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Enumeration of Cyclic Automorphic Hypergraphs

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Abstract

A counting formula for cyclic automorphic hypergraphs is given. The result can be understood as a generalization of a theorem proved by A. Nakamoto, T. Shirakura and S. Tazawa, which was originally called Royle's conjecture.

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1 Introduction

Enumerations of generalized graphs have been studied by many authors. In particular, their fundamental research was done by N. G. de Bruijn and

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D. A. Klarner [1], and the counting formula for the unlabeled general hypergraphs was given by S. Y. Wu [2], T. Ishihara [3] and Q. Jianguo [4], independently. For ordinary graphs, in 2001, G. Royle conjectured an explicit formula for the number of the unlabeled self-complementary graphs, which was described in terms of the numbers of two kinds of unlabeled graphs. Royle's conjecture was proved by A. Nakamoto, T. Shirakura and S. Tazawa [5] in 2009. After their work, the second named author [6] proved its generalization to the case of *edge colored cyclically automorphic* complete graphs, which is a generalization of the concept of self-complementary graphs to general orders. In this paper, for hypergraphs, we define cyclically automorphic graphs (cf. [7]), and give a counting formula for them.

2 Main theorem

For any positive integer n, let $V = \{v_1, v_2, \dots, v_n\}$ be a finite set of vertices. For any positive integer k, let $E = \{e_1, e_2, \dots, e_k\}$ be the set of edges of a hypergraph on V, where e_i $(i = 1, 2, \dots k)$ is a subset of V such that $\mid e_i \mid \geq 2$ and $e_i \neq e_j$ for any $i \neq j$. Thus, a hypergraph is a generalization of an ordinary graph, namely, although an edge of an ordinary graph is a set of two vertices, an edge of a hypergraph is a set of two or more vertices. However, in this paper, we only consider h-hypergraphs, for any integer $h \geq 2$, and an h-hypergraph is a hypergraph with exactly h vertices on each edge. We denote by $H_{h,n}$ the set of the only graph, named a complete h-hypergraph, which has totally $\binom{n}{h}$ edges.

For any positive integers r, let $\zeta_r = \exp\left(\frac{2\pi i}{r}\right)$ be the r-th primitive root of unity. Cyclic group $Z = \left\{\zeta_r, \zeta_r^2, \dots, \zeta_r^r = 1\right\}$ of order r is considered as the set of r distinct colors. We denote by $H_{h,n}^{(r)}$ the set of all edge colored complete unlabeled h-hypergraphs f (up to isomorphism) with n vertices in V whose edges are colored by colors in Z.

We define cyclic permutation σ as $\sigma = \begin{pmatrix} \zeta_r & \zeta_r^2 & \cdots & \zeta_r^{r-1} & \zeta_r^r \\ \zeta_r^2 & \zeta_r^3 & \cdots & \zeta_r^r & \zeta_r \end{pmatrix}$. For $f \in H_{h,n}^{(r)}$, the action of σ is the cyclic permutations of colors of each of all edges in f, i.e. $\sigma f = \zeta_r \cdot f$. An important definition is that, a graph $f \in H_{h,n}^{(r)}$ is said to be *cyclically automorphic* if it satisfies $\sigma f = f$. Let $\mathrm{sc}_h^{(r)}(n)$ denotes the number of cyclically automorphic graphs in $H_{h,n}^{(r)}$, namely $\mathrm{sc}_h^{(r)}(n) = \#\left\{f \in H_{h,n}^{(r)} | \sigma f = f\right\}$. Let $q_s(f)$ be the number of edges in f colored by ζ_r^s. The main result of the present paper is as follows:

Theorem 2.1. *For any positive integers r, h and n, we have*

$$\mathrm{sc}_h^{(r)}(n) = \sum_{f \in H_{h,n}^{(r)}} \zeta_r^{q_1(f) + 2q_2(f) + \cdots + rq_r(f)}.$$

Remark 2.2. *The second named author's theorem in [6] is the special case when $h = 2$ in the above theorem.*

3 Examples

We introduce the case $\mathrm{sc}_3^{(2)}(4)$ as an example of the main theorem. The edges of complete 3-hypergraphs of order 4 are colored by two colors in this case. First, we consider the elements of $H_{3,4}^{(2)}$ as given in Fig. 1. It has five elements, and one of them, marked by square box, is the only cyclically automorphic 3-hypergraph in $H_{3,4}^{(2)}$.

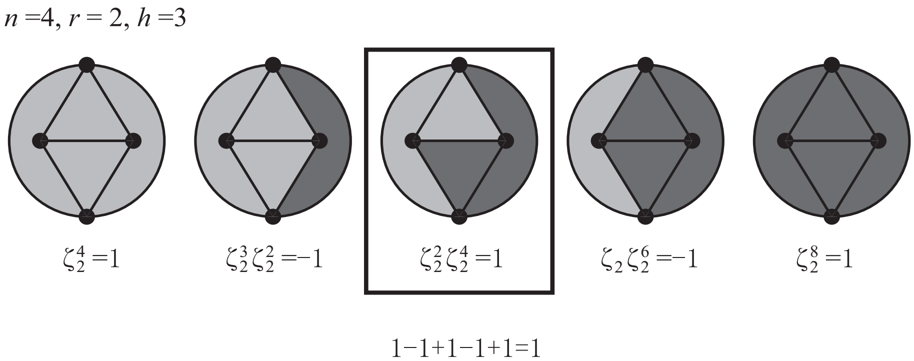


Figure 1:

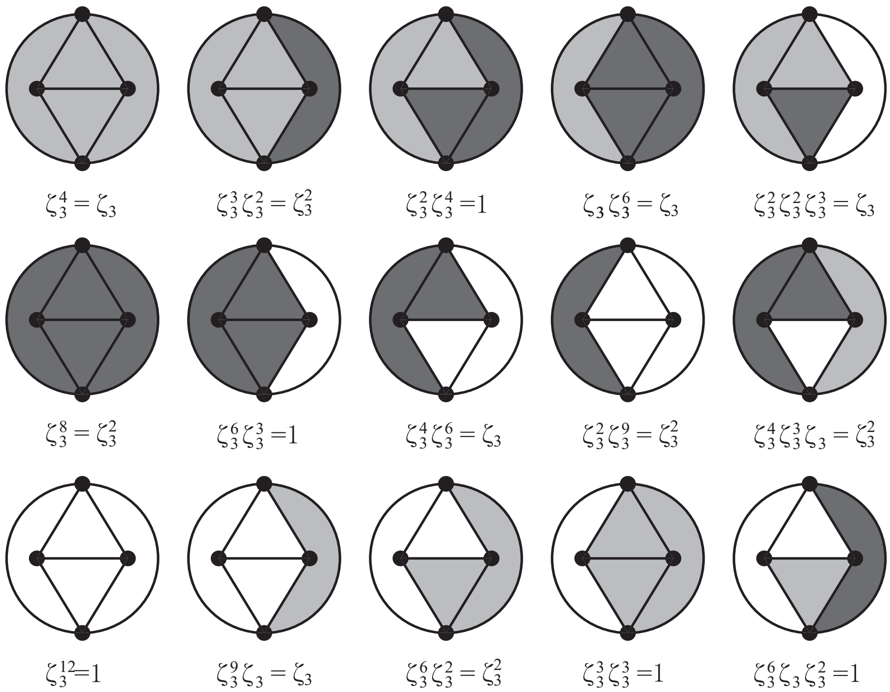
The number written under each graph is the value of the graph, in the sense of the right-hand side of Theorem 2.1, namely it is the product of the colors of its all edges. For example, let us consider the second graph from the left. This graph has three edges of the color $\zeta_2 = -1$ and one edge of the color ζ_2^2. Thus the value of the graph is $(\zeta_2)^3 \cdot (\zeta_2^2)^1 = \zeta_2 = -1$.

We obtain 1 as the total sum of numbers under five 3-hypergraphs in Fig. 1, and the number "1" coincides with the number of cyclically automorphic 3-hypergraph in $H_{3,4}^{(2)}$. This is what the theorem asserts.

Next, we introduce another case $\mathrm{sc}_3^{(3)}(4)$ as also an example of our theorem. The edges of complete 3-hypergraphs of order 4 are colored by three colors in this case. First, we consider the all elements of $H_{3,4}^{(3)}$ as given in Fig. 2. It has fifteen elements, and there are no cyclically automorphic 3-hypergraphs among them. This fact is apparent from the fact that the number of edges, 4, is relatively prime to the number of colors, 3, however it can also be derived from our theorem.

The value of each graph is written under the graph and is expressed as

$n=4$, $r=3$, $h=3$



$$5(\zeta_3+\zeta_3^2+1)=0$$

Figure 2:

an integer power of the cubic root of unity. We obtain 0 as the total sum of the values of fifteen 3-hypergraphs in Fig. 2, and the "0" coincides with the number of cyclically automorphic 3-hypergraph in $H_{3,4}^{(3)}$.

It's somewhat a puzzling fact that says that to count a special graph, you can use the edge information of all its companion graphs including itself.

4 Proof of Theorem 2.1

Let $\binom{V}{h}$ be $\{\{v_1, v_2, \ldots, v_h\} \subset V | v_i \neq v_j \text{ for } i \neq j\}$, and let $Z^{\binom{V}{h}}$ denote the set of all edge colored complete labeled h-hypergraphs with n vertices whose edges are colored by each colors in Z. A map $f^L \in Z^{\binom{V}{h}}$ can be understood as an edge colored labeled h-hypergraph, for example, when $f^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = \zeta_r^s$, it means that the edge $(v_{i_1}, v_{i_2}, \dots, v_{i_h})$ is colored by ζ_r^s. The action σf^L can be understood as the product $\zeta_r \cdot f^L$.

The set of permutations $\Xi = \{1, \sigma\}$ on the set Z is used, where 1 is the identity. We define an isomorphism γ of h-hypergraphs as follows:
Let $\Gamma = S_n$ be the symmetric group on V. We consider that $\gamma \in \Gamma$ is a permutation acts on $\binom{V}{h}$, such that, for every element $(v_{i_1}, v_{i_2}, \dots, v_{i_h})$ of $\binom{V}{h}$,

$$\gamma(v_{i_1}, v_{i_2}, \dots, v_{i_h}) = (\gamma v_{i_1}, \gamma v_{i_2}, \dots, \gamma v_{i_h}).$$

Moreover, we introduce a permutation $(\gamma : \xi)$, for $\xi \in \Xi$, on $Z^{\binom{V}{h}}$ defined by

$$(\gamma : \xi) f^L(v_{i_1}, v_{i_2}, \dots, v_{i_h}) = \xi f^L(\gamma v_{i_1}, \gamma v_{i_2}, \dots, \gamma v_{i_h}).$$

Let $Z_{\mathrm{sc}}^{\binom{V}{h}}$ be

$$Z_{\mathrm{sc}}^{\binom{V}{h}} = \left\{ f^L \in Z^{\binom{V}{h}} \mid \exists \gamma \in \Gamma,\ (\gamma : \sigma)\, f^L = f^L \right\}.$$

Let Γ_σ, Γ_1 and Γ_{f^L} be

$$\Gamma_\sigma = \left\{ \gamma \in \Gamma \mid \exists f^L \in Z^{\binom{V}{h}},\ (\gamma : \sigma)\, f^L = f^L \right\},$$

$$\Gamma_1 = \left\{ \gamma \in \Gamma \mid \exists f^L \in Z^{\binom{V}{h}},\ (\gamma : 1)\, f^L = f^L \right\},$$

and

$$\Gamma_{f^L} = \left\{ \gamma \in \Gamma \mid (\gamma : 1)\, f^L = f^L \right\} \quad \text{for } f^L \in Z^{\binom{V}{h}},$$

where 1 means the identical permutation. Let $\ell(f)$ and $\ell(f^L)$ be the numbers of ways of labeling a graph f and f^L, respectively. Namely, if f is the unlabeled graph of f^L, then $\ell(f) = \ell(f^L)$. And we have $\#\Gamma_{f^L} \cdot \ell(f^L) = n!$.

It is easy to see that

Lemma 4.1. *For $\gamma \in \Gamma$, we have the following:*

(1) f^L is a fixed point of the action of $(\gamma : 1)$ if and only if f^L is constant on every cyclic permutation in the disjoint cycle decomposition of γ.

(2) f^L is a fixed point of the action of $(\gamma : \sigma)$ if and only if in every cyclic permutation z in the disjoint cycle decomposition of γ, $f^L(z(v_{i_1}, v_{i_2}, \dots, v_{i_h})) = \zeta_r \cdot f^L(v_{i_1}, v_{i_2}, \dots, v_{i_h})$ holds for any $(v_{i_1}, v_{i_2}, \dots, v_{i_h}) \in z$.

Then, from (2) in Lemma 4.1 we have the following:

Lemma 4.2. *γ is an element of Γ_σ if and only if the length of every cyclic permutation in the disjoint cycle decomposition of γ is divisible by r.*

Next, we study the following lemma.

Lemma 4.3.

$$\mathrm{sc}_h^{(r)}(n) = \frac{1}{n!} \sum_{\gamma \in \Gamma_\sigma} \# \left\{ f^L \in Z^{\binom{V}{h}} | (\gamma : \sigma) f^L = f^L \right\}.$$

Proof.

$$\begin{aligned}
&\sum_{\gamma \in \Gamma_\sigma} \# \left\{ f^L \in Z^{\binom{V}{h}} | (\gamma : \sigma) f^L = f^L \right\} \\
&\quad = \sum_{\gamma \in \Gamma} \# \left\{ f^L \in Z^{\binom{V}{h}} | (\gamma : \sigma) f^L = f^L \right\} \\
&\quad = \sum_{\gamma \in \Gamma} \# \left\{ f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}} | (\gamma : \sigma) f^L = f^L \right\} \\
&\quad = \sum_{f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}} \# \left\{ \gamma \in \Gamma | (\gamma : \sigma) f^L = f^L \right\}.
\end{aligned}$$

For any $f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}$, we put $F = \left\{ \gamma \in \Gamma \mid (\gamma : \sigma) f^L = f^L \right\}$ and consider $\eta \in F$. We put $\{\delta_1, \delta_2, \ldots, \delta_\ell\} = \left\{ \delta \in \Gamma | (\delta : 1) f^L = f^L \right\}$, where we set $\delta_i \neq \delta_j$ for any $i \neq j$. Since $f^L = (\eta : \sigma) f^L = \eta(\sigma(f^L))$, we get $\eta^{-1}(f^L) = \sigma(f^L)$. First, we have $(\delta_i \eta : \sigma) f^L = \delta_i \eta(\sigma(f^L)) = \delta_i(\eta(\sigma(f^L))) = \delta_i(f^L) = (\delta_i : 1) f^L = f^L$. So we get $\{\delta_1 \eta, \delta_2 \eta, \ldots, \delta_l \eta\} \subset F$, namely, $\ell \leq \# F$. On the other hand, we have $(\gamma \eta^{-1} : 1) f^L = \gamma(\eta^{-1}(f^L)) = \gamma(\sigma(f^L)) = (\gamma : \sigma) f^L = f^L$. So we get $F \eta^{-1} \subset \{\delta_1, \ldots, \delta_\ell\}$, namely, $\# F \leq \ell$. Thus we obtain

$$\# \left\{ \gamma \in \Gamma | (\gamma : \sigma) f^L = f^L \right\} = \# \left\{ \gamma \in \Gamma | (\gamma : 1) f^L = f^L \right\}.$$

Hence, the right hand side of the above equality becomes

$$\begin{aligned}
\sum_{f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}} \# \left\{ \gamma \in \Gamma | (\gamma : 1) f^L = f^L \right\} &= \sum_{f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}} \# \Gamma_{f^L} \\
&= \sum_{f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}} \frac{n!}{\ell(f^L)} \\
&= n! \sum_{f^L \in Z_{\mathrm{sc}}^{\binom{V}{h}}} \frac{1}{\ell(f^L)} \\
&= n! \sum_{f \in H_{h,n}^{(r)}} \ell(f) \cdot \frac{1}{\ell(f)}.
\end{aligned}$$

The sum on the right hand side of the above equality is equal to the value $\mathrm{sc}^{(r)}(n)$. $\square$

Next, we need the following lemmas.

Lemma 4.4. *For* $\gamma \in \Gamma$,

$$\sum_{\substack{(\gamma:1)f^L=f^L \\ f^L\in Z^{\binom{V}{h}}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)}$$

$$= \begin{cases} 0 & (\gamma \notin \Gamma_\sigma), \\ \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:1)f^L = f^L\right\} & (\gamma \in \Gamma_\sigma). \end{cases}$$

Proof. For $\gamma \notin \Gamma_\sigma$, we put $\gamma = z_1 \cdot z_2 \cdot \cdots \cdot z_t$ such that the length of z_1 is k which is not divisible by r. We consider $f_1^L, f_2^L, \ldots, f_r^L \in Z^{\binom{V}{h}}$ satisfying the following two conditions:

· if $(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) \in z_1$, then
$f_1^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = \zeta_r^1$,
$f_2^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = \zeta_r^2$,
$\vdots$
$f_r^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = \zeta_r^r$,

· if $(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) \notin z_1$, then
$f_1^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = f_2^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h}) = \cdots = f_r^L(v_{i_1}, v_{i_2}, \ldots, v_{i_h})$.

It is enough to prove that the sum of the values of $f_1^L, f_2^L, \ldots, f_r^L$ is equal to 0. We calculate as follows:

$$\begin{aligned} &\sum_{j=1}^{r} \zeta_r^{q_1(f_j^L)+2q_2(f_j^L)+\cdots+rq_r(f_j^L)} \\ &= \sum_{j=1}^{r} \zeta_r^{q_1(f_1^L)+2q_2(f_1^L)+\cdots+rq_r(f_1^L)} \cdot \zeta_r^{(j-1)k} \\ &= \zeta_r^{q_1(f_1^L)+2q_2(f_1^L)+\cdots+rq_r(f_1^L)} \left(\zeta_r^{0\cdot k} + \zeta_r^{1\cdot k} + \cdots + \zeta_r^{(r-1)\cdot k}\right) \\ &= \zeta_r^{q_1(f_1^L)+2q_2(f_1^L)+\cdots+rq_r(f_1^L)} \left(\zeta_r^{0} + \zeta_r^{k} + \cdots + \zeta_r^{(r-1)k}\right). \end{aligned}$$

Since k is not divisible by r, $\zeta_r^k \neq 1$. The part $\left(\zeta_r^0 + \zeta_r^k + \cdots + \zeta_r^{(r-1)k}\right)$ is equal to 0 and the left hand side of the equality in Lemma 4.4 becomes 0 for $\gamma \notin \Gamma_\sigma$.

For $\gamma \in \Gamma_\sigma$, by Lemma 4.2, $q_1(f^L) + 2q_2(f^L) + \cdots + rq_r(f^L)$ is divisible by r. Thus,

$$\begin{aligned} \sum_{\substack{(\gamma:1)f^L=f^L \\ f\in Z^{\binom{V}{h}}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} &= \sum_{\substack{(\gamma:1)f^L=f^L \\ f\in Z^{\binom{V}{h}}}} 1 \\ &= \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:1)f^L = f^L\right\}. \end{aligned}$$

□

Lemma 4.5. *For* $\gamma \in \Gamma_\sigma$,

$$\#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:1)f^L = f^L\right\} = \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:\sigma)f^L = f^L\right\}.$$

Proof. If we fix the color of the first element of each cyclic component of γ, then the color of the remaining elements are fixed automatically. It follows from (1) and (2) of Lemma 4.1 that the numbers of graphs of both sides are equal. □

Considering

$$N = \frac{1}{n!} \sum_{\gamma \in \Gamma} \sum_{\substack{(\gamma:1)f^L = f^L \\ f^L \in Z^{\binom{V}{h}}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)},$$

we obtain the following lemma by using Lemma 4.4 and Lemma 4.5.

Lemma 4.6.

$$N = \frac{1}{n!} \sum_{\gamma \in \Gamma_\sigma} \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:\sigma)f^L = f^L\right\}.$$

Proof. Using Lemma 4.4, the right hand side can be computed as follows:

$$\begin{aligned} N &= \frac{1}{n!} \sum_{\gamma \in \Gamma} \sum_{\substack{(\gamma:1)f^L = f^L \\ f^L \in Z^{\binom{V}{h}}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \\ &= \frac{1}{n!} \sum_{\gamma \in \Gamma_\sigma} \sum_{\substack{(\gamma:1)f^L = f^L \\ f^L \in Z^{\binom{V}{h}}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \\ &= \frac{1}{n!} \sum_{\gamma \in \Gamma_\sigma} \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:1)f^L = f^L\right\}. \end{aligned}$$

The right hand side of the above equality is equal to

$$\frac{1}{n!} \sum_{\gamma \in \Gamma_\sigma} \#\left\{f^L \in Z^{\binom{V}{h}} | (\gamma:\sigma)f^L = f^L\right\}$$

by using Lemma 4.5. □

On the other hand, we have the following lemma:

Lemma 4.7.

$$N = \sum_{f \in H_{h,n}^{(r)}} \zeta_r^{q_1(f)+2q_2(f)+\cdots+rq_r(f)}.$$

Proof.

$$
\begin{aligned}
N &= \frac{1}{n!} \sum_{f^L \in Z^{\binom{V}{h}}} \sum_{\substack{(\gamma:1) f^L = f^L \\ \gamma \in \Gamma}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \\
&= \frac{1}{n!} \sum_{f^L \in Z^{\binom{V}{h}}} \sum_{\gamma \in \Gamma_{f^L}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \\
&= \frac{1}{n!} \sum_{f^L \in Z^{\binom{V}{h}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \# \Gamma_{f^L} \\
&= \sum_{f^L \in Z^{\binom{V}{h}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \frac{\# \Gamma_{f^L}}{n!}
\end{aligned}
$$

Since $\# \Gamma_{f^L} \cdot \ell(f^L) = n!$, the right hand side of the above equality is as follows.

$$
\begin{aligned}
N &= \sum_{f^L \in Z^{\binom{V}{h}}} \zeta_r^{q_1(f^L)+2q_2(f^L)+\cdots+rq_r(f^L)} \frac{1}{\ell(f^L)} \\
&= \sum_{f \in H_{h,n}^{(r)}} \zeta_r^{q_1(f)+2q_2(f)+\cdots+rq_r(f)} \cdot \ell(f) \cdot \frac{1}{\ell(f)} \\
&= \sum_{f \in H_{h,n}^{(r)}} \zeta_r^{q_1(f)+2q_2(f)+\cdots+rq_r(f)}
\end{aligned}
$$

□

Finally by adapting Lemmas 4.3, 4.6 and 4.7, we immediately obtain Theorem 2.1.

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