J. Math. Tokushima Univ.
Vol. 58 (2024), 59 − 72

Axiomatic Method of Measure and Integration (X). Definition of Curvilinear L-integral and its Fundamental Properties

By

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(Received September 30, 2024)

Abstract

In this paper, we define the concept of the length of a space curve C in the space $\boldsymbol{R}^d$ in the Lebesgue's sense. Here we assume $d \geq 2$.

Then we consider the continuous mapping $\varphi : I \to C$, which is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t), \ (a \leq t \leq b).$$

Then we define the measure space $(C, \ \mathcal{M}_C, \ \nu)$ of Lebesgue type. This is the image measure space on the curve C of $(I, \ \mathcal{M}, \ \mu)$ on the interval $I = [a, \ b]$ by the mapping φ. Then we define the curvilinear L-integral on C and we study its fundamental properties. The result is a new and exact formulation.

2000 Mathematics Subject Classification. Primary, 28Axx.

Introduction

This paper is the part X of the series of papers on the axiomatic method of measure and integration. As for the details, we refer to Ito [15]. Further we refer to Ito [1]∼[14], [16]∼[30].

In this paper, we study the concept of the length of a space curve in the space $\boldsymbol{R}^d$ in the Lebesgue's sense.

In this paper, we study the concept of the curvilinear integral of the Lebesgue type on a space curve C in the space $\boldsymbol{R}^d$ and we study their fundamental properties. Here we assume $d \geq 2$. We say simply that the curvilinear integral of the Lebesgue type is the **curvilinear L-integral**.

Since the curvilinear L-integral is a special case of the LS-integral, the properties of the LS-integral are right for the curvilinear L-integral.

The curvilinear L-integral in this paper is a generalization of the curvilinear R-integral in Ito [30].

Then we study their fundamental properties.

Here I am most grateful to my wife Mutuko for her help of typesetting of the file of this manuscript.

1 Length of a space curve

In this section, we study the concept of the length of a space curve in the space $\boldsymbol{R}^d$ in the Lebesgue's sense. Here we assume $d \geq 2$.

In this section, we choose the standard basis $\{\boldsymbol{i}_1,\ \boldsymbol{i}_2,\ \cdots,\ \boldsymbol{i}_d\}$ of the space $\boldsymbol{R}^d$ and fix it. Thereby the orthogonal coordinate $\boldsymbol{r} = {}^t(x_1,\ x_2,\ \cdots,\ x_d)$ of the point $\boldsymbol{r}$ is determined.

In general, we assume that a curve is a continuous curve in $\boldsymbol{R}^d$.

Then we study the definition of the length of the curve in the space $\boldsymbol{R}^d$ in the Lebesgue's sense and we study the condition in order that the curve in $\boldsymbol{R}^d$ has the length in the Lebesgue's sense.

Here, a countable division Δ of a closed interval $[a,\ b]$ is given by the dividing points

$$a = t_0 < t_1 < t_2 < \cdots < t_\infty = b.$$

Then we obtain the dividing points on the curve C:

$$\boldsymbol{r}_0 = \boldsymbol{r}(a),\ \boldsymbol{r}_1,\ \boldsymbol{r}_2,\ \cdots,\ \boldsymbol{r}_\infty = \boldsymbol{r}(b),$$

$$\left(\, \boldsymbol{r}_j = \boldsymbol{r}(t_j);\ j = 1,\ 2,\ \cdots,\ \infty \,\right).$$

Thus we define the length L_Δ of the polygonal line obtained by combining these dividing points on the curve C. Then L_Δ is equal to the formula

$$L_\Delta = \sum_{j=1}^{\infty} \overline{\boldsymbol{r}_{j-i}\boldsymbol{r}_j} = \sum_{j=1}^{\infty} \sqrt{\sum_{i=1}^{d} \left(x_i(t_j) - x_i(t_{j-1}) \right)^2}.$$

If L_Δ's corresponding to all countable divisions Δ of $[a,\ b]$ is bounded, we define that the supremum

$$L = \sup_{\Delta} L_\Delta$$

is the length of the curve C in the Lebesgue's sense.

If a space curve C in $\boldsymbol{R}^d$ has the length in the Lebesgue's sense, we say that this space curve C is rectifiable in the Lebesgue's sense. In this case, we simply say that the space curve C is **rectifiable**.

Then we have the theorem in the following.

Theorem 1.1 *A space curve*

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t\left(x_1(t),\ x_2(t),\ \cdots,\ x_d(t) \right),\ (a \le t \le b)$$

is rectifiable if and only if the coordinate functions $x_1 = x_1(t),\ x_2 = x_2(t),$ $\cdots,\ x_d = x_d(t)$ *are of bounded variation.*

Theorem 1.2 *We assume that a space curve*

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t) \cdots,\ x_d(t)),\ (a \le t \le b)$$

is rectifiable.

Then the lengths L_Δ*'s of all polygonal lines corresponding to all countable divisions* Δ *of the closed interval* $[a,\ b]$ *are bounded and we have its supremum*

$$L = \sup_{\Delta} L_\Delta.$$

Then we have the limit

$$L = \lim_{\Delta} L_\Delta$$

in the sense of Moore-Smith limit.

Here we remark that the length of the curve in the above does not change even if we change the parameter of the curve.

Namely, when a parameter t is expressed by a monotone increasing continuous function

$$t = t(\tau),\ (\alpha \le \tau \le \beta)$$

of another parameter τ, the length of the curve C is invariant even if we express the curve C by the formula

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t(\tau)),\ (\alpha \leq \tau \leq \beta).$$

In the sequel, we study the rectifiable space curve. Therefore, we assume that, in the curve

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t\left(\ x_1(t),\ x_2(t),\ \cdots,\ x_d(t)\ \right),\ (a \leq t \leq b),$$

the coordinate functions $x_1 = x_1(t),\ x_2 = x_2(t),\ \cdots,\ x_d = x_d(t)$ are the continuous functions of bounded variation on the closed interval $[a,\ b]$.

Then, if, for $a \leq t \leq t' \leq b$, we denote the arc combining two points $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t')$ as $C(t,\ t')$, it is evident that the curve $C(t,\ t')$ is rectifiable.

We denote the length of this curve $C(t,\ t')$ as $s(t,\ t')$.

Then, for $a \leq t < t' < t'' \leq b$, we have the equality

$$s(t,\ t') + s(t',\ t'') = s(t,\ t'').$$

Further, we give the orientation of the arc and we assume that the arc length measured in the positive direction is positive and the arc length measured in the negative direction is negative. Then we have the equality

$$s(t,\ t') = -s(t',\ t)$$

for $a \leq t < t' \leq b$.

Therefore, we have the equality

$$s(t,\ t') + s(t',\ t'') = s(t,\ t'')$$

independent of the order relation of $t,\ t',\ t'' \in [a,\ b]$.

Here, if we put

$$s = s(t) = s(a,\ t)$$

for $a \leq t \leq b$, we have the equality

$$s(t,\ t') = s(t') - s(t)$$

for $[t,\ t'] \subset [a,\ b]$, Therefore $s = s(t)$ is a monotone increasing continuous function on $[a,\ b]$.

Thus it is a continuous function of bounded variation.

Then the length $s = s(t)$ of the arc $C(a,\ t)$ is expressed by the LS-integral

$$s = \int_0^t ds(t).$$

Its value is included in the closed interval $[0,\ L]$.

Next we study the construction of the image measure space $(C,\ \mathcal{M}_C,\ \nu)$ on the curve C of the Lebesgue measure space $(I,\ \mathcal{M},\ \mu)$ on the closed interval $I = [a,\ b]$.

Here we assume that the Lebesgue measure space $(I,\ \mathcal{M},\ \mu)$ is defined on the interval $I = [a,\ b]$ in $\boldsymbol{R}$. Then we have Theorem 1.3 in the following.

Theorem 1.3 *We assume that we have $I = [a,\ b]$ and $(I,\ \mathcal{M},\ \mu)$ is the Lebesgue measure space on I. Then we have the statements* (1)~(4) *in the following*:

(1) *If $A \in \mathcal{M}$ holds, we have $0 \leq \mu(A) \leq b - a$.*

(2) *If a countable number of $A_p \in \mathcal{M}$, $(p = 1,\ 2,\ 3,\ \cdots)$ are mutually disjoint and we have the formula*

$$A = \bigcup_{p=1}^{\infty} A_p,$$

we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

(3) *For arbitrary real numbers c and d so that we have $a \leq c < d \leq b$, we have $\mu([c,\ d]) = d - c$.*

(4) *If $A,\ B \in \mathcal{M}$ are congruent, we have the equality $\mu(A) = \mu(B)$.*

We assume that a space curve C is rectifiable. Here we assume that a continuous mapping

$$\varphi:\ I \to C$$

is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (t \in I).$$

Then we can define the measure space $(C,\ \mathcal{M}_C,\ \nu)$ of Lebesgue type on the curve C by using the mapping φ.

This measure space is constructed as the image measure space on the curve C of the Lebesgue measure space $(I,\ \mathcal{M},\ \mu)$ on the interval I.

Thereby we construct the mathematical model of the concept of the length of the curve on C in the natural manner.

Namely, we have Theorem 1.4 in the following.

Theorem 1.4 *Assume that a curve C is rectifiable in the Lebesgue's sense. Then we have the Lebesgue type measure space $(C,\ \mathcal{M}_C,\ \nu)$ on C and we have the statements* (1)~(3) *in the following*:

(1) *If we have* $E \in \mathcal{M}_C$, *we have* $0 \leq \nu(E) \leq L$. *Here* L *denotes the length of the curve* C *in the Lebesgue's sense.*

(2) *If a countable number of* $E_p \in \mathcal{M}_C$, $(p = 1,\ 2,\ 3,\ \cdots)$ *are mutually disjoint and we have the equality*

$$E = \bigcup_{p=1}^{\infty} E_p,$$

we have the equality

$$\nu(E) = \sum_{p=1}^{\infty} \nu(E_p).$$

(3) *We have* $E \in \mathcal{M}_C$ *if and only if we have* $\varphi^{-1}(E) \in \mathcal{M}$. *Then we have the equality*

$$\nu(E) = \int_{\varphi^{-1}(E)} ds(t).$$

The integral of the right hand side of the equality in the above is the LS-*integral. Here* $s = s(t)$ *is the function which expresses the length of the arc* $C(a,\ t)$ *in the Lebesgue's sense.*

Therefore, by virtue of Theorem 1.4, we have the equality

$$\nu([t,\ t']) = s(t') - s(t) = s(t,\ t')$$

for $[t,\ t'] \subset [a,\ b]$.

Next, as the special case, we consider the C^1-class curve

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b)$$

in the space $\boldsymbol{R}^d$.

Then a C^1-class mapping is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b).$$

Then we can define the Lebesgue type measure space $(C,\ \mathcal{M}_C,\ \nu)$ on the curve C as the image measure space of $(I,\ \mathcal{M},\ \mu)$ by virtue of this mapping φ.

This is the mathematical model of the concept of the length of the curve in the Lebesgue sense.

Thus we have Theorem 1.5 in the following.

Theorem 1.5 *There exists the Lebesgue type measure space* $(C,\ \mathcal{M}_C,\ \nu)$ *on a* C^1*-class curve* C *and we have the statements* (1)$\sim$(3) *in the following*:

(1) *If we have* $E \in \mathcal{M}_C$, *we have* $0 \leq \nu(E) \leq L$. *Here* L *denotes the length of the curve* C *in the Lebesgue's sense.*

(2) *If a countable number of* $E_p \in \mathcal{M}_C$, $(p = 1,\ 2,\ 3,\ \cdots)$ *are mutually disjoint and we have the equality*

$$E = \bigcup_{p=1}^{\infty} E_p,$$

we have the equality

$$\nu(E) = \sum_{p=1}^{\infty} \nu(E_p).$$

(3) *We have* $E \in \mathcal{M}_C$ *if and only if we have* $\varphi^{-1}(E) \in \mathcal{M}$.
Then we have the equality

$$\nu(E) = \int_{\varphi^{-1}(E)} \|\dot{\boldsymbol{r}}(t)\| dt.$$

Here we have the formula

$$\|\dot{\boldsymbol{r}}(t)\| = \sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2 + \cdots + \dot{x}_d(t)^2}.$$

2 Definition of the curvilinear L-integral and its fundamental properties

In this section, we consider a curve C in $\boldsymbol{R}^d$. Here we assume $d \geq 2$. Then, we define the curvilinear L-integral as the integral of a scalar function defined on a curve C and we study its fundamental properties.

In this section, we call the curvilinear integral in the Lebesgue's sense as the curvilinear L-integral for the simplicity.

In general, we assume that the curve C is a rectifiable continuous curve in the Lebesgue's sense.

Here, by virtue of Theorem 1.4, we assume that the Lebesgue type measure space $(C,\ \mathcal{M}_C,\ \nu)$ is defined on the curve C.

Then, we define the function $f(\boldsymbol{r})$ is measurable in the general manner. Here, we consider a function $f(\boldsymbol{r}) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on the curve C.

We assume that a function $f(\boldsymbol{r})$ on C is an extended real-valued function. Then we denote

$$E(\infty) = \{\boldsymbol{r} \in C; |f(\boldsymbol{r})| = \infty\}.$$

Further we assume that $E(\infty) \in \mathcal{M}_C$ and $\nu(E(\infty)) = 0$.

At first, we define a simple function.

Namely, we define that a scalar function $f(\boldsymbol{r}) = f(x_1,\ x_2,\ \cdots,\ x_d)$ is a **simple function** if we have the expression

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r}),\ (a_p \in \boldsymbol{R} \cup \{\pm\infty\},\ (p \geq 1))$$

for an arbitrary countable division of C

$$\Delta : C = E_1 + E_2 + E_3 + \cdots, (E_p \in \mathcal{M}_C,\ (p \geq 1)).$$

Here a_p is equal to a real number or $\pm\infty$ and they are not necessarily different each other. $\chi_{E_p}(\boldsymbol{r})$ denotes the defining function of the set E_p.

Here we assume that we have

$$E(\infty) = \{\boldsymbol{r} \in C; |f(\boldsymbol{r})| = \infty\}$$

and

$$E(\infty) \in \mathcal{M}_C, \nu(E(\infty)) = 0.$$

Next, we define that a scalar function $f(\boldsymbol{r})$ is measurable.

Definition 2.1 Assume that a curve C is rectifiable in the Lebesgue's sense. Then we define that a scalar function $f(\boldsymbol{r})$ defined on C is **measurable** if there exist a certain sequence $\{f_n(\boldsymbol{r})\}$ of simple functions defined on C such that we have the limit

$$\lim_n f_n(\boldsymbol{r}) = f(\boldsymbol{r})$$

in the sense of point wise convergence almost everywhere on C.

If we denote

$$E_n(\infty) = \{\boldsymbol{r} \in C; |f_n(\boldsymbol{r})| = \infty\},$$

we assume that we have

$$E_n(\infty) \subset E(\infty), (n \geq 1)$$

and

$$\nu(E(\infty)) = 0.$$

Example 2.1 A simple function and a continuous function on a rectifiable curve C are measurable.

Next we define the curvilinear L-integral for a measurable function defined on C.

We define this in two steps.

(1) The case where $f(\boldsymbol{r})$ is a simple function

Then we assume that we have the expression

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r}),\ (a_p \in \boldsymbol{R} \cup \{\pm\infty\},\ (p \geq 1))$$

for a certain countable division

$$\Delta : C = E_1 + E_2 + E_3 + \cdots, (E_p \in \mathcal{M}_C,\ (p \geq 1)).$$

Then we define the curvilinear L-integral on C of $f(\boldsymbol{r})$ by the formula

$$R = \int_C f(\boldsymbol{r}) d\nu = \sum_{p=1}^{\infty} a_p \nu(E_p).$$

Here we assume that the series in the third side of the formula in the above is absolutely convergent. The value of this curvilinear L-integral is determined independent to the choice of the expression of $f(\boldsymbol{r})$ as a simple function.

(2) The case where $f(\boldsymbol{r})$ is a general measurable function

Then, by virtue of Definition 2.1, there exists a certain sequence $\{f_n(\boldsymbol{r})\}$ of simple functions such that it converges to $f(\boldsymbol{r})$ almost everywhere on C in the sense of point wise convergence.

Here, if we have the limit

$$R = \lim_n \int_C f_n(\boldsymbol{r}) d\nu,$$

we define that this limit R is a **curvilinear L-integral** of a measurable function $f(\boldsymbol{r})$ on C, and we denote it as

$$R = \int_C f(\boldsymbol{r}) d\nu.$$

This value R does not depend on the choice of a sequence $\{f_n(\boldsymbol{r})\}$ of simple functions which converges to $f(\boldsymbol{r})$ almost every where on C.

This integral in the right hand side of the formula in the above is a Lebesgue-Stieltjies integral.

Further we can define the curvilinear L-integral of a measurable vector function $\boldsymbol{a} = {}^t(a_1,\ a_2,\ \cdots,\ a_p)$ on C in the similar way. This is equivalent to define the curvilinear L-integral of a scalar function corresponding to each one of p-components $a_1,\ a_2,\ \cdots,\ a_p$. Here we assume $p \geq 2$.

By virtue of the definition of the curvilinear L-integral, we have soon the formulas in the following.

Theorem 2.1 *Assume that a curve C is rectifiable in the Lebesgue's sense. We use the notation in the above. Then we have the statements* (1) *and* (2) *in the following*:

(1) *For two integrable vector functions* $\boldsymbol{a}_1$ *and* $\boldsymbol{a}_2$ *on* C, *we have the equality*

$$\int_C (\boldsymbol{a}_1 + \boldsymbol{a}_2) d\nu = \int_C \boldsymbol{a}_1 d\nu + \int_C \boldsymbol{a}_2 d\nu.$$

(2) *For a integrable vector function* $\boldsymbol{a}$ *on* C *and a real constant* λ, *we have the equality*

$$\int_C \lambda \boldsymbol{a} d\nu = \lambda \int_C \boldsymbol{a} d\nu.$$

Further we have Theorem 2.2 in the following.

Theorem 2.2 *Assume that a rectifiable curve* C *in the Lebesgue's sense is divided into two arcs*

$$C_1 : \ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t), \ x_2(t), \ \cdots, \ x_d(t)), \ (a \leq t \leq c),$$

$$C_2 : \ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t), \ x_2(t), \ \cdots, \ x_d(t)), \ (c \leq t \leq b).$$

Then we have the equality

$$\int_C \boldsymbol{a} d\nu = \int_{C_1} \boldsymbol{a} d\nu + \int_{C_2} \boldsymbol{a} d\nu$$

for an integrable vector function $\boldsymbol{a}$ *on* C.

Especially, when a curve C in $\boldsymbol{R}^d$ is a C^1-class curve, the curvilinear L-integral can be expressed in the following.

(3) The case where $f(\boldsymbol{r})$ is a simple function

Here we have the expression

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r}), \ (a_p \in \boldsymbol{R} \cup \{\pm\infty\}, \ (p \geq 1))$$

for a certain countable division of C

$$\Delta : \ C = E_1 + E_2 + E_3 + \cdots, \ (E_p \in \mathcal{M}_C, \ (p \geq 1)).$$

Then, by virtue of Theorem 1.5, (3), we have the equality

$$\int_C f(\boldsymbol{r}) d\nu = \sum_{p=1}^{\infty} a_p \int_{\varphi^{-1}(E_p)} \|\dot{\boldsymbol{r}}(t)\| dt$$

$$= \int_a^b f(\boldsymbol{r}(t)) \|\dot{\boldsymbol{r}}(t)\| dt.$$

(4) The case where $f(\boldsymbol{r})$ is a general integrable function

The curvilinear L-integral of an integrable function $f(\boldsymbol{r})$ on C is expressed as

$$R = \int_C f(\boldsymbol{r})d\nu = \int_a^b f(\boldsymbol{r}(t))\|\dot{\boldsymbol{r}}(t))\|dt.$$

In the similar way, we can define the curvilinear L-integrals on C

$$\int_C f(\boldsymbol{r})dx_j = \int_a^b f(\boldsymbol{r}(t))\dot{x}_j(t)dt, \ (1 \le j \le d).$$

Every kind of the expression of these curvilinear L-integrals depends on the choice of the parameter of the curve C. In general, it is known that the value of the curvilinear L-integral on C does not depend on the choice of a parameter of the curve C.

Especially, for a measurable vector function $\boldsymbol{a} = {}^t(a_1,\ a_2,\ \cdots, a_d)$ on C, we have the formula of the curvilinear L-integral in the following

$$\int_C a_1dx_1 + a_2dx_2 + \cdots + a_ddx_d = \int_a^b (a_1\dot{x}_1 + a_2\dot{x}_2 + \cdots + a_d\dot{x}_d)dt$$

as the special combination of the curvilinear L-integrals of the components.

Here, if the curve C is regular, we have the condition

$$\dot{x}_1(t)^2 + \dot{x}_2(t)^2 + \cdots + \dot{x}_d(t)^2 \neq 0.$$

Hence we can take the arc length s measured from $t = a$ as a parameter.

Then, since the tangent unit vector of the curve C is $\boldsymbol{t} = {}^t(x_1'(s),\ x_2'(s),\ \cdots,\ x_d'(s))$, we can express

$$\int_C a_1dx_1 + a_2dx_2 + \cdots + a_ddx_d = \int_0^L (\boldsymbol{a}, \boldsymbol{t})ds.$$

Here L denotes the length of C. The right hand side of the formula in the above can also be expressed as

$$\int_C \boldsymbol{a}\cdot\boldsymbol{t}ds.$$

Further $\boldsymbol{a}\cdot\boldsymbol{t}$ denotes the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{t}$.

Further, if we express

$$d\boldsymbol{r} = \boldsymbol{t}ds$$

by using the vector $\boldsymbol{r}$ expressing the point $\boldsymbol{r}$ on C, the curvilinear L-integral in the above is expressed as

$$\int_C \boldsymbol{a}\cdot d\boldsymbol{r} \text{ or } \int_C (\boldsymbol{a},\ d\boldsymbol{r}).$$

Here we remark that we have

$$d\boldsymbol{r} = {}^t(dx_1,\ dx_2,\ \cdots,\ dx_d).$$

Further, the symbol $(\boldsymbol{a},\ d\boldsymbol{r})$ denotes the inner product.

If we denote the angle of $\boldsymbol{a}$ and $\boldsymbol{t}$ at each point on the curve C as θ, we have the equality

$$\int_C \boldsymbol{a} \cdot d\boldsymbol{r} = \int_0^L \|\boldsymbol{a}\| \cos\theta ds.$$

By using the notation in the above, we have the formulas of the curvilinear L-integrals in the following.

Theorem 2.3 *For a C^1-class regular curve C, we use the notation in the above. Then, we have the statements* (1) *and* (2) *in the following*:

(1) *For two measurable vector functions $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ on C, we have the equality in the following*:

$$\int_C (\boldsymbol{a}_1 + \boldsymbol{a}_2) \cdot d\boldsymbol{r} = \int_C \boldsymbol{a}_1 \cdot d\boldsymbol{r} + \int_C \boldsymbol{a}_2 \cdot d\boldsymbol{r}.$$

(2) *For a measurable vector function $\boldsymbol{a}$ on C and a real constant λ, we have the equality in the following*:

$$\int_C \lambda\boldsymbol{a} \cdot d\boldsymbol{r} = \lambda \int_C \boldsymbol{a} \cdot d\boldsymbol{r}.$$

Further we have Theorem 2.4 in the following.

Theorem 2.4 *If a C^1-class regular curve C is divided into two arcs*

$$C_1 :\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \le t \le c),$$

$$C_2 :\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (c \le t \le b),$$

we have the equality

$$\int_C \boldsymbol{a} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{a} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{a} \cdot d\boldsymbol{r}$$

for a measurable vector function $\boldsymbol{a}$ on C.

Although the curve C in the above is continuous at $t = c$ but it is not differentiable at $t = c$, we can extend the definition of the curvilinear L-integral so that we have the similar formula as above. Since the curvilinear L-integral is a special case of the LS-integral, we refer to Ito [14], Chapters 7 and 11.

References

[1] Yoshifumi Ito, *Analysis* I, Science House, 1991. (In Japanese).

[2] ————-, *Analysis* II, Science House, 1998. (In Japanese). (Out of print).

[3] ————-, *Axioms of Arithmetic*, Science House, 1999. (In Japanese).

[4] ————-, *Method of Renormalization and Distributions*, J Math. Univ. Tokushima, **35**(2001), 35-55.

[5] ————, *Foundation of Analysis*, Science House, 2002. (In Japanese).

[6] ————, *Theory of Measure and Integration*, Science House, 2002. (In Japanese).

[7] ————-, *Analysis* II, (Rev. ed.), Science House, 2002. (In Japanese).

[8] ———, *Why the area is obtained by the integration*, Mathematics Seminar, **44**, **no.6**(2005), pp.50-53. (In Japanese).

[9] —————, *New Meanings of Conditional Convergence of the Integrals*, Real Analysis Symposium 2007, Osaka, pp.41-44. (In Japanese).

[10] —————-, *Definition and Existence Theorem of Jordan Measure*, Real Analysis Symposium 2010, Kitakyushu, pp.1-4. (In Japanese).

[11] —————-, *Differential and Integral Calculus*, Vol.II, −*Theory of* Riemann *Integral*–, preprint, 2010. (In Japanese).

[12] —————-, *Theory of* Lebesgue *Integral*, preprint, 2010. (In Japanese).

[13] —————-, *Differential and Integral Calculus*, Vol.I, −*Theory of Differentiation*–, preprint, 2011. (In Japanese).

[14] —————-, RS-*integral and* LS-*integral*, preprint, 2011. (In Japanese).

[15] —————-, *Curvilinear Integral and Surface Integral. Measure and Integration on a Riemannian Manifold*, preprint, 2011. (In Japanese).

[16] —————-, *Introduction to Analysis*, preprint, 2014. (In Japanese).

[17] —————-, *Differentiation of the Lebesgue-Stieltjes Measure*, Real Analysis Symposium 2016, Nara, pp.7-12. (In Japanese).

[18] —————-, *Axiomatic Method of Measure and Integration* (I). *Definition and Existence Theorem of the* Jordan *Measure*, J.Math, Tokushima Univ., **52**(2018), 1-16.

[19] ——————-, *Axiomatic Method of Measure and Integration* (II). *Definition of the* Riemann *Integral and its Fundamental Properties*, J. Math. Tokushima Univ., **52**(2018), 17-38.

[20] ——————, *Axiomatic Method of Measure and Integration* (III). *Definition and Existence Theorem of the* Lebesgue *Measure*, J. Math. Tokushima Univ., **53**(2019), 5-26.

[21] ——————-, *Axiomatic Method of Measure and Integration* (IV). *Definition of the* Lebesgue *Integral and its Fundamental Properties*, J. Math. Tokushima Univ., **53**(2019), 27-53.

[22] ——————-, *Axiomatic Method of Measure and Integration* (V). *Definition and Existence Theorem of the* RS-*measure*, J. Math. Tokushima Univ., **54**(2020), 19-40.

[23] ——————-, *Axiomatic Method of Measure and Integration* (VI). *Definition of the* RS-*integral and its Fundamental Properties*, J. Math. Tokush ima Univ., **54**(2020), 41-55.

[24] ——————-, *Axiomatic Method of Measure and Integration* (XIV). *The Measure and the Integration on a* Rieamannian *Manifold*, J. Math. Tokushima Univ., **55**(2021), 19-49.

[25] ——————-, *Study on the Spectra of Hydrogen Type Ions*, J. Math. Tokushima Univ., **55**(2021), 51-73.

[26] ——————-, *Axiomatic Method of Measure and Integration* (VII). *Definition and Existence Theorem of* LS-*measure*, J. Math. Tokushima Univ., **56**(2022).

[27] ——————-, *Axiomatic Method of Measure and Integration* (VIII). *Definition of the* LS-*integral and its Fundamental Properties*, J. Math. Tokushima Univ., **56**(2022)

[28] ——————-, *Axiomatic Set Theory*, J. Math. Tokushima Univ., **57**(2023), 1-18.

[29] ——————-, *System of Axioms of Euclidean Geometry*, J. Math. Tokushima Univ., **57**(2023), 93-100.

[30] ——————-, *Axiomatic Method of Measure and Integration* (IX). *Definition of the Curvilinear* R-*Integral and its Fundamental Properties*, J.Math. Tokushima Univ., **58**(2024), 33-57.