

A Lemma for a Strong Comparison Principle of Nonlinear Parabolic Equations

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Abstract

In this note, we prove a lemma for a strong comparison principle of nonlinear parabolic equations. We shall prove a function which is a viscosity subsolution minus a viscosity supersolution of the equation becomes a viscosity subsolution of a parabolic equation which may not coincide with the original equation. Thanks to a strong maximum principle of nonlinear parabolic equations we have a strong comparison principle.

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Introduction

We consider nonlinear parabolic equations of the form

$$(1) \quad u_t + F(Du, D^2u) = 0 \quad \text{in } Q_T := (0, T) \times \Omega,$$

where $u : \overline{Q_T} \rightarrow \mathbf{R}$ is an unknown function, $F = F(q, X)$ is a given function, $T > 0$ and Ω is a domain in \mathbf{R}^N . Here $u_t = \partial u / \partial t$, Du and D^2u denote, respectively, the time derivative of u , the gradient of u and the Hessian of u in space variables.

Our goal is that when u and v are, respectively, a subsolution and supersolution of (1), $u - v$ becomes a subsolution of a parabolic equation which may not coincide with (1). For uniformly parabolic equations we know that a strong maximum principle holds to (1). If $u - v \leq 0$ in Q_T and there exists a point in Q_T that satisfies $u - v = 0$, then a strong maximum principle yields $u \equiv v$ in Q_T . This means a strong comparison principle holds to (1). So our goal is important to prove a strong comparison principle. For nonlinear parabolic equations we may not expect existence of classical solutions. So we deal with this problem using viscosity solutions (cf. [2], [6]).

In the study of a strong comparison principle with viscosity solutions, there are a few papers. Trudinger [9] proved a strong comparison principle for Lipschitz continuous viscosity solutions of uniformly elliptic equations. Ishii and Yoshimura [5] proved a strong comparison principle for semicontinuous viscosity solutions to uniformly elliptic equations. At the same time Giga and the second author [4] studied a strong comparison principle. Their proof [4, Proof of 3.1, p175-177] works for uniformly elliptic equations of the form $F(D^2u) = 0$ but it does not work for non-uniformly elliptic equations of the form $F(Du, D^2u) = 0$. As a special case the second author and Sakaguchi [8] proved a strong comparison principle for semicontinuous viscosity solutions to the prescribed mean curvature equation.

For parabolic problems there is a result by the second author [7]. Since the proof [7, Proof of Lemma 3.4, p159-162] is based on that of [4], it works for uniformly parabolic equations of the form $u_t + F(D^2u) = 0$ but it does not work for (1). To nonlinear parabolic equations Da Lio [3] proved a strong maximum principle for semicontinuous viscosity solutions. Once our goal is proved, thanks to the strong maximum principle we can show that a strong comparison principle holds to (1).

1 Proof of Lemma

We shall study nonlinear parabolic equations of form

$$(1.1) \quad u_t + F(Du, D^2u) = 0 \quad \text{in } Q_T.$$

We list assumptions on $F = F(p, X)$.

(F1) F is lower semicontinuous in $\mathbf{R}^N \times \mathbf{S}^N$.

(F2) F is degenerate elliptic, i.e.,

$$\text{if } X \geq Y \quad \text{then } F(p, X) \leq F(p, Y) \quad \text{for all } p \in \mathbf{R}^N.$$

We introduce F_0 as follows

$$(1.2) \quad F_0(p, X) := \inf\{F(p + q, X + Y) - F(q, Y); (q, Y) \in \mathbf{R}^N \times \mathbf{S}^N\}.$$

This function F_0 is introduced in [5], [6]. To consider our problem we will assume lower boundedness of F_0 .

(F3) $F_0(p, X) > -\infty$ for all $p \in \mathbf{R}^N$ and $X \in \mathbf{S}^N$.

We easily see a following property about F_0 .

Proposition 1.1. *If F satisfies (F1), then F_0 is lower semicontinuous.*

Proof. We fix $(\hat{p}, \hat{X}) \in \mathbf{R}^N \times \mathbf{S}^N$. Since F is lower semicontinuous, we see that for all $\varepsilon > 0$ there exists $\delta > 0$ that satisfies

$$\text{if } (p, X) \in B_\delta(\hat{p}, \hat{X}) \text{ then } -\varepsilon + F(\hat{p}, \hat{X}) < F(p, X).$$

Here

$$B_\delta(\hat{p}, \hat{X}) = \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N; \{|p - \hat{p}|^2 + \|X - \hat{X}\|^2\}^{1/2} < \delta\},$$

where $\|X\| := \max\{|X\xi|; \xi \in \mathbf{R}^N, |\xi| = 1\}$ for $X \in \mathbf{S}^N$. By the definition of F_0 we have that for $(r, Z) \in \mathbf{R}^N \times \mathbf{S}^N$

$$\begin{aligned} F_0(\hat{p}, \hat{X}) - F_0(r, Z) &= \inf\{F(\hat{p} + q, \hat{X} + Y) - F(q, Y); (q, Y) \in \mathbf{R}^N \times \mathbf{S}^N\} \\ &\quad - \inf\{F(r + q, Z + Y) - F(q, Y); (q, Y) \in \mathbf{R}^N \times \mathbf{S}^N\}. \end{aligned}$$

Since F_0 is bounded from below, there exists $(\hat{q}, \hat{Y}) \in \mathbf{R}^N \times \mathbf{S}^N$ that satisfies

$$F_0(r, Z) = F(r + \hat{q}, Z + \hat{Y}) - F(\hat{q}, \hat{Y}).$$

Then we observe that

$$\begin{aligned} F_0(\hat{p}, \hat{X}) - F_0(r, Z) &\leq F(\hat{p} + \hat{q}, \hat{X} + \hat{Y}) - F(\hat{q}, \hat{Y}) - F(r + \hat{q}, Z + \hat{Y}) + F(\hat{q}, \hat{Y}) \\ &\leq F(\hat{p} + \hat{q}, \hat{X} + \hat{Y}) - F(r + \hat{q}, Z + \hat{Y}). \end{aligned}$$

Now we have that if $(r, Z) \in B_\delta(\hat{p}, \hat{X})$ then $F_0(\hat{p}, \hat{X}) - F_0(r, Z) < \varepsilon$. \square

Now we are in a position to state our main result.

Lemma 1.2. *Assume that (F1), (F2) and (F3) hold. Let $u \in USC([0, \infty) \times \mathbf{R}^N)$ and $v \in LSC([0, \infty) \times \mathbf{R}^N)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (1.1). We set $w = u - v$. Then w is a viscosity subsolution of*

$$(1.3) \quad u_t + F_0(Du, D^2u) = 0 \quad \text{in } Q_T.$$

Proof. Let $\phi \in C^2((0, \infty) \times \mathbf{R}^N)$ and $(\hat{t}, \hat{x}) \in Q_T$ satisfy $(w - \phi)(\hat{t}, \hat{x}) \geq (w - \phi)(s, y)$ for all $(s, y) \in B_r(\hat{t}, \hat{x})$ for some $r > 0$, where $B_r(\hat{t}, \hat{x})$ denotes an open ball in \mathbf{R}^{N+1} centered at (\hat{t}, \hat{x}) with a radius r . We may assume that $w - \phi$ takes its locally strict maximum at (\hat{t}, \hat{x}) . For $\varepsilon > 0$ we set

$$\Phi(t, x, s, y) := u(t, x) - v(s, y) - \phi(t, x) - \frac{1}{2\varepsilon}(|x - y|^2 + |t - s|^2).$$

Since Φ is an upper semicontinuous function, Φ takes its maximum on $\overline{B_r(\hat{t}, \hat{x})} \times \overline{B_r(\hat{t}, \hat{x})}$ for some $r > 0$. Let $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ be a maximizer of Φ on $\overline{B_r(\hat{t}, \hat{x})} \times \overline{B_r(\hat{t}, \hat{x})}$.

Step 1. We shall show $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ converges to $(\hat{t}, \hat{x}, \hat{t}, \hat{x})$ as $\varepsilon \rightarrow 0$.

Since Φ takes its maximum on $\overline{B_r(\hat{t}, \hat{x})} \times \overline{B_r(\hat{t}, \hat{x})}$ at $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$, we see that

$$\Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \geq \Phi(\hat{t}, \hat{x}, \hat{t}, \hat{x}) = (w - \phi)(\hat{t}, \hat{x}).$$

As usual we may assume that $(w - \phi)(\hat{t}, \hat{x}) = 0$. Since $u, -v \in USC([0, \infty) \times \mathbf{R}^N)$ and $\phi \in C^2((0, \infty) \times \mathbf{R}^N)$, there exists a constant C that satisfies

$$(1.4) \quad \frac{1}{2\varepsilon} (|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2) \leq u(t_\varepsilon, x_\varepsilon) - v(s_\varepsilon, y_\varepsilon) - \phi(t_\varepsilon, x_\varepsilon) \leq C$$

Then we see that

$$(1.5) \quad \lim_{\varepsilon \rightarrow 0} (x_\varepsilon - y_\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} (t_\varepsilon - s_\varepsilon) = 0.$$

Note that $(t_\varepsilon, x_\varepsilon) \in \overline{B_r(\hat{t}, \hat{x})}$, the Bolzano-Weierstrass theorem yields that there exists a sequence $\{\varepsilon_k\}$ which decreases to 0 as $k \rightarrow \infty$ and $(t_0, x_0) \in \overline{B_r(\hat{t}, \hat{x})}$ satisfying

$$(t_{\varepsilon_k}, x_{\varepsilon_k}) \rightarrow (t_0, x_0) \quad \text{as } k \rightarrow \infty.$$

By (1.5) we observe that

$$\lim_{k \rightarrow \infty} (y_{\varepsilon_k} - x_0) = \lim_{k \rightarrow \infty} (y_{\varepsilon_k} - x_{\varepsilon_k}) + \lim_{k \rightarrow \infty} (x_{\varepsilon_k} - x_0) = 0.$$

So we have $\lim_{k \rightarrow \infty} x_{\varepsilon_k} = \lim_{k \rightarrow \infty} y_{\varepsilon_k} = x_0$. By a similar way we have $\lim_{k \rightarrow \infty} t_{\varepsilon_k} = \lim_{k \rightarrow \infty} s_{\varepsilon_k} = t_0$. Concerning (1.4) we know that $u \in USC([0, \infty) \times \mathbf{R}^N)$ and $v \in LSC([0, \infty) \times \mathbf{R}^N)$. Then we observe that

$$\begin{aligned} 0 \leq \liminf_{k \rightarrow \infty} \frac{|x_{\varepsilon_k} - y_{\varepsilon_k}|^2 + |t_{\varepsilon_k} - s_{\varepsilon_k}|^2}{2\varepsilon_k} &\leq \limsup_{k \rightarrow \infty} \frac{|x_{\varepsilon_k} - y_{\varepsilon_k}|^2 + |t_{\varepsilon_k} - s_{\varepsilon_k}|^2}{2\varepsilon_k} \\ &\leq u(t_0, x_0) - v(t_0, x_0) - \phi(t_0, x_0) \\ &\leq (w - \phi)(\hat{t}, \hat{x}) = 0. \end{aligned}$$

These inequalities yield

$$\lim_{k \rightarrow \infty} \frac{|x_{\varepsilon_k} - y_{\varepsilon_k}|^2}{\varepsilon_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{|t_{\varepsilon_k} - s_{\varepsilon_k}|^2}{\varepsilon_k} = 0.$$

Recall that $w - \phi$ takes its locally strict maximum at (\hat{t}, \hat{x}) we see that

$$\lim_{k \rightarrow \infty} x_{\varepsilon_k} = \lim_{k \rightarrow \infty} y_{\varepsilon_k} = \hat{x}, \quad \lim_{k \rightarrow \infty} t_{\varepsilon_k} = \lim_{k \rightarrow \infty} s_{\varepsilon_k} = \hat{t}.$$

Recall again that $w - \phi$ takes its locally strict maximum at (\hat{t}, \hat{x}) we observe that these convergences are independent of taking a subsequence. Finally we have that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} x_\varepsilon = \lim_{\varepsilon \rightarrow 0} y_\varepsilon = \hat{x}, \quad \lim_{\varepsilon \rightarrow 0} t_\varepsilon = \lim_{\varepsilon \rightarrow 0} s_\varepsilon = \hat{t}, \\ \lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{|t_\varepsilon - s_\varepsilon|^2}{\varepsilon} = 0. \end{aligned}$$

Step 2. We shall show $w = u - v$ is a viscosity subsolution of (1.3).

We set

$$\Psi(t, x, s, y) := \frac{1}{2\varepsilon}(|x - y|^2 + |t - s|^2).$$

Since Φ takes its maximum at $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$, $(u - \phi)(t, x) - v(s, y) - \Psi(t, x, s, y)$ takes its maximum at the same point. By step 1 we may assume that

$$(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in B_r(\hat{t}, \hat{x}) \times B_r(\hat{t}, \hat{x}).$$

Applying Crandall-Ishii's Lemma [1] we see that for each $\alpha > 1$ there exist $X, Y \in \mathbf{S}^N$ that satisfy

$$\begin{aligned} (\Psi_t(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), D_x \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), X) &\in \overline{\mathcal{P}^{2,+}}(u - \phi)(t_\varepsilon, x_\varepsilon), \\ (\Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), D_y \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), Y) &\in \overline{\mathcal{P}^{2,+}}(-v)(s_\varepsilon, y_\varepsilon), \\ (\Leftrightarrow (-\Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), -D_y \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), -Y) &\in \overline{\mathcal{P}^{2,-}}v(s_\varepsilon, y_\varepsilon)), \\ (1.6) \quad -(\alpha + \|A\|) I_{2N} &\leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \frac{1}{\alpha} A^2. \end{aligned}$$

Here

$$A = D^2 \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \begin{pmatrix} D_{xx}^2(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) & D_{xy}^2(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \\ D_{yx}^2(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) & D_{yy}^2(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \end{pmatrix},$$

$\overline{\mathcal{P}^{2,+}}$ and $\overline{\mathcal{P}^{2,-}}$, respectively, denote closure of a set of parabolic super 2-jets $\mathcal{P}^{2,+}$ and a set of parabolic sub 2-jets $\mathcal{P}^{2,-}$ (cf. [2],[6]). By calculations we have

$$\begin{aligned} \Psi_t(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) &= \frac{t_\varepsilon - s_\varepsilon}{\varepsilon}, \quad \Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = -\frac{t_\varepsilon - s_\varepsilon}{\varepsilon}, \\ D_x \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) &= \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \quad D_y \Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = -\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \\ (1.7) \quad A &= \frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{aligned}$$

By the definition of $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ we observe that

$$\begin{aligned} \left(\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \phi_t(t_\varepsilon, x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + D\phi(t_\varepsilon, x_\varepsilon), X + D^2\phi(t_\varepsilon, x_\varepsilon) \right) &\in \overline{\mathcal{P}}^{2,+} u(t_\varepsilon, x_\varepsilon), \\ \left(\frac{t_\varepsilon - s_\varepsilon}{\varepsilon}, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y \right) &\in \overline{\mathcal{P}}^{2,-} v(s_\varepsilon, y_\varepsilon). \end{aligned}$$

Hereafter we may suppress a point $(t_\varepsilon, x_\varepsilon)$ of ϕ_t and ϕ . Since u and v are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1), we see that

$$(1.8) \quad \phi_t + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F \left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X \right) \leq 0,$$

$$(1.9) \quad \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y \right) \geq 0.$$

Subtracting (1.9.) from (1.8), we get

$$(1.10) \quad \phi_t + F \left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X \right) - F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y \right) \leq 0.$$

From (1.6) and (1.7) we have $X + Y \leq O$. By (F2) and the definition of F_0 (1.2) we observe that

$$\begin{aligned} &F \left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X \right) - F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y \right) \\ &= F \left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X \right) - F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X \right) \\ &\quad + F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X \right) - F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y \right) \\ &\geq F \left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X \right) - F \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X \right) \\ &\geq \inf \{ F(D\phi + q, D^2\phi + Z) - F(q, Z); (q, Z) \in \mathbf{R}^N \times \mathbf{S}^N \} \\ &= F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon)) \end{aligned}$$

Combining (1.10) and the lower semicontinuity of F_0 we see that

$$\begin{aligned} 0 &\geq \phi_t(t_\varepsilon, x_\varepsilon) + F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon)) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \{ \phi_t(t_\varepsilon, x_\varepsilon) + F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon)) \} \\ &\geq \phi_t(\hat{t}, \hat{x}) + F_0(D\phi(\hat{t}, \hat{x}), D^2\phi(\hat{t}, \hat{x})) \end{aligned}$$

This means w is a viscosity subsolution of $u_t + F_0(Du, D^2u) = 0$. \square

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