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# Length of Integer Solutions of Linear Diophantine Equations and Related Problems

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**Abstract**

We shall introduce a length of the integer solutions of linear diophantine equations and investigate the fundamental properties of this length. We will also give an application of this length to a famous mathematical puzzle called *three jug problem*.

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## 1 Introduction

Probably the simplest diophantine equation may be the following linear diophantine equation of two valuables $x, y$,

$$E_{(a,b;c)} : ax + by = c, \text{ where } a, b, c \in \mathbb{Z}.$$

We shall denote the integer solutions of $E_{(a,b;c)}$ as $S_{(a,b;c)}$. It is well known the above equation has the integer solutions $(x, y)$ if and only if $\mathrm{GCD}(a, b)|c$ and all the solutions are explicitly obtained by using the Euclidean Algorithm.

Let us start an example $E_{(5,3;38)} : 5x + 3y = 38$. We shall explain the usual way of writing down the integer solutions of this equation. Firstly, from the Euclidean Algorithm, one can find the special integer solutions $(x, y) = (-1, 2)$ of the equation $E_{(5,3;1)}$. Multiplying both sides of the equation $E_{(5,3;1)}$ by 38, one obtains the solutions $(x, y) = (-38, 78)$ of the equation $E_{(5,3;38)}$. Then all the integer solutions $S_{(5,3;38)}$ of the equation $E_{(5,3,38)}$ are written as follows;

$$S_{(5,3;38)} = \{(x, y) \mid x = -38 + 3k, y = 78 - 5k, \text{where } k \in \mathbb{Z}\}.$$

We note that this set of integer solutions $S_{(5,3;38)}$ is a residue class of $\mathbb{Z}^2$ modulo $\{k(3, -5) \mid k \in \mathbb{Z}\}$, where $\{k(3, -5) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$. Therefore $(-38, 76)$ is a representative of the residue class $S_{(5,3;38)}$. But, taking $k = 15$, we can choose another "small" representative $(x, y) = (7, 1)$. In the next section, we shall introduce the length of integer solutions and $(7, 1)$ are really the smallest integer solutions and suitable for the representative of this residue class (see Theorem 2.6 and Remark 2.7).

## 2 The length of integer solutions

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$. Then the following $d(\mathbf{a}, \mathbf{b})$ defines a different way of measuring the distance of $\mathbb{R}^n$ which is called the Manhattan distance,

$$d(\boldsymbol{a}, \boldsymbol{b}) = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

Let us denote $(0, 0, \ldots, 0) \in \mathbb{R}^n$ by $\mathbf{0}$. Now treat the linear diophantine equation

$$E_{(a_1,a_2,\ldots,a_n;c)} : a_1x_1 + a_2x_2 + \cdots + a_nx_n = c, \text{where } a_1, a_2, \ldots, a_n, c \in \mathbb{Z}.$$

We shall define the length $L(\boldsymbol{x})$ of the integer solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of the above linear diophantine equation $E_{(a_1,a_2,\ldots,a_n;c)}$ by putting

$$L(x_1, x_2, \ldots, x_n) = d(\boldsymbol{x}, \mathbf{0}) = |x_1| + |x_2| + \cdots + |x_n|.$$

**Remark 2.1** *The Manhattan distance for the case $n = 2$ is named after the grid pattern of the streets and avenues in Manhattan.*

In the following, we shall restrict ourselves to the simplest case $n = 2$, i.e.,

$$E = E_{(a,b;c)} : ax + by = c.$$

Then there exist the solutions $(x, y) \in S_{(a,b;c)}$ with the length $L(x, y) = |x|+|y|$ of the minimal value. We denote this minimal value $\min\{L(x, y) \mid (x, y) \in S_{(a,b;c)}\}$ by $L_E$, and call the value $L_E$ the *minimal length* of the integer solutions of the linear diophantine equation $E = E_{(a,b;c)}$. We also call the solutions $(x, y)$ with the minmal length $L_E$ the *minimal integer solutions.*

Firstly, we shall begin the distribution of the length of integer solutions of an example of the equation $E_{(5,3;38)}$.

Tabel 1

| $k$ | $(x = -38 + 3k, y = 76 - 5k)$ | The length $L(x, y)$ |
|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ |
| 0 | $(-38, 76)$ | 114 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $k$ | $(-38 + 3k, 76 - 5k)$ | $114 - 8k$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 11 | $(-5, 21)$ | 26 |
| 12 | $(-2, 16)$ | 18 |
| 13 | $(1, 11)$ | 12 |
| 14 | $(4, 6)$ | 10 |
| 15 | $(7, 1)$ | 8 |
| 16 | $(10, -4)$ | 14 |
| 17 | $(13, -9)$ | 22 |
| 18 | $(16, -14)$ | 30 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $k$ | $(-38 + 3k, 76 - 5k)$ | $8k - 114$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Then the above length $L(x, y)$ is classified into the following three arithmetic progressions, which will be abbreviated to AP in the following;

$\{18 + 8k | k \geq 0\}$ AP with the initial term 18 and the common difference 8,

$\{8, 10, 12\}$, Finite AP with the common difference 2,

$\{14 + 8k | k \geq 0\}$ AP with the initial term 14 and the common difference 8.

Now we will generalize the above results to the equation $E_{(a,b;c)} : ax + by$, where $a > b > 0$ and $\mathrm{GCD}(a, b) = 1$ and $c > 0$. Then the length $L(x, y)$ of the integer solutions $S_{(a,b;c)}$ is classified into the following three classes:

Infinite AP with the common difference $a + b$, for $(x, y) \in S_2 = \{(x, y) | x < 0\}$,

Finite AP with the common difference $a - b$, for $(x, y) \in S_0 = \{(x, y) | x, y \geq 0\}$,

Infinite AP with the common difference $a + b$, for $(x, y) \in S_1 = \{(x, y) | y < 0\}$.

Let $L_i$ be the minimal length of the minimal integer solutions in $S_i$, $(0 \leq i \leq 2)$. We note the case $S_0 = \emptyset$ may happen. For example, any $(x, y) \in S_{(a,b;1)}$ with $a > b \geq 2$ satify $xy < 0$ and hence $S_0 = \emptyset$ for this case.

**Theorem 2.2** *Assume* $a > b > 0$, $\mathrm{GCD}(a,b) = 1$ *and* $c > 0$. *Then the minimal length* $L_E = \min(L_0, L_1, L_2)$. *In case* $S_0 = \emptyset$, $L_E = \min(L_1, L_2)$. *In case* $S_0 \neq \emptyset$, $L_E = \min(L_0, L_1)$.

## 2.1 Algorithm for finding the minimal solutions 1

We shall recall the Euclidean algorithm for $a > b > 0$ with $n$ steps;

$$\begin{array}{lcl} a & = & a_0 b + r_1,\ (0 < r_1 < b), \\ b & = & a_1 r_1 + r_2,\ (0 < r_2 < r_1), \\ r_1 & = & a_2 r_2 + r_3,\ (0 < r_3 < r_2), \\ & \vdots & \\ r_{n-2} & = & a_{n-1} r_{n-1} + r_n,\ (0 < r_n < r_{n-1}), \\ r_{n-1} & = & a_n r_n, \\ r_n & = & d = \mathrm{GCD}(a,b). \end{array}$$

Put $r_{-1} = a$, and $r_0 = b$. Then the binary recurrence sequences $X_i, Y_i$ are defined by putting

$$X_i = a_{i-1} X_{i-1} + X_{i-2}, \quad Y_i = a_{i-1} Y_{i-1} + Y_{i-2},$$

with initial terms $X_{-1} = 1, X_0 = 0$ and $Y_{-1} = 0, Y_0 = 1$, One obtains, by induction,

$$a(-1)^{i-1} X_i + b(-1)^i Y_i = r_i, (-1 \leq i \leq n).$$

Assume $n \geq 2$, i.e., $b \not| a$. $d$ denotes $\mathrm{GCD}(a,b)$. Then, from the extended Euclidean algorithm, $E_{(a,b;d)}$ has the minimal integer solutions

$$(x, y) = ((-1)^{n-1} X_n, (-1)^n Y_n).$$

Then $(-1)^n X_{n+1} = (-1)^n b, (-1)^{n+1} Y_{n+1} = (-1)^{n+1} a$, and hence $X_i \leq \dfrac{b}{2d}$ and $Y_i \leq \dfrac{a}{2d}$. Therefore the minimal length $L_E$ satisfies

$$L_E = X_n + Y_n < \frac{a+b}{2d}.$$

Since $X_i + Y_i < X_n + Y_n$ for any $-1 \leq i \leq n-1$, one can generalize this result as follows.

**Theorem 2.3** *For the case* $c = r_i$ $(-1 \leq i \leq n)$, *the minimal integer solutions of the equation* $E = E_{(a,b;r_i)} : ax + by = r_i$ *and the minimal length* $L_E$ *are given by*

$$(x, y) = ((-1)^{i-1} X_i, (-1)^i Y_i), \quad L_E = X_i + Y_i, (-1 \leq i \leq n).$$

## 2.2 Continued fraction

Let $\frac{a}{b}$ be a reduced fraction satisfying the following $n$ steps.

$$\begin{aligned}
a &= a_0 b + r_1,\ (0 < r_1 < b)\\
b &= a_1 r_1 + r_2,\ (0 < r_2 < r_1)\\
r_1 &= a_2 r_2 + r_3,\ (0 < r_3 < r_2)\\
&\vdots\\
r_{n-2} &= a_{n-1} r_{n-1} + r_n,\ (0 < r_n < r_{n-1})\\
r_{n-1} &= a_n r_n\\
r_n &= 1 = GCD(a,b)
\end{aligned}$$

Then the continued fraction expansion of the rational number $\frac{a}{b}$ is denoted by

$$\frac{a}{b} = [a_0; a_1, a_2 \ldots, a_n].$$

The $k-$th intermediate continued fraction is defined by putting

$$\frac{P_k}{Q_k} = [a_0; a_1, \ldots, a_k].$$

Put $P_{-1} = 0, P_0 = 1, Q_{-1} = 1, Q_0 = 0$. The recurrence sequences $P_k, Q_k$ are defied by putting

$$P_{k+1} = a_k P_k + P_{k-1}, Q_{k+1} = a_k Q_k + Q_{k-1},\ \text{for } k \geq 0.$$

These recurrence sequences can be written by the use of matrices,

$$\begin{pmatrix} P_{k+1} & P_k \\ Q_{k+1} & Q_k \end{pmatrix} = \begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $a = P_{n+1}, Q_{n+1} = b$, one gets,

$$\begin{pmatrix} a & P_n \\ b & Q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $Q_k = X_k,\ P_k = Y_k$, where $X_k, Y_k$ are those of the extended Euclidean algorithm for $a > b$.

## 2.3 The Frobenius coin problem

To investigate the algorithm of finding the minimal integer solutions for larger $c$, we shall recall the Frobenius coin problem, which states the existence of the non-negative integer solutions for given linear diophantine equations. Let $\{a_1, a_2, \ldots, a_n\}$ be coprime positive integers. Fron Schur's theorem, there exists the largest positive integer $c = g(a_1, a_2, \ldots, a_n)$ for which the linear diophantine equation $E_{(a_1,a_2,,\ldots,a_n;c)} : a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ has no non-negative integer solutions $x_1, x_2, \ldots, x_n$. The number $g(a_1, a_2, \ldots, a_n)$ is called the *Frobenius number*. From the definition of the Frobenius number, the equation $E_{(a_1,a_2,\ldots,a_n;c)}$ has non-negative integer solutions for any $c > g(a_1, a_2, \ldots, a_n)$. Though the explicit closed form of the Frobenius number for $n \geq 3$ is still an open problem, the following case $n = 2$ is well known.

**Theorem 2.4 (Frobenius number for the case $n = 2$)** *Let $a, b$ be coprime positive integers. Then the Frobenius number $g(a, b)$ for the equation $ax+by = c$ is*

$$g(a, b) = ab - a - b.$$

*When $c$ varies $0 \leq c \leq g(a, b) = (a - 1)(b - 1) - 1$, there exist exactly $\dfrac{(a-1)(b-1)}{2}$ equations $E_{(a,b;c)}$ with non-negative integer solutions $x, y$.*

**Remark 2.5** *Last part of the above theorem is easily proved from the following property;*

$$\begin{aligned} &E_{(a,b;c)} \text{ has non-negative integer solutions} \\ \Longleftrightarrow\ &E_{(a,b;g(a,b)-c)} \text{ has no non-negative integer solutions.} \end{aligned}$$

## 2.4 Algorithm for finding minimal solutions 2

Assume the positive integers $a, b$ satisfy $a > b > 0$ and $\mathrm{GCD}(a, b) = 1$. Then the condition for the equation $E_{(a,b;c)} : ax + by = c$ has non-negative integer solutions $(x, y)$ is the following. Consider the following linear congruence $by \equiv c \pmod{a}$. Then there exists $y = y_0$ with $0 \leq y_0 < a$. If $c - by_0 \geq 0$, the integer $x_0 = (c - by_0)/a$ satisfies $ax_0 + by_0 = c$ with $x_0, y_0 \geq 0$, i.e., the equation has the non-negative integer solutions $(x_0, y_0)$. Moreover $(x_0, y_0)$ are the minimal integer solutions of $S_0$ and hence the length $L(x_0, y_0) = x_0 + y_0$ is $L_0$. Then the solutions $(x_0 + b, y_0 - a)$ are the minimal solutions of $S_1$ and the length $L(x_0 + b, y_0 - a) = x_0 - y_0 + a + b$ is the minimal length $L_1$. Thus

$$L_0 \leq L_1 \Longleftrightarrow y_0 \leq \frac{a+b}{2}.$$

On the contrary, if $c - by_0 < 0$, the integer $x_0 = (c - by_0)/a$ satisfy $ax_0 + by_0 = c$ with $x_0 < 0, y_0 \geq 0$, and the equation does not have non-negative integer solutions. The length $L(x_0, y_0) = -x_0 + y_0$ is $L_2$ for this case. Hence the solutions $(x_0 + b, y_0 - a)$ are the minimal solutions of $S_1$ and the length $L(x_0 + b, y_0 - a) = x_0 - y_0 + a + b$ is the minimal length $L_1$. Thus we have

$$L_2 \leq L_1 \Longleftrightarrow -x_0 + y_0 \leq \frac{a+b}{2}.$$

**Theorem 2.6** *Under the above notations, the minimal integer solutions and the minimal length* $L_E$ *are the following:*
*If* $c \geq by_0$ *and* $y_0 \leq \dfrac{a+b}{2}$*, then the minimal integer solutions are* $(x_0, y_0)$ *and the minimal length is* $L_E = x_0 + y_0$*.*
*If* $c \geq by_0$ *and* $y_0 > \dfrac{a+b}{2}$*, then the minimal integer solutions are* $(x_0+b, y_0-a)$ *and the minimal length is* $L_E = x_0 - y_0 + a - b$*.*
*If* $c < by_0$ *and* $-x_0+y_0 \leq \dfrac{a+b}{2}$*, then the minimal integer solutions are* $(x_0, y_0)$ *and the minimal length is* $L_E = -x_0 + y_0$*.*
*If* $c < by_0$ *and* $-x_0 + y_0 > \dfrac{a+b}{2}$*, then the minimal integer solutions are* $(x_0 + b, y_0 - a)$ *and the minimal length is* $L_E = x_0 - y_0 + a + b$*.*

**Remark 2.7** *Given a equation* $E_{(a,b;c)} : ax+by = c$*, with* $a > b > 0$*,* $c > 0$ *and* $\mathrm{GCD}(a, b) = 1$*. From this theorem, one can find the minimal integer solutions* $(X, Y)$ *and any integer solutions are written in the form* $(X+kb, Y-ka), k \in \mathbb{Z}$*. When* $a = 5, b = 3$ *and* $c = 37$*, one can verifies that* $(7, 1)$ *are the minimal integer solutions as mentioned in the firsst section.*

# 3 Related problems

## 3.1 Equivalence classes of the linear diophantine equation

Let $V$ be the set of integer vectors $(a, b, c) \in \mathbb{Z}^3$ which satisfy the condition $\mathrm{GCD}(a, b)|c$. Then $(a, b, c) \in V$ is nothing but the integer solutions $S_{(a,b;c)} \neq \emptyset$ of the corresponding diophantine equation $E_{(a,b;c)}$. We denote $(a_1, b_1, c_1) \cong (a_2, b_2, c_2)$ if there exists integers $p, q, pq \neq 0$ which satisfy

$$p(a_1, b_1, c_1) = q(a_2, b_2, c_2).$$

Then one can see

$$(a_1, b_1, c_1) \cong (a_2, b_2, c_2) \Longleftrightarrow S_{(a_1,b_1;c_1)} = S_{(a_2,b_2;c_2)}.$$

Therefore, for any $(a, b, c) \in V$, there exists $(a_0, b_0, c_0) \cong (a, b, c)$ with $\mathrm{GCD}(a_0, b_0) = 1$.

Let $\varepsilon_a, \varepsilon_b, \varepsilon_c \in \{-1, 1\}$ and put $a' = \varepsilon_a a, b' = \varepsilon_b b, c' = \varepsilon_c c$. The map $\phi$ from $E_{(a,b;c)}$ to $E_{(a',b';c')}$ is defined by putting

$$\phi : (x, y) \in S_{(a,b;c)} \to (x', y') \in S_{(a',b';c')},$$

where $x' = \phi(x) = \varepsilon_c\varepsilon_a x, y' = \phi(y) = \varepsilon_c\varepsilon_b y$. Then $\phi$ defines a bijection from $S_{(a,b;c)}$ to $S_{(a',b';c')}$ which preserves the length of integer solutions

$$L(x, y) = |x| + |y| = |x'| + |y'| = L(x', y') = L(\phi(x), \phi(y)),$$

where

$$ax+by=c \Longleftrightarrow (a\varepsilon_a)(\varepsilon_c\varepsilon_a x)+(b\varepsilon_b)(\varepsilon_c\varepsilon_b y)=(c\varepsilon_c) \Leftrightarrow a'x'+b'y'=c''.$$

Thus, to investigate the distributions of the integer solutions of the given equation $E_{(a,b;c)}$, we may restrict ourselves to the case $a>b>0, c>0$ and $\mathrm{GCD}(a,b)=1$, without loss of generality.

**Remark 3.1** *We denote* $(a,b,c)\sim(a',b',c')$ *if there exists the above map* $\phi: E_{(a,b;c)} \to E_{a',b',c')}$. *There are examples* $(a,b,c)\not\sim(a',b',c')$ *and* $(a,b,c)\ncong(a',b',c')$, *but have the same set of the length of integer solutions. For example, consider the equations* $E_{(11,3;7)}$ *and* $E_{(9,5;7)}$. *Then* $(11,3,7)\not\sim(9,5,7)$ *and* $(11,3,7)\ncong(9,5,7)$, *but both equations have the same set of the length of integer solutions* $\{7,21,28,\ldots\}$, *i.e., AP with the initial term* 7 *and the common difference* 14.

## 3.2 Three Jug Problem

Originally, *three jug problem* is the following mathematical puzzle. Let $a$, $b$ are positive integers with $a>b$. Given three jugs, the first jug $A$ with $a$ pints, the second jug $B$ with $b$ pints, and the third jug $C$ with $a+b$ pints. Make two jugs $A$ and $C$ with the same amount $(a+b)/2$, by only completely filling up and/or emptying vessels into others. It is known that this problem can be solved by using the solution of the linear diophantine equation $E_{(a,b;(a+b)/2))}$.

This problem is slightly modified and generalized as follows. Given two empty buckets $A$ and $B$ of positive integer capacities $a$ and $b$, respectively and a well containing an inexhautible supply of water. Moreover $a>b$ and $\mathrm{GCD}(a,b)=1$. One is asked to obtain a fixed quantity of liquid $c$ using only two initially empty buckets $A$ and $B$ by only completely filling up and/or emptying buckets into others and also utilizing the well. In the film "Die Hard: With a Vengeance" (1995), this problem of the case $a=5$, $b=3$ and $c=4$ has been treated.

We shall explain this example using the symbol $[p,q]$, where $p$ represents the amount of water in the first bucket $A$ with the capacity 5 and $q$ represents the amount of water in the second bucket $B$ with the capacity 3.

(1)

$$[0,0]\to[5,0]\to[2,3]\to[2,0]\to[0,2]\to[5,2]\to[4,3]$$

(2)

$$[0,0]\to[0,3]\to[3,0]\to[3,3]\to[5,1]\to[0,1]\to[1,0]\to[1,3]\to[4,0]$$

Here (1) is the procedure corresponding the integer solutions $(2,-2)$ of $E_{(5,3;4)}$ : $5x+3y=4$, and (2) is the procedure corresponding the integer solutions $(-1,3)$ of $E_{(5,3;4)}$.

Let $(x, y)$ be the integer solutions of $E_{(a,b;c)} : ax + by = c$. $N$ denotes the number of times to need to amount $c$ with the buckets $a$ and $b$ corresponding to these solutios $(x, y)$. Then $N$ is formulated as follows.

**Theorem 3.2 (modified three jug problem)** *Assume* $a > b$ *and* $\mathrm{GCD}(a, b) = \mathrm{GCD}(a, c) = \mathrm{GCD}(b, c) = 1$. *Then, using the length of the integer solution* $L(x, y) = |x| + |y|$, *the number of times* $N$ *is expressed by;*

*If* $1 \leq c < b$, *then* $N = 2L(x, y) - 2$.
*If* $b < c < a$ *and* $x > 0$, *then* $N = 2L(x, y) - 2$.
*If* $b < c < a$ *and* $x < 0$, *then* $N = 2L(x, y)$.

Assume $a > b > 0$ with $\mathrm{GCD}(a, b) = \mathrm{GCD}(a, c) = \mathrm{GCD}(b, c) = 1$ as above. Then, very roughly speaking, to determine the minimal number of times $N$ for the above modified three jug problem is nothing but to determine the minimal length $L_E$ of the equation $E_{(a,b;c)}$. Assume the additional condition $a > c > 0$, then there exists only one couple $(x, y)$ of minimal integer solutions for the cases $c \neq \dfrac{a+b}{2}$ from Theorem 2.5. Moreover the case $c = \dfrac{a+b}{2}$ has exactly 2 minimal integer solutions;

$$(x, y) = \left( \frac{-b+1}{2}, \frac{a+1}{2} \right), \text{ and } \left( \frac{b+1}{2}, \frac{-a+1}{2} \right).$$

**Theorem 3.3** *Let* $a$ *be coprime positive integers with* $a > b$. *For any* $c$ $(1 \leq c < a)$, *the equation* $E_{(a,b;c)} : ax + by = c$ *has exactly one couple of minimal integer solutions except for the case* $c = \dfrac{a+b}{2}$. $E = E_{(a,b;(a+b)/2)} : ax + by = \dfrac{a+b}{2}$ *has exactly two minimal integer solutions*

$$(x, y) = \left( \frac{-b+1}{2}, \frac{a+1}{2} \right), \text{ and } \left( \frac{b+1}{2}, \frac{-a+1}{2} \right),$$

*where the both minimal length is* $L_E = \dfrac{a+b}{2}$.

**Remark 3.4** *The exceptional case* $c = \dfrac{a+b}{2}$ *occurs only when* $a \equiv b \equiv 1 \pmod 2$.

**Corollary 3.5** *Consider three jug problem for the case* $a, b$, *where* $a$ *and* $b$ *are coprime odd positive integers with* $a > b$. *Then the number of times* $N$ *corresponding to the solutions* $\left( \dfrac{b+1}{2}, \dfrac{-a+1}{2} \right)$ *is* $a + b - 1$.

*The number of times* $N$ *corresponding to the solutions* $\left( \dfrac{-b+1}{2}, \dfrac{a+1}{2} \right)$ *is* $a + b$.

**Remark 3.6** *Three jug problem is sometimes called the decanter problem, where the liquid is wine. In Japan, Mitsuyoshi Yoshida published a book "Jinkouki" in 1627. In this book he treated three jug problem of the case* $[a, b] = [7, 3]$, *where the liquid is oil.*

## 3.3 Examples of minimal integer solutions

In this section, we shall treat the special class of equations $E_{(a,b;c)}$. Let $F_n$ and $L_n$ be $n$-th Fibonacci and Lucas numbers, respectively. Fibonacci numbers $F_n$ and Lucas numbers $L_n$ are defined by putting,

$$F_{n+1} = F_n + F_{n-1}, \text{ and } L_{n+1} = L_n + L_{n-1},$$

with inital terms $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$. For the sake of readers, we shall list Fibonacci numbers and Lucas numbers for small indices $n$.

| $n$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| $L_n$ | $-1$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 |

Here we will give the minimal solutions of the following equations for small $c$.

$$F_{n+1}x + F_n y = c, \text{ and } L_{n+1}x + L_n y = c, \text{where } 1 \leq c \leq 5.$$

The following well known formula is called Cassini's identity, which played the key role in our old paper [3].

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \text{i.e.,} \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

Thus one obtains

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = \begin{vmatrix} F_{n+1} & F_n - F_{n+1} \\ F_n & F_{n-1} - F_n \end{vmatrix} = \begin{vmatrix} F_{n+1} & -F_{n-1} \\ F_n & -F_{n-2} \end{vmatrix} = - \begin{vmatrix} F_{n+1} & F_{n-1} \\ F_n & F_{n-2} \end{vmatrix}.$$

Hence we have shown the equation $E_{(F_{n+1},F_n,1)} : F_{n+1}x + F_n y = 1$ have integer solutions $(x, y) = ((-1)^{n-1}F_{n-2}, (-1)^n F_{n-1})$. Actually these solutions are the minimal integer solutions and the minimal length is $L_E = F_n (= F_{n-1} + F_{n-2})$. Hence we have shown the following result.

(3.1) The equation $E_{(F_{n+1},F_n;1)}$
Minimal integer solutions are $(x, y) = ((-1)^{n-1}F_{n-2}, (-1)^n F_{n-1})$. The minimal length is $L_E = F_n$.
Similarly, one can easily verify the following examples.
(3.2) The equation $E_{(F_{n+1},F_n;2)}$
Minimal integer solutions are $x = (-1)^n F_{n-3}, y = (-1)^{n-1}F_{n-2}$. The minimal

length is $L_E = F_{n-1}$.
(3.3) The equation $E_{(F_{n+1},F_n;3)}$
Minimal integer solutions are $x = (-1)^{n-1}F_{n-4}, y = (-1)^n F_{n-3}$. The minimal length is $L_E = F_{n-2}$.
(3.4) The equation $E_{(F_{n+1},F_n;4)}$
Minimal integer solutions are $x = (-1)^n 2F_{n-3}, y = (-1)^{n-1}2F_{n-2}$, The minimal length is $L_E = 2F_{n-1}$.
(3.5) The equation $E_{(F_{n+1},F_n,;5)}$
Minimal integer solutions are $x = (-1)^n F_{n-5}, y = (-1)^{n-1}F_{n-4}$. The minimal length is $L_E = F_{n-3}$.
(3.6) The equation $E_{(L_{n+1},L_n;1)}$
Minimal integer solutions are $x = (-1)^n F_{n-1}, y = (-1)^{n+1}F_n$. The minimal length is $L_E = F_{n+1}$.
(3.7) The equation $E_{(L_{n+1},L_n;2)}$
Minimal integer solutions are $x = (-1)^{n+1}F_n, y = (-1)^n F_{n+1}$. The minimal length is $L_E = F_{n+2}$.
(3.8) The equation $E_{(L_{n+1},L_n;3)}$
Minimal integer solutions are $x = (-1)^{n-1}F_{n-2}, y = (-1)^n F_{n-1}$. The minimal length is $L_E = F_n$.
(3.9) The equation $E_{(L_{n+1},L_n;4)}$
Minimal integer solutionsa re $x = (-1)^n F_{n-3}, y = (-1)^{n-1}F_{n-2}$. The minimal length is $L_E = F_{n-1}$.
(3.10) The equation $E_{(L_{n+1},L_n;5)}$
Minimal integer solutions are $x = (-1)^n L_{n-2}, y = (-1)^{n-1}L_{n-1}$. The minimal length is $L_E = L_n$.

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