchester's square law :

$$x(t)^2 - Ey(t)^2 = x_0^2 - Ey_0^2\,. \tag{1.2}$$

Furthermore, through fundamental calculations, we derive the following representation formula for solutions $x = x(t)$ and $y = y(t)$:

$$x(t) = \frac{1}{2}\left\{(x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}(t-t_0)} + (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}(t-t_0)}\right\} \tag{1.3}$$

and

$$y(t) = \frac{1}{2}\left\{(x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}(t-t_0)} - (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}(t-t_0)}\right\} \tag{1.4}$$

(see [3] and [4]).

For simplicity, we assume this battle to be a battle of annihilation, meaning it ends when the number of Y force's combatants reaches zero at which point X force achieves victory.

2 Results

We compare the following two types of battles where the initial number of combatants for X and Y forces are $x_0 > 0$ and $y_0 > 0$, respectively.

Type I : We consider the initial value problem (1.1) for $0 \leq t \leq T$ with initial conditions $x(0) = x_0$ and $y(0) = y_0$. In this scenario, X force wins the battle at time $t = T$, meaning $y(T) = 0$ (while $x(T) > 0$).

Type II : X force divides Y force's initial combatants into n parts and engages them sequentially. X force ultimately wins the battle (see [1] for $n = 2$). Let the proportion of each division be $\theta_1, \theta_2, \cdots, \theta_n$, where $\theta_1 + \theta_2 + \cdots + \theta_n = 1$ and $0 < \theta_k < 1$. For each $k = 1, 2, \cdots, n$, we consider the initial value problem (1.1) for $T_{k-1} \leq t \leq T_k$ (where $T_0 = 0$) with initial conditions $x(T_{k-1}) = x_{k-1}$ and $y(T_{k-1}) = \theta_k y_0$, and $y(T_k) = 0$ (while $x_k \equiv x(T_k) > 0$). In this scenario, X force wins the battle at $t = T_n$, meaning $y(T_n) = 0$ (while $x(T_n) > 0$), and in addition, X force's aim is to achieve victory with the smallest possible number of initial combatants

Our main results are as follows.

Theorem 1 (1) *For the battle of Type I, the victory condition for X force is*

$$x_0 > \sqrt{E}y_0\,. \tag{2.1}$$

The end time T of the battle is given by

$$T = \frac{1}{2\sqrt{\alpha\beta}}\log\left(\frac{x_0 + \sqrt{E}y_0}{x_0 - \sqrt{E}y_0}\right), \tag{2.2}$$

where the remaining X force at time T is $x(T) = \sqrt{x_0^2 - Ey_0^2}$.

(2) *For the battle of Type II* (*with* n *divisions*), *the optimal victory condition for* X *force is*

$$x_0 > \sqrt{\frac{E}{n}} y_0 \,. \tag{2.3}$$

The end time T_n *of the battle is given by*

$$T_n = \frac{1}{2\sqrt{\alpha\beta}} \log \left(\prod_{k=0}^{n-1} \frac{\sqrt{x_0^2 - k(\frac{\sqrt{E}}{n} y_0)^2} + \frac{\sqrt{E}}{n} y_0}{\sqrt{x_0^2 - k(\frac{\sqrt{E}}{n} y_0)^2} - \frac{\sqrt{E}}{n} y_0} \right) , \tag{2.4}$$

where the remaining X *force at time* T_n *is* $x(T_n) = \sqrt{x_0^2 - \frac{E}{n} y_0^2}$.

(3) *Furthermore, under condition* (2.1), *the following inequality holds*

$$T > T_n \tag{2.5}$$

for $n \geq 2$.

Proof. (1) We consider the initial value problem (1.1) for $0 \leq t \leq T$ with initial conditions $x(0) = x_0$ and $y(0) = y_0$. When X force wins against Y force at time $t = T$ (i.e. $y(T) = 0$ while $x(T) > 0$), we obtain from (1.2) with $t_0 = 0$ that $x(T) = \sqrt{x_0^2 - E y_0^2}$. Thus, the victory condition of X force is $x_0 > \sqrt{E} y_0$ which implies (2.1).

When $y(T) = 0$, using the representation formula (1.4) with $t_0 = 0$ for solution y, we have

$$(x_0 + \sqrt{E} y_0) e^{-\sqrt{\alpha\beta} T} - (x_0 - \sqrt{E} y_0) e^{\sqrt{\alpha\beta} T} = 0 \,.$$

Solving for the end time T of the battle, we obtain

$$T = \frac{1}{2\sqrt{\alpha\beta}} \log \left(\frac{x_0 + \sqrt{E} y_0}{x_0 - \sqrt{E} y_0} \right)$$

which implies (2.2).

(2) First, we consider the initial value problem (1.1) for $0 \leq t \leq T_1$ with initial conditions $x(0) = x_0$ and $y(0) = \theta_1 y_0$. When X force wins against Y force at time $t = T_1$ (i.e. $y(T_1) = 0$ while $x(T_1) > 0$), we obtain from (1.2) with $t_0 = 0$ that

$$x_1 \equiv x(T_1) = \sqrt{x_0^2 - E \theta_1^2 y_0^2} \,. \tag{2.6}$$

When $y(T_1) = 0$, using (1.4) with $t_0 = 0$ and initial conditions $x(0) = x_0$ and $y(0) = \theta_1 y_0$, we have

$$(x_0 + \sqrt{E} \theta_1 y_0) e^{-\sqrt{\alpha\beta} T_1} - (x_0 - \sqrt{E} \theta_1 y_0) e^{\sqrt{\alpha\beta} T_1} = 0 \,.$$

Solving for T_1, we obtain

$$T_1 = \frac{1}{2\sqrt{\alpha\beta}} \log\left(\frac{x_0 + \sqrt{E}\theta_1 y_0}{x_0 - \sqrt{E}\theta_1 y_0}\right). \tag{2.7}$$

Next, we consider the initial value problem (1.1) for $T_1 \le t \le T_2$ with initial conditions $x(T_1) = x_1$ and $y(T_1) = \theta_2 y_0$. When X force wins against Y force at time $t = T_2$ (i.e. $y(T_2) = 0$ while $x(T_2) > 0$), we obtain from (1.2) with $t_0 = T_1$ that

$$\begin{aligned} x_2 \equiv x(T_2) &= \sqrt{x_1^2 - E\theta_2^2 y_0^2} \\ &= \sqrt{x_0^2 - E(\theta_1^2 + \theta_2^2)y_0^2}\,, \end{aligned} \tag{2.8}$$

where we used (2.6).

When $y(T_2) = 0$, using (1.4) with $t_0 = T_1$ and initial conditions $x(T_1) = x_1$ and $y(T_1) = \theta_2 y_0$, we have

$$(x_1 + \sqrt{E}\theta_2 y_0)e^{-\sqrt{\alpha\beta}(T_2 - T_1)} - (x_1 - \sqrt{E}\theta_2 y_0)e^{\sqrt{\alpha\beta}(T_2 - T_1)} = 0\,.$$

Solving for T_2, we obtain

$$\begin{aligned} T_2 &= T_1 + \frac{1}{2\sqrt{\alpha\beta}} \log\left(\frac{x_1 + \sqrt{E}\theta_2 y_0}{x_1 - \sqrt{E}\theta_2 y_0}\right) \\ &= \frac{1}{2\sqrt{\alpha\beta}} \log\left(\frac{x_0 + \sqrt{E}\theta_1 y_0}{x_0 - \sqrt{E}\theta_1 y_0} \cdot \frac{x_1 + \sqrt{E}\theta_2 y_0}{x_1 - \sqrt{E}\theta_2 y_0}\right), \end{aligned} \tag{2.9}$$

where we used (2.7).

By repeating a similar argument, we finally consider the initial value problem (1.1) for $T_{n-1} \le t \le T_n$ with initial conditions $x(T_{n-1}) = x_{n-1}$ and $y(T_{n-1}) = \theta_n y_0$.

When X force wins against Y force at time $t = T_n$ (i.e. $y(T_n) = 0$ while $x(T_n) > 0$), we obtain from (1.2) with $t_0 = T_{n-1}$ that

$$\begin{aligned} x_n \equiv x(T_n) &= \sqrt{x_{n-1}^2 - E\theta_n^2 y_0^2} \\ &= \sqrt{x_0^2 - E(\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2)y_0^2} \end{aligned} \tag{2.10}$$

(see (2.8)).

When $y(T_n) = 0$, using (1.4) with $t_0 = T_{n-1}$ and initial conditions $x(T_{n-1}) = x_{n-1}$ and $y(T_{n-1}) = \theta_n y_0$, we have

$$(x_{n-1} + \sqrt{E}\theta_n y_0)e^{-\sqrt{\alpha\beta}(T_n - T_{n-1})} - (x_{n-1} - \sqrt{E}\theta_n y_0)e^{\sqrt{\alpha\beta}(T_n - T_{n-1})} = 0\,.$$

Solving for T_n, we obtain

$$T_n = T_{n-1} + \frac{1}{2\sqrt{\alpha\beta}} \log\left(\frac{x_{n-1} + \sqrt{E}\theta_n y_0}{x_{n-1} - \sqrt{E}\theta_n y_0}\right)$$
$$= \frac{1}{2\sqrt{\alpha\beta}} \log\left(\prod_{k=0}^{n-1} \frac{x_k + \sqrt{E}\theta_{k+1} y_0}{x_k - \sqrt{E}\theta_{k+1} y_0}\right), \tag{2.11}$$

where we used (2.7) and (2.9).

Noting the relationship between the arithmetic and geometric means, we see that $\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2$ becomes the smallest when $\theta_1 = \theta_2 = \cdots = \theta_n = 1/n$.

Then, we have from (2.10) with $\theta_1 = \theta_2 = \cdots = \theta_n = 1/n$ that

$$x_n \equiv x(T_n) = \sqrt{x_0^2 - \frac{E}{n} y_0^2} > 0\,,$$

and hence, we see that the optimal victory condition of X force is

$$x_0 > \sqrt{\frac{E}{n}}\, y_0 \tag{2.12}$$

which implies (2.3).

Moreover, when $\theta_1 = \theta_2 = \cdots = \theta_n = 1/n$, from (2.11) and

$$x_k = \sqrt{x_0^2 - k\Big(\frac{\sqrt{E}}{n} y_0\Big)^2}$$

for $k = 1, 2, \cdots, n-1$ (see (2.6), (2.8) and (2.10)), we obtain the end time T_n of the battle such that

$$T_n = \frac{1}{2\sqrt{\alpha\beta}} \log \prod_{k=0}^{n-1} \frac{\sqrt{x_0^2 - k(\frac{\sqrt{E}}{n} y_0)^2} + \frac{\sqrt{E}}{n} y_0}{\sqrt{x_0^2 - k(\frac{\sqrt{E}}{n} y_0)^2} - \frac{\sqrt{E}}{n} y_0} \tag{2.13}$$

which implies (2.4).

(3) Lastly, we will investigate the relationship between T given by (2.2) and T_n given by (2.4).

When $n \geq 2$ and $x > ny > 0$, the following inequality holds

$$\prod_{k=0}^{n-1} \frac{\sqrt{x^2 - ky^2} + y}{\sqrt{x^2 - ky^2} - y} < \frac{x + ny}{x - ny}\,. \tag{2.14}$$

To verify this, note that for $k = 0, 1, 2, \cdots, n-1$,

$$\begin{aligned}\frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\cdot\frac{x+ky}{x-ky} &< \frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\cdot\frac{\sqrt{x^2-ky^2}+ky}{\sqrt{x^2-ky^2}-ky}\\ &= \frac{x^2+(k+1)\sqrt{x^2-ky^2}y}{x^2-(k+1)\sqrt{x^2-ky^2}y}\\ &< \frac{x^2+(k+1)xy}{x^2-(k+1)xy}\\ &= \frac{x+(k+1)y}{x-(k+1)y}.\end{aligned}$$

Applying this inequality iteratively, we obtain

$$\begin{aligned}&\prod_{k=0}^{n-1}\frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\\ &= \prod_{k=2}^{n-1}\frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\cdot\frac{\sqrt{x^2-y^2}+y}{\sqrt{x^2-y^2}-y}\cdot\frac{x+y}{x-y}\\ &< \prod_{k=2}^{n-1}\frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\cdot\frac{x+2y}{x-2y}\\ &< \prod_{k=3}^{n-1}\frac{\sqrt{x^2-ky^2}+y}{\sqrt{x^2-ky^2}-y}\cdot\frac{x+3y}{x-3y}\\ &< \cdots < \frac{\sqrt{x^2-(n-1)y^2}+y}{\sqrt{x^2-(n-1)y^2}-y}\cdot\frac{x+(n-1)y}{x-(n-1)y}\\ &< \frac{x+ny}{x-ny}.\end{aligned}$$

Thus, from inequality (2.14) with $x = x_0$ and $y = \frac{\sqrt{E}}{n}y_0$, we obtain

$$\log\left(\prod_{k=0}^{n-1}\frac{\sqrt{x_0^2-k(\frac{\sqrt{E}}{n}y_0)^2}+\frac{\sqrt{E}}{n}y_0}{\sqrt{x_0^2-k(\frac{\sqrt{E}}{n}y_0)^2}-\frac{\sqrt{E}}{n}y_0}\right) < \log\left(\frac{x_0+\sqrt{E}y_0}{x_0-\sqrt{E}y_0}\right),$$

and hence, we conclude that $T_n < T$ which implies (2.5). □

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