J. Math. Tokushima Univ.
Vol. 58 (2024), 73 − 81

# On Asymptotic Forms of Solutions for Lanchester Type Differential Equations with Time Dependent Coefficients

By

Nanaka Mishima and Kosuke Ono†

*Department of Mathematical Sciences,*
*Tokushima University, Tokushima 770-8502, JAPAN*
*e-mail* : *k.ono@tokushima-u.ac.jp*†

(Received March 1, 2024)

**Abstract**

We consider an ordinary differential system which is a so-called Lanchester type model with time dependent coefficients. We study on asymptotic forms of solutions that decay to the origin $(0, 0)$.

2010 Mathematics Subject Classification. 34A34, 34D05

## 1 Introduction

In this paper we investigate on the ordinary differential system of the form :

$$\begin{cases} x'(t) = -a(t)y(t) \\ y'(t) = -b(t)x(t) \end{cases} \tag{1.1}$$

where $a(t)$ and $b(t)$ are positive continuous functions on $[0, \infty)$, and satisfy

$$A(t) = \int_0^t a(s)\,ds \to \infty \quad \text{and} \quad B(t) = \int_0^t b(s)\,ds \to \infty \tag{1.2}$$

as $t \to \infty$.

Throughout this paper we will consider non-negative solutions for (1.1) with positive initial data

$$x(0) = x_0 > 0 \quad \text{and} \quad y(0) = y_0 > 0\,. \tag{1.3}$$

System (1.1) is known as one of Lanchester type models, which describes many phenomena appearing in economics, logistics, biology, and so on.

F.W.Lanchester [5] first proposed system (1.1) to describe combat situations (see [2], [3], [11] for a review). Although the equations were never directly listed, Taylor and others suggest that Lanchester could mathematically represent his statements of the linear ancient battle as system (1.1) (see [4], [10], [12], [13] for Lanchester linear law models, and also see [1], [6], [7], [8], [9] for Lanchester square law models and generalized Lanchester models).

First we consider the system with constant coefficients $a(t) = \alpha > 0$ and $b(t) = \beta > 0$, that is,

$$\begin{cases} x'(t) = -\alpha y(t) \\ y'(t) = -\beta x(t) \end{cases} \tag{1.4}$$

with initial data (1.3) (see [10]).

Since it holds that

$$\begin{cases} (x(t)^2)' = -2\alpha x(t)y(t) \\ (y(t)^2)' = -2\beta x(t)y(t)\,, \end{cases} \tag{1.5}$$

we can see easily that $(x(t)^2 - Ey(t)^2)' = 0$ by using the exchange radio $E = \alpha/\beta$ of (1.1), and hence, $x(t)^2 - Ey(t)^2$ is a constant value which is denoted by symbol $M$, that is,

$$x(t)^2 - Ey(t)^2 = x_0^2 - Ey_0^2 = M \tag{1.6}$$

Thus, we have from (1.4) and (1.6) that

$$x'(t) = -\alpha y(t) = -\sqrt{\alpha\beta}\sqrt{x(t)^2 - M}\,, \tag{1.7}$$

and moreover, by fundamental calculation we obtain the following representation formula of solution $(x(t), y(t))$ :

(i) When $M = 0$ (i.e. $x_0 = \sqrt{E}y_0$),

$$x(t) = x_0 e^{-\sqrt{\alpha\beta}t} \quad \text{and} \quad y(t) = y_0 e^{-\sqrt{\alpha\beta}t} \tag{1.8}$$

for $t \geq 0$.

(ii) When $M \neq 0$ (i.e. $x_0 \neq \sqrt{E}y_0$), putting $z - x = \sqrt{x^2 - M}$, since it follows from (1.7) that $(z - x)(z' + \sqrt{\alpha\beta}z) = 0$, function $z(t)$ satisfies $z'(t) = -\sqrt{\alpha\beta}z(t)$ and then

$$z(t) = z_0 e^{-\sqrt{\alpha\beta}t}\,, \quad z_0 = x_0 + \sqrt{x_0^2 - M} = x_0 + \sqrt{E}y_0\,,$$

that is,

$$x(t) = \frac{1}{2}\left(z_0 e^{-\sqrt{\alpha\beta}t} + \frac{M}{z_0} e^{\sqrt{\alpha\beta}t}\right) \tag{1.9}$$
$$= \frac{1}{2}\left((x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}t} + (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}t}\right)$$

and

$$y(t) = \frac{1}{2\sqrt{E}}\left(z_0 e^{-\sqrt{\alpha\beta}t} - \frac{M}{z_0} e^{\sqrt{\alpha\beta}t}\right) \tag{1.10}$$
$$= \frac{1}{2\sqrt{E}}\left((x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}t} - (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}t}\right)$$

for $t \geq 0$.

**Remark.** When the time dependent coefficients $a(t)$ and $b(t)$ satisfy $a(t)/b(t) = const. > 0$ for $t \geq 0$, we can obtain the similar representation formula of solution $(x(t), y(t))$ of (1.1) replaced $\sqrt{\alpha\beta}t$ in (1.8)–(1.10) by $\int_0^t \sqrt{a(s)b(s)}ds$.

The notations we use in this paper are standard. Positive constants will be denoted by $C$ and will change from line to line.

## 2 Non-negative solutions

It is easy to see that $(x(t), y(t)) \equiv (0, 0)$ is a solutions of (1.1) and the initial value problem (1.1)–(1.3) has the uniqueness of solutions. Moreover, there exists $0 < T \leq \infty$ such that one of the followings holds:

$$\lim_{t\to T} x(t) > 0 \quad \text{and} \quad \lim_{t\to T} y(t) = 0\,;$$
$$\lim_{t\to \infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to \infty} y(t) = 0\,;$$
$$\lim_{t\to T} x(t) = 0 \quad \text{and} \quad \lim_{t\to T} y(t) > 0\,.$$

Indeed, if it holds that $\lim_{t\to\infty} x(t) = x_\infty > 0$ and $\lim_{t\to\infty} y(t) = y_\infty > 0$, then from $x' = -a(t)y < 0$ and $y' = -b(t)x < 0$ we obtain that

$$x_0 \geq -x(t) + x_0 = \int_0^t a(s)y(s)\,ds \geq y_\infty A(t)$$
$$y_0 \geq -y(t) + y_0 = \int_0^t b(s)d(s)\,ds \geq x_\infty B(t)$$

which is a contradiction because of assumption (1.2).

In [8], they have derived about relations between behavior of solutions of (1.1) and their initial data.

For each $x_0 > 0$, we define the set $S_{x_0} \subset \mathbb{R}^2$ by

$$S_{x_0} = \{(x, 0)\,|\, 0 < x < x_0\} \cup \{(0, 0)\} \cup \{(0, y)\,|\, y > 0\}\,.$$

**Proposition 2.1 ([8])** *Suppose that $a(t)$ and $b(t)$ satisfy* (1.2) *and*

$$0 < \inf_{t\geq 0} \frac{a(t)}{b(t)} \leq \sup_{t\geq 0} \frac{a(t)}{b(t)} < \infty \,.$$

*Then for any $(C_1, C_2) \in S_{x_0}$, there is one and only one solution $(x(t), y(t))$ of* (1.1) *satisfying $x(0) = x_0$ and*

$$\lim_{t\to T} (x(t), y(t)) = (C_1, C_2) \,,$$

*where $T$ is some positive number or $T = \infty$.*

**Remark.** In Appendix, we will give another simple proof of a so-called comparison principle for solutions which plays an important rule in the proof of Proposition 2.1.

In the following section we discuss how solutions decaying to the origin $(0,0)$ behave at $+\infty$.

## 3 Asymptotic forms of solutions

In what follows, "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$ for positive functions $f(t)$ and $g(t)$ defined near $\infty$. Similarly, for vector-valued functions "$(f_1(t), f_2(t)) \sim (g_1(t), g_2(t))$ as $t \to \infty$" means that $f_i(t) \sim g_i(t)$ as $t \to \infty$, $i = 1, 2$.

**Theorem 3.1** *Suppose that $a(t)$ and $b(t)$ satisfy* (1.2) *and*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = const. > 0 \,. \tag{3.1}$$

*Then every solution $(x(t), y(t))$ of* (1.1) *decaying to the origin $(0,0)$ has the asymptotic form*

$$(\log x(t)\,,\, \log y(t)) \sim \left( -\int_0^t \sqrt{a(s)b(s)}\, ds \,,\, -\int_0^t \sqrt{a(s)b(s)}\, ds \right) \tag{3.2}$$

*as $t \to \infty$.*

**Proof.** Let $\lim_{t\to\infty} a(t)/b(t) = E > 0$ and $\lim_{t\to\infty}(x(t), y(t)) = (0,0)$. By L'Hospital's rule together with (1.5), we have

$$\lim_{t\to\infty} \frac{x(t)^2}{y(t)^2} = \lim_{t\to\infty} \frac{(x(t)^2)'}{(y(t)^2)'} = \lim_{t\to\infty} \frac{a(t)}{b(t)} = E \,. \tag{3.3}$$

Thus, since $(\log x)' = x'/x = -a(t)y/x$, we have

$$\lim_{t\to\infty} \frac{\log x(t)}{-\int_0^t \sqrt{a(s)b(s)}\, ds} = \lim_{t\to\infty} \frac{(\log x(t))'}{\sqrt{a(t)b(t)}} = \lim_{t\to\infty} \sqrt{\frac{a(t)}{b(t)}} \frac{y(t)}{x(t)} = 1 \,, \tag{3.4}$$

which implies that $\log x(t) \sim -\int_0^t \sqrt{a(s)b(s)}\,ds$ as $t \to \infty$.

On the other hand, since $(\log y)' = y'/y = -b(t)x/y$, we have from (3.3) that

$$\lim_{t\to\infty} \frac{\log x(t)}{\log y(t)} = \lim_{t\to\infty} \frac{(\log x(t))'}{(\log y(t))'} = \lim_{t\to\infty} \frac{a(t)}{b(t)} \frac{y(t)^2}{x(t)^2} = 1\,. \tag{3.5}$$

Thus, we obtain from (3.3)–(3.5) that

$$\lim_{t\to\infty} \frac{\log y(t)}{-\int_0^t \sqrt{a(s)b(s)}\,ds} = \lim_{t\to\infty} \frac{\log x(t)}{-\int_0^t \sqrt{a(s)b(s)}\,ds} \frac{\log x(t)}{\log y(t)} = 1\,,$$

which implies that $\log y(t) \sim -\int_0^t \sqrt{a(s)b(s)}\,ds$ as $t \to \infty$. □

**Theorem 3.2** *Suppose that $a(t)$ and $b(t)$ are of class $C^1$ and satisfy (1.2) and*

$$\left(\frac{a(t)}{b(t)}\right)' \le 0 \quad \textit{for large } t. \tag{3.6}$$

*Then every solution $(x(t), y(t))$ of (1.1) decaying to $(0,0)$ has*

$$x(t) = O\left(e^{-\int_0^t \sqrt{a(s)b(s)}\,ds}\right)\,. \tag{3.7}$$

*In addition, if*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = const. > 0\,, \tag{3.8}$$

*then*

$$y(t) = O\left(e^{-\int_0^t \sqrt{a(s)b(s)}\,ds}\right)\,. \tag{3.9}$$

**Proof.** Since $x(t) \to 0$ as $t \to \infty$ and $(-x(t)^2)' = (a(t)/b(t))(-y(t)^2)'$ and $y(t) \to 0$ as $t \to \infty$, it follows that

$$\begin{aligned} x(t)^2 &= \int_t^\infty (-x(s)^2)'\,ds = \int_t^\infty \frac{a(s)}{b(s)}(-y(s)^2)'\,ds \\ &= \frac{a(t)}{b(t)}y(t)^2 + \int_t^\infty \left(\frac{a(s)}{b(s)}\right)' y(s)^2\,ds \end{aligned}$$

for large $t$, and from (3.6) that there exists $t_1 > 0$ such that

$$y(t) \ge \sqrt{\frac{b(t)}{a(t)}}x(t) \quad \text{for } t \ge t_1.$$

Then we have

$$(x(t)^2)' = -2a(t)x(t)y(t) \le -2\sqrt{a(t)b(t)}x(t)^2 \quad \text{for } t \ge t_1.$$

Solving this differential inequality on $[t_1, t]$, we obtain

$$x(t)^2 \le x_1^2 e^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} \quad \text{for } t \ge t_1\,, \tag{3.10}$$

where $x_1^2 = x(t_1)^2 e^{2\int_0^{t_1}\sqrt{a(s)b(s)}\,ds} \ge 0$, which gives (3.7).

Moreover, since $y(t) \to 0$ as $t \to \infty$ and $(-y(t)^2)' = (b(t)/a(t))(-x(t)^2)'$ and $x(t) \to 0$ as $t \to \infty$, it follows that

$$\begin{aligned} y(t)^2 &= \int_t^\infty (-y(s)^2)'\,ds = \int_t^\infty \frac{b(s)}{a(s)}(-x(s)^2)'\,ds \\ &= \frac{b(t)}{a(t)}x(t)^2 + \int_t^\infty \left(\frac{b(s)}{a(s)}\right)' x(s)^2\,ds \end{aligned} \tag{3.11}$$

for large $t$. Here, since it follows from (3.8) that

$$0 \le \frac{b(t)}{a(t)} e^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} \le Ce^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} \tag{3.12}$$

for large $t$, and from $(b(t)/a(t))' = -(b(t)/a(t))^2(a(t)/b(t))' \ge 0$ for large $t$ and (3.8) that

$$\begin{aligned} 0 &\le \int_t^\infty \left(\frac{b(s)}{a(s)}\right)' e^{-2\int_0^s \sqrt{a(r)b(r)}\,dr} ds \\ &= -\frac{b(t)}{a(t)} e^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} - \int_t^\infty \frac{b(s)}{a(s)} \left(-2\sqrt{a(s)b(s)}e^{-2\int_0^s \sqrt{a(r)b(r)}\,dr}\right) ds \\ &\le -\frac{b(t)}{a(t)} e^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} + C\int_t^\infty \left(-e^{-2\int_0^s \sqrt{a(r)b(r)}\,dr}\right)' ds \\ &\le Ce^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} \quad \text{for large } t, \end{aligned} \tag{3.13}$$

we obtain from (3.10)–(3.13) that

$$y(t)^2 \le Ce^{-2\int_0^t \sqrt{a(s)b(s)}\,ds} \quad \text{for large } t,$$

which gives (3.9). □

**Remark.** By (3.1) or (3.6), we see that $\sqrt{a(t)b(t)} \ge Ca(t)$ with some positive constant $C$ for large $t$.

## 4 Appendix

It is easy to see that a vector function $(x(t), y(t))$ is the solution of initial value problem (1.1)–(1.3) if and only if it solves the system of integral equations

$$\begin{cases} x(t) = x_0 - \displaystyle\int_0^t a(s)y(s)\,ds \\ y(t) = y_0 - \displaystyle\int_0^t b(s)x(s)\,ds\,, \end{cases} \tag{4.1}$$

or

$$\begin{cases} x(t) = x_0 - \displaystyle\int_0^t a(s)\left(y_0 - \int_0^s b(r)x(r)\,dr\right)ds \\ y(t) = y_0 - \displaystyle\int_0^t b(s)\left(x_0 - \int_0^s a(r)y(r)\,dr\right)ds\,. \end{cases} \tag{4.2}$$

We discuss a comparison principle of solutions which plays an important rule in the proof of Proposition 2.1, and we will give another simple proof.

**Proposition 4.1 (Comparison principle)** *Let* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *be solutions of initial value problem* (1.1) *with initial data* $(x_1(0), y_1(0))$ *and* $(x_2(0), y_2(0))$*, respectively.*

(i) *If* $x_1(0) \geq x_2(0)$ *and* $y_1(0) \leq y_2(0)$*, then* $x_1(t) \geq x_2(t)$ *and* $y_1(t) \leq y_2(t)$ *for* $t \geq 0$.

(ii) *If* $x_1(0) \geq x_2(0)$ *and* $y_1(0) \leq y_2(0)$ *and* $(x_1(0), y_1(0)) \neq (x_2(0), y_2(0))$*, then* $x_1(t) > x_2(t)$ *and* $y_1(t) < y_2(t)$ *for* $t > 0$.

**Proof.** Let $x_1(0) \geq x_2(0)$ and $y_1(0) < y_2(0)$. We will show that $y_1(t) < y_2(t)$ for $t > 0$ by a contradiction.

If there exists a number $T$ such that $y_1(t) < y_2(t)$ $(0 \leq t < T)$ and $y_1(T) = y_2(T)$, then we have from (4.2) that

$$\begin{aligned} y_1(T) &= y_1(0) - \int_0^T b(s)\left(x_1(0) - \int_0^s a(r)y_1(r)\,dr\right)ds \\ &< y_2(0) - \int_0^T b(s)\left(x_2(0) - \int_0^s a(r)y_2(r)\,dr\right)ds = y_2(T) \end{aligned}$$

which is a contradiction, and hence we obtain that $y_1(t) < y_2(t)$ for $t \geq 0$.

Moreover, we have from (4.1) that

$$\begin{aligned} x_1(t) &= x_1(0) - \int_0^t a(s)y_1(s)\,ds \\ &> x_2(0) - \int_0^t a(s)y_2(s)\,ds = x_2(t) \end{aligned}$$

for $t \geq 0$.

By the similar way, if $x_1(0) > x_2(0)$ and $y_1(0) \leq y_2(0)$, we can also show that $x_1(t) > x_2(t)$ and $y_1(t) < y_2(t)$ for $t \geq 0$, and hence, we conclude the claim (ii).

Moreover, by uniqueness of solutions and the claim (ii), we obtain the claim (i). □

## References

[1] M.J. Artelli and R.F. Deckro, Modeling the Lanchester laws with system dynamics, J. Defence Modeling and Simulation, **5** (2008) 1–20.

[2] D.R. Howes and R.M. Thrall, A theory of ideal linear weights for heterogeneous combat forces, Naval Research Logistics Quartery, **20** (1973) 645–659.

[3] G. Isac and A. Gosselin, A military application of viability: winning cones, differential inclusions, and Lanchester type models for combats, Pareto Optimality, Game Theory and Equilibria, 759–797, Springer Optim. Appl., 17, Springer, New York 2008.

[4] T. Ito, T.Ogiwara, and H. Usami, Asymptotic properties of solutions of a Lanchester-type model, Differential Equations & Applications **12** (2020) 1–12

[5] F.W. Lanchester, Aircraft in Warfare: The Dawn of the Fourth Arm, 1916 (Reprinted by Forgotten Books, 2015).

[6] K.Y. Lin and N. J. MacKay, The optimal policy for the one-against-many heterogeneous Lanchester, Oper. Res. Lett. **42** (2014) 473–477.

[7] N.J. MacKay, Lanchester models for mixed forces with semi-dynamical target allocation, J. Oper. Res. Soc. **60** (2009) 1421–1427.

[8] T. Ogiwara and H. Usami, On the behavior of solutions for Lanchester square-law models with time-dependent coefficients, Josai Mathematical Monographs, **11** (2018) 15–25.

[9] K. Ono, Asymptotic properties of solutions for Lanchester type models with time dependent coefficients, J. Math. Tokushima Univ., **57** (2023)

[10] F. Sato, Mathematics of nature and mathematics of society I & II - Analysis using differential equations (Japanese), Nippon Hyoron Sha Co.,LTD. Publishers 1984 & 1987.

[11] J.G. Taylor, Lanchester Models of Warfare, I and II, Operations Research Society of America 1983.

[12] J.G. Taylor, Battle-outcome prediction for an extended system of Lanchester-type differential equations, J. Math. Anal. Appl., **103** (1984) 371–379.

[13] H.T. Tran and H. Usami, Asymptotic behavior of positive solutions of a Lanchester-type model, Differential Equations and Applications, **9** (2017) 241–252.