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Axiomatic Method of Measure and Integration (IX). Definition of Curvilinear R-integral and its Fundamental Properties

By

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Abstract

In this paper, we define the concept of the length of a plane curve C. Then we define the measure space $(C, \mathcal{B}_C, \nu)$ as the image measure space on the curve C of the Jordan measure space $(I, \mathcal{B}, \mu)$ on the interval I. These measure spaces are the measure spaces of Jordan type.

Then we define the curvilinear R-integral of a measurable function on C and we study its fundamental properties by using the axiomatic method of measure and integration.

The result is a new and exact formulation.

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Introduction

This paper is the part IX of the series of papers on the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to Ito [15]. Further we refer to Ito [1] $\sim$ [14], [16] $\sim$ [29].

In this paper, we define the concept of the length of a plane curve.

Then we define the curvilinear integral of the Riemann type on the curve C. We happen to say that the curvilinear integral of the Riemann type is the **curvilinear R-integral**.

Since the curvilinear R-integral is a special case of the RS-integral, the properties of the RS-integral are still true for the curvilinear R-integral.

Then we study their fundamental properties.

Here I am most grateful to my wife Mutuko for her help of typesetting of the file of this manuscript.

1 Length of a plane curve

In this section, we study the concept of the length of a plane curve of Riemann type and its fundamental properties.

In the parameter expression of a plane curve C, we have a formula

$$C : x = x(t),\ y = y(t),\ (a \leq t \leq b).$$

Here a and b are two real numbers so that $a < b$ holds. Here we assume that the plane curve C is a continuous curve.

Then the plane curve C has the length if and only if the coordinate functions $x(t)$ and $y(t)$ are two functions of bounded variation. In this case, the plane curve C is said to be **rectifiable**.

1.1 Function of bounded variation

In this subsection, we define the concept of a function of bounded variation and study its fundamental properties.

Now we assume that a function $y = f(x)$ is defined on a closed interval $I = [a,\ b]$. Here we have $a < b$. Then a finite division Δ defined by the subintervals of the closed interval $I = [a,\ b]$ is given by the dividing points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Here, for this division Δ, we define the sum

$$V_\Delta = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|.$$

We say that V_Δ is the **variation** corresponding to the finite division Δ of the closed interval I .

Then we say that the function $y = f(x)$ is **of bounded variation** on the closed interval I if the set of V_Δ is bounded for all finite divisions Δ of the closed interval I. We say that such a function $y = f(x)$ is a **function of bounded variation**.

Here we put

$$V = \sup_\Delta V_\Delta.$$

Here V_Δ's are defined for all finite divisions Δ of the closed interval I. Then we say that this supremum V is the **total variation** of the function $y = f(x)$ on the closed interval I.

Then we have the following proposition.

Proposition 1.1 *A function $y = f(x)$ of bounded variation defined on a closed interval $I = [a, b]$ is bounded on the closed interval I.*

Here, in the defining formula of the variation V_Δ of $y = f(x)$ for a finite division Δ of $I = [a, b]$, we denote the partial sum of the nonnegative terms $f(x_j) - f(x_{j-1})$ as P_Δ and the partial sum of the nonpositive terms $f(x_j) - f(x_{j-1})$ as $-N_\Delta$. Then we have the equalities

$$V_\Delta = P_\Delta + N_\Delta, \ f(b) - f(a) = P_\Delta - N_\Delta.$$

Therefore, if $y = f(x)$ is of bounded variation on $I = [a, b]$, $\{P_\Delta\}_\Delta$ denotes the set of P_Δ corresponding to all finite divisions Δ of $I = [a, b]$. As for $\{N_\Delta\}_\Delta$, we have the same as above. Then $\{P_\Delta\}_\Delta$ and $\{N_\Delta\}_\Delta$ are bounded.

Since $\{P_\Delta\}_\Delta$ and $\{N_\Delta\}_\Delta$ are bounded, we denote their supremum as

$$P = \sup_\Delta P_\Delta, \ N = \sup_\Delta N_\Delta.$$

Then we have the equality

$$V = P + N, \ f(b) - f(a) = P - N.$$

Then we say that, P, N and V are the **positive variation**, the **negative variation** and the **total variation** of $y = f(x)$ on $I = [a, b]$ respectively.

We assume that a function $y = f(x)$ is of bounded variation on the closed interval $I = [a, b]$. Then, for an arbitrary point x in $[a, b]$, $y = f(x)$ is of bounded variation on the closed interval $[a, x]$. Therefore, since V, P and N for the interval $[a, x]$ are three functions of x respectively. Then we have the equalities

$$V(x) = P(x) + N(x), \ f(x) - f(a) = P(x) - N(x).$$

Then we say that $P(x)$, $N(x)$ and $V(x)$ are the **positive variation**, the **negative variation** and the **total variation** of $y = f(x)$ on each interval $[a, x]$ respectively.

By virtue of the definitions, we have the following proposition.

Proposition 1.2 *We use the notation in the above. Then* $P(x)$ *and* $N(x)$ *are two monotone increasing functions in the wider sense on* $[a,\ b]$.

In general, we happen to say that a monotone increasing function in the wider sense on $[a,\ b]$ is a monotone nondecreasing function.

By virtue of the definitions of $P(x)$ and $N(x)$, we have Theorem in the following.

Theorem 1.1 *A function of bounded variation defined on the closed interval* $[a,\ b]$ *is equal to the difference of two monotone increasing functions in the wider sense.*

The inverse of Theorem 1.1 in the above is also true. Namely we have Theorem 1.2 in the following.

Theorem 1.2 *We assume that a function* $y = f(x)$ *is equal to a difference of two monotone increasing functions in the wider sense defined on the closed interval* $[a,\ b]$. *Then the function* $y = f(x)$ *is of bounded variation on the interval* $[a,\ b]$.

Theorem 1.3 *If two functions* $f(x)$ *and* $g(x)$ *are of bounded variation on the closed interval* $[a,\ b]$, *the following functions* (1)$\sim$(4) *are also of bounded variation on the interval* $[a,\ b]$:

(1) $\alpha f(x)$. *Here* α *is a nonzero real constant.* (2) $f(x) \pm g(x)$.

(3) $f(x)g(x)$. (4) $f(x)/g(x)$. *Here there exists a positive constant* $m > 0$ *such that we have the condition* $|g(x)| \geq m > 0,\ (x \in [a,\ b])$.

Here, if a function $f(x)$ is of bounded variation on $[a,\ b]$, we denote its total variation as $V(a,\ b)$. Then we have the following Theorem 1.4.

Theorem 1.4 *We use the notation in the above. Then, for* $c \in [a,\ b]$, *we have the equality*

$$V(a,\ b) = V(a,\ c) + V(c,\ b).$$

Here, if a function $y = f(x)$ is of bounded variation on $[a,\ b]$, we denote the positive variation and the negative variation of $y = f(x)$ on $[a,\ b]$ as $P(a,\ b)$ and $N(a,\ b)$ respectively.

Then we have the following Corollary 1.1.

Corollary 1.1 *We use the notation in the above. Then, for* $c \in [a,\ b]$, *we have the equalities*:

$$P(a,\ b) = P(a,\ c) + P(c,\ b),$$

$$N(a,\ b) = N(a,\ c) + N(c,\ b).$$

Theorem 1.5 *If a function $y = f(x)$ is continuous and of bounded variation on the closed interval $[a,\ b]$, $P(x)$, $N(x)$ and $V(x)$ are also continuous on $[a,\ b]$.*

Example 1.1 A piece-wise monotone function on a closed interval $[a,\ b]$ is of bounded variation.

Example 1.2 The function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & (0 < x \leq \dfrac{1}{\pi}), \\ 0, & (x = 0) \end{cases}$$

is not of bounded variation on the closed interval $[0,\ \dfrac{1}{\pi}]$.

By virtue of Example 1.1, a function of bounded variation need not be continuous.

Further, by virtue of Example 1.2, a continuous function need not be of bounded variation.

Therefore, the conditions such as the continuity and the property of bounded variation for some functions do not have any direct relation.

Proposition 1.3 *We assume that a function $f(x)$ is* R*-integrable on a closed interval $[a,\ b]$. Then the integral function*

$$F(x) = \int_a^x f(t)dt,\ (x \in [a,\ b])$$

is of bounded variation on $[a,\ b]$.

Corollary 1.2 *If a function $f(x)$ is a C^1-function on $[a,\ b]$, $f(x)$ is of bounded variation on $[a,\ b]$.*

At last, we show that we can construct a RS-measure space on $[a,\ b]$ by using a left-continuous function of bounded variation or a right-continuous function of bounded variation. Therefore we can define the RS-integral for a RS-measurable function on $[a,\ b]$.

As for the details of the RS-integral, we refer to Ito [14].

1.2 Definition of the length of a plane curve

In this subsection, we define the concept of the length of a plane curve.

As for the length of a plane curve, we construct the measure space of Jordan type on the plane curve as a mathematical model of the length of Jordan type of a plane curve.

Now we fix a standard basis $\{\boldsymbol{i},\ \boldsymbol{j}\}$ on the plane $\boldsymbol{R}^2$. Thereby we determine the orthogonal coordinates $\boldsymbol{r} = {}^t(x,\ y)$ of a point $\boldsymbol{r}$.

We consider a continuous plane curve

$$C : x = x(t),\ y = y(t),\ (a \leq t \leq b).$$

Namely, if two functions $x = x(t)$ and $y = y(t)$ are the continuous functions defined on the closed interval $[a,\ b]$, we say that the locus C of a point $\boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x(t),\ y(t))$ in $\boldsymbol{R}^2$ for $t \in [a,\ b]$ is a **continuous curve**.

We happen to say that this continuous curve is simply a **curve**. We say that the variable t is the parameter of the curve C.

We say that the curve C is a **closed curve** if we have the condition $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. Further we say that the curve C is an open curve if it is not a closed curve. We say that, for certain $t_1,\ t_2 \in [a,\ b]$ such as $t_1 \neq t_2$ holds, the point $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$ is a double point if we have the condition $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$. We say that a curve C is a **simple curve** or a **Jordan curve** if it does not contain any double point other than both end points $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$.

In the sequel, for the simplicity of the statement, we assume that a considered curve is a Jordan curve.

Here, if a finite division Δ of the closed interval $[a,\ b]$ is defined by the dividing points

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

we obtain the dividing points on the curve C

$$\boldsymbol{r}_0 = \boldsymbol{r}(a),\ \boldsymbol{r}_1,\ \boldsymbol{r}_2 \cdots,\ \boldsymbol{r}_n = \boldsymbol{r}(b),$$

$$\big(\ \boldsymbol{r}_j = \boldsymbol{r}(t_j),\ j = 0,\ 1,\ 2,\ \cdots n\ \big)$$

and we define the length of the line graph obtained by combining these dividing points by some line segments by virtue of the formula

$$\begin{aligned} L_\Delta &= \sum_{j=1}^{n} \overline{\boldsymbol{r}_{j-1}\boldsymbol{r}_j} \\ &= \sum_{j=1}^{n} \sqrt{(x(t_j) - x(t_{j-1}))^2 + (y(t_j) - y(t_{j-1}))^2}. \end{aligned} .$$

If L_Δ's for all finite divisions Δ of $[a,\ b]$ is bounded, we define that its supremum

$$L = \sup_{\Delta} L_{\Delta}$$

is the **length of the curve** C.

Then we have the following Theorem.

Theorem 1.6 *A plane curve*

$$C: \ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x(t),\ y(t)),\ (a \le t \le b)$$

has the length if and only if two coordinate functions $x = x(t)$ *and* $y = y(t)$ *are of bounded variation on* $[a,\ b]$.

Theorem 1.7 *We assume that a plane curve*

$$C: \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x(t),\ y(t)),\ (a \le t \le b)$$

has the length. Then, the lengths L_{Δ} *of the line graphs corresponding for all finite divisions* Δ *of the closed interval* $[a,\ b]$ *are bounded and we have its supremum*

$$L = \sup_{\Delta} L_{\Delta}.$$

Then we have the limit

$$L = \lim_{\Delta} L_{\Delta}$$

in the sense of Moore-Smith limit.

In Theorem 1.7 in the above, we can prove that the set $\boldsymbol{\Delta}$ of all direct sum division

$$[a,\ b] = [a,\ t_1) + [t_1,\ t_2) + \cdots + [t_{n-1}, b) + \{b\}$$

by using the subintervals of the closed interval $[a,\ b]$ is a direct set with respect to the subdivision $\le$.

Therefore, $\{L_{\Delta};\ \Delta \in \boldsymbol{\Delta}\}$ is a direct family and we can consider the limit

$$\lim_{\Delta} L_{\Delta}$$

in the sense of Moor-Smith. Then we have the equality

$$L = \lim_{\Delta} L_{\Delta}$$

by virtue of Theorem 1.7.

Here we remark that the length of the curve in the above does not change by the transformation of parameter of the curve.

Namely, when the parameter t is expressed by a monotone continuous function

$$t = t(\tau),\ (\alpha \le \tau \le \beta)$$

of an another parameter τ, the length of the curve is invariant even if we express the curve C as the formula

$$C : \boldsymbol{r} = \boldsymbol{r}(t(\tau)),\ (\alpha \leq \tau \leq \beta).$$

We say that a plane curve is a **rectifiable** if it has the length.

In the sequel, we consider a rectifiable plane curve.

Therefore, in the curve

$$C :\ \boldsymbol{r}(t) = {}^t(x(t),\ y(t)),\ (a \leq t \leq b),$$

we assume that two coordinate functions $x = x(t)$ and $y = y(t)$ are the continuous functions of bounded variation on the closed interval $[a,\ b]$.

Then, for t and t' such as $a \leq t \leq t' \leq b$, we express the arc of the curve combining two points $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t')$ as $C(t,\ t')$. Then it is evident that the curve $C(t,\ t')$ is rectifiable.

We denote the length of this curve as $s(t,\ t')$. Then, for $t,\ t',\ t''$ such as $a \leq t < t' < t'' \leq b$, we have evidently the equality

$$s(t,\ t') + s(t',\ t'') = s(t,\ t'').$$

Further, we assume that, by the orientation of the arc of the curve, the arc length measured in the positive direction is positive and the arc length measured in the negative direction is negative. Then we have the equality

$$s(t,\ t') = -s(t',\ t)$$

for t and t' such as $a \leq t < t' \leq b$.

Therefore we have the equality

$$s(t,\ t') + s(t',\ t'') = s(t,\ t'')$$

independent to the order relation of $t,\ t',\ t'' \in [a,\ b]$.

Here, if we put

$$s = s(t) = s(a,\ t)$$

for $a \leq t \leq b$, we have the equality

$$s(t,\ t') = s(t') - s(t)$$

for $[t,\ t'] \subset [a,\ b]$. Therefore $s = s(t)$ is a continuous monotone increasing function on $[a,\ b]$. Therefore, it is a continuous function of bounded variation.

Then the length $s = s(t)$ of the arc $C(a,\ t)$ of the curve is expressed by the RS-integral

$$s = \int_0^t ds(t).$$

Its value belongs to the closed interval $[0,\ L]$.

Next we study the construction of the image measure space $(C,\ \mathcal{B}_C,\ \nu)$ on the curve C of the Jordan measure space $(I,\ \mathcal{B},\ \mu)$ on the closed interval $I = [a,\ b]$.

Then we have Theorem 1.8 in the following.

Theorem 1.8 *Put $I = [a,\ b]$. Let $(I,\ \mathcal{B},\ \mu)$ be the Jordan measure space on I. Then we have the statements* (1)$\sim$(4) *in the following*:

(1) *If $A \in \mathcal{B}$ holds, we have $0 \leq \mu(A) \leq b - a$.*

(2) *If at most countable number of $A_p \in \mathcal{B}$, $(p = 1,\ 2, \cdots)$ are mutually disjoint and we have the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p \in \mathcal{B},$$

we have the formula

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(3) *For arbitrary c and d such as $a \leq c \leq d \leq b$ holds, we have the equality $\mu([c,\ d]) = d - c$.*

(4) *If A and B in $\mathcal{B}$ are congruent, we have the equality*

$$\mu(A) = \mu(B).$$

In Theorem 1.8, the symbols $\bigcup_{p=1}^{(\infty)} A_p$ and $\sum_{p=1}^{(\infty)} \mu(A_p)$ show either one of the finite sum or the countable sum.

In the sequel, we do not always give the similar remark.

We assume that the plane curve C is rectifiable. Then the continuous mapping

$$\varphi :\ I \longrightarrow C$$

is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t) = {}^t(x(t),\ y(t)),\ (a \leq t \leq b).$$

Then we can define the measure space $(C,\ \mathcal{B}_C,\ \nu)$ of Jordan type on the curve C by using the Jordan measure space $(I,\ B,\ \mu)$.

This measure space $(C,\ \mathcal{B}_C,\ \nu)$ is constructed so that it is the image measure space on the curve C of the Jordan measure space $(I,\ \mathcal{B},\ \mu)$ on the interval I.

Thereby, we construct the mathematical model of the concept of the length of Jordan type on C.

Namely we have Theorem 1.9 in the following.

Theorem 1.9 *There exists a measure space* $(C,\ \mathcal{B}_C,\ \nu)$ *of Jordan type on the curve* C *and we have the statements* (1)$\sim$(3) *in the following*:

(1) *If* $E \in B_C$ *holds, we have* $0 \leq \nu(E) \leq L$. *Here* L *denotes the length of the curve* C.

(2) *If at most countable number of* $E_p \in \mathcal{B}_C,\ (p = 1,\ 2, \cdots)$ *are mutually disjoint and we have the condition*

$$E = \bigcup_{p=1}^{(\infty)} E_p \in \mathcal{B}_C,$$

we have the formula

$$\nu(E) = \sum_{p=1}^{(\infty)} \nu(E_p).$$

(3) $E \in \mathcal{B}_C$ *if and only if* $\varphi^{-1}(E) \in \mathcal{B}$. *Then we have the formula*

$$\nu(E) = \int_{\varphi^{-1}(E)} ds(t).$$

The integral on the right hand side denotes the RS-*integral. Here* $s = s(t)$ *is the function which is the arc length of the arc* $C(a,\ t)$ *of the curve* C.

Therefore, for $[t,\ t'] \subset [a,\ b]$, we have the equality

$$\nu([t,\ t']) = s(t') - s(t) = s(t,\ t')$$

by virtue of Theorem 1.9.

Theorem 1.10 *A* C^1*-curve on the plane*

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x(t),\ y(t)),\ (a \leq t \leq b)$$

is rectifiable. Its length L *is given by the equality*

$$L = \int_a^b \|\dot{\boldsymbol{r}}(t)\| dt.$$

Here we have the equality

$$\|\dot{\boldsymbol{r}}(t)\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}.$$

Assume that a simple C^1-curve is defined by the formula

$$C:\ x = x(t),\ y = y(t),\ (a \leq t \leq b).$$

We denote the arc length $s(a,\ t)$ of the arc $C(a,\ t)$ of the curve C as $s = s(t)$.

Then, by virtue of Theorem 1.10, we have the equality

$$s(t) = \int_a^t \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt.$$

Therefore we have the equality

$$\frac{ds}{dt} = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = \sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dx}{dt}\Big)^2}.$$

Here, if a curve

$$C:\ x = x(t),\ y = y(t),\ (a \leq t \leq b)$$

is a regular C^1-curve, we have the formula

$$\dot{x}(t)^2 + \dot{y}(t)^2 > 0.$$

Therefore, since $\frac{ds}{dt} > 0$ holds, the function $s = s(t)$ has the inverse function $t = t(s)$. Then the function $t = t(s)$ is of class C^1. Then we can obtain the arc length s as a parameter of the curve C. Therefore, when we take s as a parameter of the curve C, we have the equality

$$\Big(\frac{dx}{ds}\Big)^2 + \Big(\frac{dy}{ds}\Big)^2 = 1.$$

This is proved by the formula

$$\Big(\frac{dx}{ds}\Big)^2 + \Big(\frac{dy}{ds}\Big)^2 = \Big\{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2\Big\}\Big(\frac{dt}{ds}\Big)^2 = 1$$

by the calculation by using the method of differentiation of a composite function.

Then we have the following Theorem.

Theorem 1.11 *In a regular C^1-curve*

$$C:\ x = x(t),\ y = y(t),\ (a \leq t \leq b),$$

the ratio of the length of the chord combining the both end points $P(x,\ y)$ and $Q(x + \Delta x,\ y + \Delta y)$ of the arc $C(t,\ t + \Delta t)$ of the curve and the arc length Δs converges to 1 when $\Delta s \to 0$ holds. Namely we have the limit

$$\frac{\text{the chord}}{\text{the arc length}} = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta s} \to 1,\ (\Delta s \to 0).$$

Corollary 1.3 *Assume that a function* $y = f(x)$ *is of class* C^1 *on the closed interval* $[a,\ b]$. *Then the curve*

$$C:\ y = f(x),\ (a \leq x \leq b)$$

is rectifiable and its length is given by the formula

$$s = \int_a^b \sqrt{1 + \dot{f}(x)^2} dx.$$

Corollary 1.4 *Assume that a function* $f(\theta)$ *is of class* C^1 *on the closed interval* $[\alpha,\ \beta]$. *Then the curve*

$$C:\ r = f(\theta),\ (\alpha \leq \theta \leq \beta)$$

defined by the polar coordinates is rectifiable and its length s *is given by the formula*

$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Example 1.3 The lengths of the quadratic curves are given in the following :

(1) **Parabola**. In the parabola $y^2 = 4lx,\ (l > 0)$, the arc length from the vertex $(0,\ 0)$ to an arbitrary points $(x,\ y)$ is given by the formula

$$s = \sqrt{x(x+l)} - l \log \left(\sqrt{\frac{x+l}{l}} - \sqrt{\frac{x}{l}} \right).$$

Here assume $y > 0$.

(2) **Ellipse**. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is expressed by the formula

$$x = a \sin \theta,\ y = b \cos \theta,\ (0 \leq \theta \leq 2\pi)$$

assuming that the point $(0,\ b)$ is the starting point. Then the arc length s of the ellipse is given by the formula

$$s = a \int_0^\theta \sqrt{1 - e^2 \sin^2 \theta} d\theta = \int_0^x \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}} dx.$$

Here we assume that $a \geq b > 0$ holds and $e = \frac{\sqrt{a^2 - b^2}}{a}$ denotes the eccentricity.

(3) **Hyperbola**. The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is expressed by the formula

$$x = a\sec\theta,\ y = b\tan\theta,\ (-\frac{\pi}{2} < \theta < \frac{\pi}{2})$$

assuming that the point $(a,\ 0)$ is the starting point. Then the arc length s of the hyperbola is given by the formula

$$s = \int_0^\theta \sqrt{(a^2+b^2)\sec^4\theta - a^2\sec^2\theta}\,d\theta = \int_a^x \sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}dx.$$

Here we assume that $0 < \theta < \dfrac{\pi}{2}$ holds and $e = \sqrt{\dfrac{a^2+b^2}{a}}$ denotes the eccentricity.

Remark 1.1 When $k^2 \neq 0,\ 1$ holds, we say that the three types of the integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\ \int \sqrt{\frac{1-k^2x^2}{1-x^2}}dx,$$
$$\int \frac{dx}{(1-a^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

are the elliptic integral of the first, second, third kind respectively.

If we put $x = \sin\theta$ in these integrals, they are expressed in the following three kind of integrals

$$\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\ \int \sqrt{1-k^2\sin^2\theta}\,d\theta,$$
$$\int \frac{d\theta}{(1-a^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

respectively.

These three kind of elliptic integrals are not expressed as the elementary functions.

We see that the length of quadratic curves obtained in Example 1.3 are expressed by using the elliptic integrals in the other cases than the circles and the parabolas.

2 Length of a space curve

In this section, we construct a measure space of Jordan type on a curve as a mathematical model for the length of Jordan type of a curve in the space $\boldsymbol{R}^d$. Here we assume $d \geq 2$.

In this section, we obtain a standard basis $\{\boldsymbol{i}_1,\ \boldsymbol{i}_2,\ \cdots,\ \boldsymbol{i}_d\}$ on the space $\boldsymbol{R}^d$ and fix it. Thus we assume that the orthogonal coordinate $\boldsymbol{r} = {}^t(x_1,\ x_2,\ \cdots,\ x_d)$ of a point $\boldsymbol{r}$ is determined.

In general, we assume that a considered curve in $\boldsymbol{R}^d$ is a continuous curve.

Then we give the definition of the length of a curve in $\boldsymbol{R}^d$ and the condition that a curve in $\boldsymbol{R}^d$ has the length in the similar way as in the case of the plane curve.

Here, when a finite division Δ of a closed interval $[a,\ b]$ is given by the dividing points

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

we take the dividing points on the curve C:

$$\boldsymbol{r}_0 = \boldsymbol{r}(a),\ \boldsymbol{r}_1,\ \boldsymbol{r}_2,\ \cdots,\ \boldsymbol{r}_n = \boldsymbol{r}(b),$$

$$\Big(\, \boldsymbol{r}_j = \boldsymbol{r}(t_j),\ j = 0,\ 1,\ 2,\ \cdots,\ n \,\Big)\,.$$

Then, we define the length of the polygonal line obtained by combining these dividing points by the line segment by the formula:

$$L_\Delta = \sum_{j=1}^{n} \overline{\boldsymbol{r}_{j-1}\boldsymbol{r}_j} = \sum_{j=1}^{n} \sqrt{\sum_{i=1}^{d} (x_i(t_j) - x_i(t_{j-1}))^2}.$$

If the set of all L_Δ's corresponding to all finite divisions Δ of $[a,\ b]$ is bounded, we define that its supremum

$$L = \sup_{\Delta} L_\Delta$$

is the **length** of the curve C.

Then we have the following Theorem.

Theorem 2.1 *A space curve*

$$C : \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b)$$

has the length if and only if the d coordinate functions $x_1 = x_1(t),\ x_2 = x_2(t),\ \cdots, x_d = x_d(t)$ are of bounded variation on $[a,\ b]$.

Theorem 2.2 *We assume that a space curve*

$$C : \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b)$$

has the length. Then the set of all lengths L_Δ's of the polygonal lines corresponding to all finite divisions Δ of the closed interval $[a,\ b]$ is bounded and we have its supremum

$$L = \sup_{\Delta} L_\Delta.$$

Then we have the limit

$$L = \lim_{\Delta} L_\Delta$$

in the sense of Moore-Smith *limit.*

In Theorem 2.2 in the above, the limit in the sense of Moor-Smith is considered in the similar way to Theorem 1.7.

Here we remark that the length of the curve in the above is independent of the transformation of the parameter of the curve.

Namely, when the parameter t is expressed by the monotone increasing continuous function of another parameter τ, the length of the curve C does not change even if we express the curve C by the formula

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t(\tau)),\ (\alpha \leq \tau \leq \beta).$$

In the sequel, we consider the rectifiable space curve. Therefore, on the curve

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b),$$

the d coordinate functions $x_1 = x_1(t),\ x_2 = x_2(t),\ \cdots,\ x_d = x_d(t)$ are assumed to be the continuous functions of bounded variation on the closed interval $[a,\ b]$.

Here we denote the arc of the curve combining two points $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t')$ for $a \leq t < t' \leq b$ as $C(t,\ t')$.

Then it is evident that the curve $C(t,\ t')$ has the length. We denote the length of this curve as $s(t,\ t')$.

Then it is evident that, for $a \leq t < t' < t'' \leq b$, we have the equality

$$s(t,\ t') + s(t',\ t'') = s(t',\ t'').$$

Further, by orienting the arc of the curve, we assume that the arc length measured in the positive direction is positive and the arc length measured in the negative direction is negative. Then, for $a \leq t < t' \leq b$, we have the equality

$$s(t,\ t') = -s(t',\ t).$$

Therefore we have the equality

$$s(t,\ t') + s(t',\ t'') = s(t,\ t'')$$

independent of the order of $t,\ t',\ t'' \in [a,\ b]$.

Here, for $a \leq t \leq b$, we put

$$s = s(t) = s(a,\ t).$$

Then, for $[t,\ t'] \subset [a,\ b]$, we have the equality

$$s(t,\ t') = s(t') - s(t).$$

Therefore $s = s(t)$ is a monotone increasing continuous function.

Thus it is a continuous function of bounded variation.

Then the length $s = s(t)$ of the arc of the curve $C(a,\ t)$ is expressed as the RS-integral

$$s = \int_0^t ds(t).$$

Its value is included in the closed interval $[0,\ L]$.

Next we consider the construction of the image measure space $(C,\ \mathcal{B}_C,\ \nu)$ on the curve C of the Jordan measure space $(I,\ \mathcal{B},\ \mu)$ on the closed interval $I = [a,\ b]$.

Now, by virtue of Theorem 1.8, we assume that we have the Jordan measure space $(I,\ \mathcal{B},\ \mu)$ on the closed interval $I = [a,\ b]$.

We assume that the space curve C is rectifiable. Then, since a continuous mapping $\varphi:\ I \to C$ is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t) = {}^t(x_1(t),\ x_2(t),\ \cdots,\ x_d(t)),\ (a \leq t \leq b),$$

we can define the measure space $(C,\ \mathcal{B}_C,\ \nu)$ of Jordan type on the curve C by using the measure space $(I,\ \mathcal{B},\ \mu)$.

This measure space is constructed so that it is the image measure space on the curve C of the Jordan measure space on the interval I.

Thereby, we construct the mathematical model of the concept of the length of Jordan type of the curve C.

Namely we have Theorem 2.3 in the following.

Theorem 2.3 *There exists the measure space $(C,\ \mathcal{B}_C,\ \nu)$ of Jordan type on a rectifiable curve C such that it satisfies the following statements* (1) $\sim$ (3):

(1) *If we have $E \in \mathcal{B}_C$, we have $0 \leq \nu(E) \leq L$. Here L denotes the length of the curve C.*

(2) *If at most countable number of $E_p \in \mathcal{B}_C,\ (p = 1,\ 2,\ \cdots)$ are mutually disjoint and we have the condition*

$$E = \bigcup_{p=1}^{(\infty)} E_p \in \mathcal{B}_C,$$

we have the equality

$$\nu(E) = \sum_{p=1}^{(\infty)} \nu(E_p).$$

(3) *We have* $E \in \mathcal{B}_C$ *if and only if we have* $\varphi^{-1}(E) \in \mathcal{B}$. *Then we have the equality*

$$\nu(E) = \int_{\varphi^{-1}(E)} ds(t).$$

Here the integral on the right hand side of the equality in the above denotes the RS-*integral. Here* $s = s(t)$ *is the arc length of the arc* $C(a,\ t)$ *of the curve* C.

Therefore, by virtue of Theorem 2.3, we have the equality

$$\nu([t,\ t']) = s(t') - s(t) = s(t,\ t')$$

for $[t,\ t'] \subset [a,\ b]$.

Next, as a special case, we consider a certain C^1-curve

$$C:\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t\left(\ x_1(t),\ x_2(t),\ \cdots,\ x_d(t)\ \right),\ (a \le t \le b)$$

in the space $\boldsymbol{R}^d$.

Now we assume that we have the Jordan measure space $(I,\ \mathcal{B},\ \mu)$ on an interval $I = [a,\ b]$ in $\boldsymbol{R}$.

Here we construct a measure space of Jordan type on C.

We construct this measure space as the image measure space on the curve C of the Jordan measure space on the interval I.

Thereby we can construct naturally the mathematical model of the concept of the length of Jordan type on C.

Then, since the C^1-mapping $\varphi:\ I \to C$ is defined by the formula

$$\varphi(t) = \boldsymbol{r}(t) = {}^t\left(\ x_1(t),\ x_2(t),\ \cdots,\ x_d(t)\ \right),\ (a \le t \le b),$$

we can define the measure space $(C,\ \mathcal{B}_C,\ \nu)$ of Jordan type on the curve C by using the measure space $(I,\ \mathcal{B},\ \mu)$ and we have Theorem 2.4 in the following.

Theorem 2.4 *There exists the measure space* $(C,\ \mathcal{B}_C,\ \nu)$ *of Jordan type on a* C^1-*curve* C *such that we have the statements* $(1) \sim (3)$ *in the following*:

(1) *If we have* $E \in B_C$, *we have* $0 \le \nu(E) \le L$. *Here* L *denotes the length of the curve* C.

(2) *If at most countable number of* $E_p \in \mathcal{B}_C,\ (p = 1,\ 2,\ \cdots)$ *are mutually disjoint and we have the condition*

$$E = \bigcup_{p=1}^{(\infty)} E_p \in \mathcal{B}_C,$$

we have the equality

$$\nu(E) = \sum_{p=1}^{(\infty)} \nu(E_p).$$

(3) *We have* $E \in \mathcal{B}_C$ *if and only if we have* $\varphi^{-1}(E) \in \mathcal{B}$. *Then we have the equality*:

$$\nu(E) = \int_{\varphi^{-1}(E)} \|\dot{\boldsymbol{r}}(t)\| dt.$$

Here we have the formula

$$\|\dot{\boldsymbol{r}}(t)\| = \sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2 + \cdots + \dot{x}_d(t)^2}.$$

3 Definition of the curvilinear R-integral and its fundamental properties

In this section, we define the curvilinear R-integral as the integral of Riemann type of a scalar function defined on a curve C in the space $\boldsymbol{R}^d$ and we study its fundamental properties. Here we assume $d \geq 2$.

In this section, we say the curvilinear R-integral as the integral for simplicity.

In general, we assume that the curve C in $\boldsymbol{R}^d$ is a rectifiable continuous curve.

Here, by virtue of Theorem 2.4, we assume that the measure space $(C,\ \mathcal{B}_C,\ \nu)$ of Jordan type is defined on the curve C.

Then, at first, we define that a scalar function defined on C is measurable.

At first, we define a simple function.

Namely we say that a scalar function $f(\boldsymbol{r}) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on C is a **simple function** if it is expressed as

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r})$$

for an arbitrary countable division of C

$$\Delta:\ C = E_1 + E_2 + \cdots,\ (E_p \in \mathcal{B}_C,\ p = 1,\ 2, \cdots).$$

Here a_p denotes a real number and they are not necessarily different each other. $\chi_{E_p}(\boldsymbol{r})$ denotes the defining function of the set E_p. Then we denote the simple function $f(\boldsymbol{r})$ as $f_\Delta(\boldsymbol{r})$.

Here we define that $\boldsymbol{\Delta}_C$ is the set of all countable divisions Δ of the curve C by using the measurable sets in $\mathcal{B}_C$. Then we remark that $\boldsymbol{\Delta}_C$ is a direct set with respect to the relation of the subdivision $\leq$.

Definition 3.1 We define that a scalar function $f(\boldsymbol{r})$ defined on a rectifiable curve C is **measurable** if there exists a direct family $\{f_\Delta(\boldsymbol{r});\ \Delta \in \boldsymbol{\Delta}_C\}$ of simple functions defined on C such that we have the limit

$$\lim_{\Delta} f_\Delta(\boldsymbol{r}) = f(\boldsymbol{r})$$

uniformly on C in the sense of Moore-Smith limit.

Namely, for an arbitrary $\varepsilon > 0$, there exists a certain division $\Delta_0 \in \boldsymbol{\Delta}_C$ such that, for an arbitrary division $\Delta \in \boldsymbol{\Delta}_C$ such as $\Delta_0 \leq \Delta$, we have the condition

$$|f_\Delta(\boldsymbol{r}) - f(\boldsymbol{r})| < \varepsilon,\ \left(\ \boldsymbol{r} = {}^t(x_1,\ x_2,\ \cdots,\ x_d) \in C\ \right).$$

Example 3.1 A simple function and a continuous function defined on a rectifiable curve C are measurable.

Next we define the integral of Riemann type of a measurable function $f(\boldsymbol{r})$ defined on a rectifiable curve C.

We define this integral in the two steps.

(1) In the case where a function $f(\boldsymbol{r})$ is a simple function

Here we assume that $f(\boldsymbol{r})$ is expressed as

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r}),\ (a_p \in \boldsymbol{R},\ p \geq 1)$$

for a countable division of C.

Then we define the integral of Riemann type on C of $f(\boldsymbol{r})$ by the formula

$$R = \int_C f(\boldsymbol{r}) d\nu = \sum_{p=1}^{\infty} a_p \nu(E_p).$$

Here we assume that the sum of the series in the right hand side of the equality in the above converges absolutely. The value of this integral is determined independently to the choice of the expression of $f(\boldsymbol{r})$ as a simple function.

(2) In the case where a function $f(\boldsymbol{r})$ is a general measurable function

Then, by virtue of Definition 3.1, there exists a direct family $\{f_\Delta(\boldsymbol{r}) :\ \Delta \in \boldsymbol{\Delta}_C\}$ of simple functions such that it converges to $f(\boldsymbol{r})$ uniformly on C.

Then, if we have the limit

$$R = \lim_{\Delta} \int_C f_\Delta(\boldsymbol{r}) d\nu$$

in the sense of Moore-Smith limit, we say that this limit R is the **integral of Riemann type** of the measurable function $f(\boldsymbol{r})$ on C and denote it as

$$R = \int_C f(\boldsymbol{r}) d\nu.$$

This value does not depend on the choice of the direct family $\{f_\Delta(\boldsymbol{r});\ \Delta \in \boldsymbol{\Delta}_C\}$ of simple functions which coverges to $f(\boldsymbol{r})$ uniformly on C.

We say that this integral is a **curvilinear R-integral** on C and we denote it as

$$R = \int_C f(\boldsymbol{r}) ds.$$

Here ds denotes the **line element**.

This integral on the right hand side of the equality in the above is considered to be a Riemann-Stieltjes integral.

Further, we can define the curvilinear integral of a measurable vector function $\boldsymbol{a} = {}^t(a_1,\ a_2,\ \cdots,\ a_d)$ on C is the similar way. This is equivalent to the definitions of the curvilinear integrals of the scalar functions corresponding to each one of d components $a_1,\ a_2,\ \cdots,\ a_d$ of $\boldsymbol{a}$.

We obtain the formulas in the following directly from the definition of the curvilinear integral.

Theorem 3.1 *We use the notation in the above for a rectifiable curve C. Then we have the statements* (1) *and* (2) *in the following* :

(1) *For two measurable vector functions $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, we have the formula in the following* :

$$\int_C (\boldsymbol{a}_1 + \boldsymbol{a}_2) d\nu = \int_C \boldsymbol{a}_1 d\nu + \int_C \boldsymbol{a}_2 d\nu.$$

(2) *For a measurable vector function $\boldsymbol{a}$ on C and a real constant λ, we have the formula in the following* :

$$\int_C \lambda \boldsymbol{a} d\nu = \lambda \int_C \boldsymbol{a} d\nu.$$

Further we have Theorem 3.2 in the following.

Theorem 3.2 *Assume that a rectifiable curve C is divided into two arcs*

$$C_1 :\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t\left(\ x_1(t),\ x_2(t),\ \cdots,\ x_d(t)\ \right),\ (a \le t \le c),$$

$$C_2 :\ \boldsymbol{r} = \boldsymbol{r}(t) = {}^t\left(x_1(t),\ x_2(t),\ \cdots,\ x_d(t) \right),\ (c \le t \le b).$$

Then, for a measurable vector function $\boldsymbol{a}$, *we have the equality*

$$\int_C \boldsymbol{a} d\nu = \int_{C_1} \boldsymbol{a} d\nu + \int_{C_2} \boldsymbol{a} d\nu.$$

Especially, in the case where a curve C in $\boldsymbol{R}^d$ is C^1-curve, we can express the curvilinear integral in the following.

(3) In the case where $f(\boldsymbol{r})$ is a simple function

Here $f(\boldsymbol{r})$ is expressed by the formula

$$f(\boldsymbol{r}) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(\boldsymbol{r}),\ (a_p \in \boldsymbol{R},\ p \ge 1)$$

for a countable division of C

$$\Delta :\ C = E_1 + E_2 + \cdots,\ (E_p \in \mathcal{B}_C,\ p \ge 1).$$

Then, by virtue of Theorem 2.4, (B), we have the equality

$$\begin{aligned} \int_C f(\boldsymbol{r}) d\nu &= \sum_{p=1}^{\infty} a_p \int_{\varphi^{-1}(E_p)} \|\dot{\boldsymbol{r}}(t)\| dt \\ &= \int_a^b f(\boldsymbol{r}(t)) \|\dot{\boldsymbol{r}}(t)\| dt. \end{aligned}$$

(4) In the case where $f(\boldsymbol{r})$ is a general measurable function

The curvilinear integral of a measurable function $f(\boldsymbol{r})$ on C is expressed as

$$R = \int_C f(\boldsymbol{r}) d\nu = \int_a^b f(\boldsymbol{r}(t)) \|\dot{\boldsymbol{r}}(t)\| dt.$$

In the similar way, as the curvilinear integral, we can define the curvilinear integrals as follows:

$$\int_C f(\boldsymbol{r}) dx_j = \int_a^b f(\boldsymbol{r}(t)) \dot{x}_j(t) dt,\ (1 \le j \le d).$$

Every type of the expression of these curvilinear integrals depends on the choice of a parameter of the curve C. In general, it is known that the value of the curvilinear integral on C does not depend on the choice of a parameter of the curve C.

Especially, for a measurable vector function $\boldsymbol{a} = {}^t(a_1,\ a_2,\ \cdots,\ a_d)$ on C, we have the formula of a curvilinear integral in the following :

$$\int_C a_1 dx_1 + a_2 dx_2 + \cdots + a_d dx_d$$

$$= \int_a^b (a_1 \dot{x}_1 + a_2 \dot{x}_2 + \cdots + a_d \dot{x}_d) dt.$$

This is the special combination of the curvilinear integrals of the components.

Here, if a curve C is regular, we have the condition

$$\dot{x}_1(t)^2 + \dot{x}_2(t)^2 + \cdots + \dot{x}_d(t)^2 \neq 0.$$

Therefore we can take the arc length s measured from the point $t = a$ as a parameter.

Then, since the unit vector of the curve C is $\boldsymbol{t} = {}^t\left(x_1'(s),\ x_2'(s),\ \cdots,\ x_d'(s) \right)$, we have the formula

$$\int_C a_1 dx_1 + a_2 dx_2 + \cdots + a_d dx_d = \int_0^L \boldsymbol{a} \cdot \boldsymbol{t} ds.$$

Here L denotes the length of C, and we can express also in the form

$$\int_C \boldsymbol{a} \cdot \boldsymbol{t} ds.$$

Here $\boldsymbol{a} \cdot \boldsymbol{t}$ denotes the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{t}$.

Further, when we express as $d\boldsymbol{r} = \boldsymbol{t} ds$ by using the vector $\boldsymbol{r}$ which express the point $\boldsymbol{r}$ on C, we can express the curvilinear integral in the above as the formulas

$$\int_C \boldsymbol{a} \cdot d\boldsymbol{r}$$

or

$$\int_C (\boldsymbol{a},\ d\boldsymbol{r}).$$

Here we remark that we have the symbol

$$d\boldsymbol{r} = {}^t(dx_1,\ dx_2,\ \cdots,\ dx_d).$$

Further the symbol $(\boldsymbol{a},\ d\boldsymbol{r})$ denotes the inner product.

If, at each point on the curve C, we denote the angle of $\boldsymbol{a}$ and $\boldsymbol{t}$ as θ, we have the equality

$$\int_C \boldsymbol{a} \cdot d\boldsymbol{r} = \int_\theta^L \|\boldsymbol{a}\| \cos\theta ds.$$

By using the notation in the above, we have the formula of the curvilinear integral.

Theorem 3.3 *We use the notation in the above for a C^1-class regular curve C. Then we have the statements* (1) *and* (2) *in the following*:

(1) *For two measurable vector functions* $\boldsymbol{a}_1$ *and* $\boldsymbol{a}_2$ *on* C, *we have the formula in the following*:

$$\int_C (\boldsymbol{a}_1 + \boldsymbol{a}_2) \cdot d\boldsymbol{r} = \int_C \boldsymbol{a}_1 \cdot d\boldsymbol{r} + \int_C \boldsymbol{a}_2 \cdot d\boldsymbol{r}.$$

(2) *For a measurable vector function* $\boldsymbol{a}$ *on* C *and a real constant* λ, *we have the formula in the following*:

$$\int_C \lambda \boldsymbol{a} \cdot d\boldsymbol{r} = \lambda \int_C \boldsymbol{a} \cdot d\boldsymbol{r}.$$

Further we have Theorem 3.4 in the following.

Theorem 3.4 *If a* C^1*-class regular curve* C *is divided into two arcs*

$$C_1 : \boldsymbol{r} = \boldsymbol{r}(t) = {}^t \left(x_1(t),\ x_2(t),\ \cdots,\ x_d(t) \right),\ (a \le t \le c),$$

$$C_1 : \boldsymbol{r} = \boldsymbol{r}(t) = {}^t \left(x_1(t),\ x_2(t),\ \cdots,\ x_d(t) \right),\ (c \le t \le b),$$

we have the formula

$$\int_C \boldsymbol{a} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{a} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{a} \cdot d\boldsymbol{r}$$

for a measurable vector function $\boldsymbol{a}$ *on* C.

We can extend the definition of the curvilinear integral so that we have the similar formula when the curve C in the above is continuous at $t = c$ and it is not differentiable at $t = c$.

Since the curvilinear R-integral is the special case of the RS-integral, as for the details of the properties of the curvilinear R-integral, we refer to the paragraph of the fundamental properties of the RS-integral in Ito [14].

At last, we show the example of the curvilinear R-integral on $\boldsymbol{R}^2$.

Example 3.1 If a rectifiable curve AB is devided into the arcs $AA_1,\ A_1A_2,\ A_2B$ and they are expressed as $y = \varphi_1(x),\ y = \varphi_2(x)$ and $y = \varphi_3(x)$ respectively, we have the equality

$$\begin{aligned} S &= \int_{AA_1} f(x,\ y)dx + \int_{A_1A_2} f(x,\ y)dx + \int_{A_2B} f(x,\ y)dx \\ &= \int_a^{a_1} f(x,\ \varphi_1(x))dx + \int_{a_1}^{a_2} f(x,\ \varphi_2(x))dx + \int_{a_2}^{b} f(x,\ \varphi_3(x))dx \end{aligned}$$

for the curvilinear R-integral on the curve C. Thus we have the equality

$$S = \int_{AB} f(x,\ y)dx.$$

Example 3.2 Assume that the xy-coordinate system has a positive orientation. Assume that a Jordan closed curve C has a positive orientation so that we see the inner side at the left hand side when we move along the curve.

Then, if the area of the inside of the curve C is S, we have the equality

$$S = \int_C xdy = -\int_C ydx = \frac{1}{2}\int_C (xdy - ydx).$$

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