

**Axiomatic Method of Measure
and Integration (VIII).
Definition of the LS-integral
and its Fundamental Properties**

(Y. Ito, “RS-integral and LS-integral”, Chap.10–11)

By

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Abstract

In this paper, we define the LS-integral of the LS-measurable functions on \mathbf{R}^d , ($d \geq 1$).

Then we study the fundamental properties of the LS-integral. These facts are the new results.

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Introduction

This paper is the part VIII of the series of the papers on the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to Ito [6], [14]. Further we refer to Ito [1] ~ [5], [7] ~ [13] and [15] ~ [24].

In this paper, we study the definition of the d -dimensional LS-integral on the d -dimensional Euclidean space \mathbf{R}^d and their fundamental properties. Here we assume $d \geq 1$. In the sequel, we happen to omit the adjective “ d -dimensional”.

We assume that the d -dimensional Euclidean space \mathbf{R}^d is the d -dimensional LS-measure space $(\mathbf{R}^d, \mathcal{M}, \mu)$. Then we define the class of LS-measurable functions adapting to this LS-measure and we define the LS-integral for these LS-measurable functions.

Then we can define that a function $f(x)$ is LS-measurable if the level set $\{x; f(x) < \alpha\}$ is a LS-measurable set for an arbitrary real number α similarly to the theory of Lebesgue integral.

Nevertheless, in my theory of Riemann integral, we define that $f(x)$ is Jordan measurable if it is the limit of a direct family of simple functions in the sense of uniform convergence in the wider sense outside the set of singular points of $f(x)$.

Similarly to this, in this paper, we define that $f(x)$ is LS-measurable if it is the limit of a sequence of simple functions in the sense of pointwise convergence outside the set of singular points of $f(x)$.

The LS-integral is defined for a LS-measurable function. Therefore, in order to study the relations between the LS-integral and the operations of functions, we must prepare the properties with proofs that the LS-measurability of functions is preserved as relating to the operations of four fundamental rules of calculation, the supremum, the infimum, and the limit.

We give these results as the theorems of the properties of LS-measurable functions.

We define the LS-integral for such a LS-measurable function.

Then the concept of the pointwise convergence outside the set of singular points conforms well with the class of the LS-measurable functions and the class of the LS-integrable functions. Namely the limit function $f(x)$ of a sequence of functions in one of these classes in the sense of pointwise convergence outside the set of singular points of $f(x)$ belongs to the same class.

By using the similar expressions as above, it is reasonable that, in the theory of the RS-integral, the convergence of a sequence of functions is defined by using the uniform convergence in the wider sense outside the set of singular points. Similarly it is reasonable that, in the theory of the LS-integral, the convergence of a sequence of functions is defined by using the pointwise convergence outside the set of singular points.

In the following, we can clarify that an integral domain E should be a LS-measurable set in the theory of the LS-integral. If we assume that a considered subset E of \mathbf{R}^d is not a LS-measurable set, even a constant function defined on E is not a LS-measurable function. After all, it is meaningless that we consider a LS-measurable function defined on such a set E in itself. Therefore the definition of a LS-integral on a LS-nonmeasurable set E becomes also meaningless.

Therefore, it is meaningless that we consider a LS-nonmeasurable set and

a LS-nonmeasurable function in the theory of the LS-integral. The study on them is not a subject of the theory of the LS-integral.

Further, because a LS-measure is a complete measure, we cannot define a wider theory of measure including the LS-nonmeasurable sets by the extension of the theory of LS-measure.

By virtue of this point, it is also meaningless to consider the LS-nonmeasurable sets.

Here I express my heartfelt gratitude to my wife Mutuko for her help of typesetting of the T_EX-file of this manuscript.

1 d -dimensional LS-measurable functions

In this section, we define the concept of the LS-measurable functions and we study their fundamental properties.

Assume that a d -dimensional LS-measure space $(\mathbf{R}^d, \mathcal{M}, \mu)$ is defined in a d -dimensional Euclidean space \mathbf{R}^d .

We assume that a subset E of \mathbf{R}^d is a **LS-measurable set**. In the sequel, for the simplicity, we say that E is **measurable**.

Now we consider a measurable function defined on a measurable set E .

We denote the family of all LS-measurable sets included in E as \mathcal{M}_E and we denote the restricted measure on \mathcal{M}_E of the LS-measure μ on \mathbf{R}^d as the same symbol μ . Then we say that the measure space (E, \mathcal{M}_E, μ) is a **d -dimensional LS-measure space** on E .

In the sequel, we consider this LS-measure space (E, \mathcal{M}_E, μ) when we study the LS-integral of a LS-measurable function $f(x)$ on E .

Further, we denote \mathcal{M}_E as \mathcal{M} as the abbreviation.

In the sequel, we assume that a considered function $f(x)$ is an extended real-valued function defined on E . Then, when we denote $E(\infty) = \{x \in E; |f(x)| = \infty\}$, we say that a point of $E(\infty)$ is a **singular point** of $f(x)$.

At first, we define a simple function.

Definition 1.1 We say that an extended real-valued function $f(x)$ defined on a measurable set E in \mathbf{R}^d is a **simple function** if, for a countable division Δ of E such as

$$(\Delta) : E = \sum_{p=1}^{\infty} E_p = E_1 + E_2 + \cdots, \quad (E_p \in \mathcal{M}_E, (p \geq 1)), \quad (1.1)$$

we have the expression

$$f(x) = \sum_{p=1}^{\infty} \alpha_p \chi_{E_p}(x). \quad (1.2)$$

Here α_p is equal to a real number or $\pm\infty$ and they need not be different each other. $\chi_{E_p}(x)$ denotes the defining function of a set E_p . Then we denote the simple function $f(x)$ as $f_\Delta(x)$.

Here we assume that we have $E(\infty) \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.

Here we assume that all the subsets E_1, E_2, \dots of E are the LS-measurable sets and they are mutually disjoint.

In Definition 1.1, the defining function $\chi_A(x)$ of a set A is defined in the following:

$$\chi_A(x) = \begin{cases} 1, & (x \in A), \\ 0, & (x \notin A). \end{cases}$$

Since a simple function $f(x)$ is a function, its range is determined. Namely the range of a simple function is at most countable set in the extended real number space $\overline{\mathbf{R}} = [-\infty, \infty]$.

Especially, if the range of a simple function is a finite set in $\overline{\mathbf{R}}$, we say that this simple function is a **step function**

Nevertheless, there are the infinitely many varieties of the choices of the expressions in the forms of the formula (1.2) for a simple function $f(x)$ because the forms of the divisions Δ of E in the formulas (1.1) have the infinitely many varieties.

Thus, even if a simple function $f(x)$ has a fixed range, we use the symbol $f_\Delta(x)$ in order to distinguish the simple functions whose expressions in the formula (1.2) are different.

Then we define the concept of a LS-measurable function in the following.

Definition 1.2 We say that an extended real-valued function $f(x)$ defined on a measurable set E in \mathbf{R}^d is a **LS-measurable function** if we have the conditions (i) and (ii) in the following:

- (i) We have $E(\infty) \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.
- (ii) There exists a sequence of simple functions $\{f_n(x); n \geq 1\}$ such that we have the limit

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \tag{1.3}$$

in the sense of pointwise convergence on $E \setminus E(\infty)$.

Here, if we put $E_n(\infty) = \{x \in E; |f_n(x)| = \infty\}$ for $n \geq 1$, we assume that we have the relations

$$E_n(\infty) \subset E(\infty), \quad (n \geq 1).$$

The condition (ii) in Definition 1.2 means that we have the condition (iii) in the following.

- (iii) At each point x in $E \setminus E(\infty)$ and for an arbitrary $\varepsilon > 0$, there exists a certain natural number n_0 such that, for an arbitrary natural number n so that $n \geq n_0$ holds, we have the inequality

$$|f_n(x) - f(x)| < \varepsilon. \quad (1.4)$$

When, in the set E excluding a null set e , we have a certain property (P) for a measurable function $f(x)$ or a sequence of measurable functions $\{f_n(x)\}$, we say that this property (P) holds **almost everywhere** for the function $f(x)$ or the sequence of functions $\{f_n(x)\}$.

For example, if we have the equality

$$f(x) = 0, \quad (x \in E \setminus e, \mu(e) = 0),$$

we say that $f(x)$ is equal to 0 almost everywhere in E .

We denote this as

$$f(x) = 0, \quad (\text{a.e. } x \in E).$$

Further, if we have the limit

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad (x \in E \setminus e, \mu(e) = 0),$$

we say that $f_n(x)$ converges to $f(x)$ almost everywhere in E .

We denote this as

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad (\text{a.e. } x \in E).$$

Then the values of the limit function $f(x)$ happen to be undetermined on the null set e .

But we determine one value of $f(x)$ at each point in such a null set e and we fix it.

By virtue of this fact, we determine the domain of this function in the fixed manner. Namely, when the domains of functions are different from each other, it is meaningless that we state somewhat proposition for these functions.

In this case, even if we give $f(x)$ any value on the null set e , it does not influence the value of the LS-integral of $f(x)$. This is the idea that we express the proposition clearly.

Remark 1.1 We use the expression in the above. Then the condition (iii) in the above may be rephrased in the following (iii)':

(iii)' At almost every point x in E and for arbitrary $\varepsilon > 0$, there exists a certain natural number n_0 such that, for an arbitrary natural number n such as $n \geq n_0$, we have the inequality

$$|f_n(x) - f(x)| < \varepsilon.$$

For simplicity, we happen to say that a LS-measurable function $f(x)$ is a **measurable function** or **measurable**.

Example 1.1 Assume that E is a measurable set in \mathbf{R}^d . Then a simple function $f(x)$ and a continuous function $f(x)$ defined on E are measurable.

Theorem 1.1 Assume that E is a measurable set on \mathbf{R}^d and assume that two functions f and g defined on E are the measurable functions. Then the functions (1) ~ (10) in the following are the measurable functions defined on E :

$$(1) \quad f + g. \quad (2) \quad f - g. \quad (3) \quad fg.$$

$$(4) \quad f/g. \text{ Here we assume that we have } g(x) \neq 0, (x \in E).$$

$$(5) \quad \alpha f. \text{ Here we assume that } \alpha \text{ is a real constant.}$$

$$(6) \quad |f|^p. \text{ Here we assume that } p \neq 0 \text{ is a real number.}$$

$$(7) \quad \sup(f, g). \quad (8) \quad \inf(f, g).$$

$$(9) \quad f^+ = \sup(f, 0). \quad (10) \quad f^- = -\inf(f, 0).$$

We define the functions $\sup(f, g)$ and $\inf(f, g)$ in Theorem 1.1 in the following:

$$\sup(f, g)(x) = \sup(f(x), g(x)), (x \in E),$$

$$\inf(f, g)(x) = \inf(f(x), g(x)), (x \in E).$$

Further we have the formulas

$$f = f^+ - f^-, |f| = f^+ + f^-.$$

Theorem 1.2 If a function $f(x)$ is measurable on E and we have $F \in \mathcal{M}_E$, the restriction $f_F(x) = f(x)|_F$ of $f(x)$ to F is measurable on F .

Now we use the notation in the following. Assume that α and β are two arbitrary real numbers or $\pm\infty$.

Then we put

$$E(f > \alpha) = \{x \in E; f(x) > \alpha\},$$

$$E(f \leq \alpha) = \{x \in E; f(x) \leq \alpha\},$$

$$E(f = \alpha) = \{x \in E; f(x) = \alpha\},$$

$$E(\alpha < f \leq \beta) = \{x \in E; \alpha < f(x) \leq \beta\}, (\alpha < \beta).$$

Theorem 1.3 *Assume that $f(x)$ is a function defined on E . Then the four statements in the following are equivalent:*

- (1) *For an arbitrary real number α , we have $E(f > \alpha) \in \mathcal{M}_E$.*
- (2) *For an arbitrary real number α , we have $E(f \leq \alpha) \in \mathcal{M}_E$.*
- (3) *For an arbitrary real number α , we have $E(f \geq \alpha) \in \mathcal{M}_E$.*
- (4) *For an arbitrary real number α , we have $E(f < \alpha) \in \mathcal{M}_E$.*

Corollary 1.1 *For a function $f(x)$ defined on E , the statements (1) and (2) in the following are equivalent:*

- (1) *For an arbitrary real number α , we have $E(f > \alpha) \in \mathcal{M}_E$.*
- (2) *For an arbitrary rational number r , we have $E(f > r) \in \mathcal{M}_E$.*

Corollary 1.2 *Assume that a function $f(x)$ defined on E satisfies the conditions in Theorem 1.3. Then every set in the following belongs to \mathcal{M}_E :*

- (1) $E(f = \alpha)$. Here α is an arbitrary real number.
- (2) $E(f < \infty)$.
- (3) $E(f = \infty)$.
- (4) $E(f > -\infty)$.
- (5) $E(f = -\infty)$.

Theorem 1.4 *For a function $f(x)$ defined on E , the following two assertions (1) and (2) are equivalent:*

- (1) *$f(x)$ is measurable in E . Namely, there exists a sequence of simple functions which converges to $f(x)$ at every point on $E \setminus E(\infty)$.*
- (2) *For an arbitrary real number α , we have $E(f > \alpha) \in \mathcal{M}_E$.*

If a function $f(x)$ is measurable on E , there exists a sequence of simple functions which converges to $f(x)$ at each point outside the set of all singular points of $f(x)$, by virtue of the definition.

Nevertheless, this theorem means that we have a method of constructing such a sequence of simple functions concretely.

We give this results as the Corollary 1.3 in the following.

Corollary 1.3 *Assume that a function $f(x)$ is measurable on E . Then, for each natural number $n \geq 1$, we put*

$$E_n^p = E\left(\frac{p}{n} \leq f < \frac{p+1}{n}\right), \quad (p = 0, \pm 1, \pm 2, \dots)$$

and we denote the defining function of E_n^p as

$$C_n^p(x) = \chi_{E_n^p}(x).$$

Then, if we define the simple function $f_n(x)$ by the formula

$$f_n(x) = \sum_{p=-\infty}^{\infty} \frac{p}{n} C_n^p(x), \quad (x \in E),$$

the sequence of simple functions $\{f_n(x)\}$ converges to $f(x)$ at each point in $E \setminus E(\infty)$.

Theorem 1.5 *If a function $f(x)$ is measurable on E and we have $f(x) \geq 0$, we can choose a sequence of simple functions $\{f_n(x)\}$ such that we have $f_n(x) \geq 0$, ($n \geq 1$) and it converges to $f(x)$ at each point in $E \setminus E(\infty)$.*

Theorem 1.6 *If the functions $f_n(x)$, ($n \geq 1$) are measurable on E , the functions (1)~(5) in the following are also measurable on E :*

- (1) $\sup_{n \geq 1} f_n(x)$. (2) $\inf_{n \geq 1} f_n(x)$. (3) $\overline{\lim}_{n \rightarrow \infty} f_n(x)$. (4) $\underline{\lim}_{n \rightarrow \infty} f_n(x)$.
 (5) *If there exists $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $f(x)$ is also measurable on E .*

Theorem 1.7 (Egorov's Theorem) *Assume that $(\mathbf{R}^d, \mathcal{M}, \mu)$ is a positive LS-measure space. Assume that E is a measurable set in \mathbf{R}^d and we have $\mu(E) < \infty$. Assume that $f_n(x)$, ($n \geq 1$) are the finite valued measurable functions almost everywhere in E .*

Further assume that we have the finite limit $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ almost everywhere on E . Then, for an arbitrary $\varepsilon > 0$, there exists a set $F \in \mathcal{M}_E$ such that we have the results (1) and (2) in the following:

- (1) *We have $F \subset E$ and $\mu(E \setminus F) < \varepsilon$.*

(2) $f_n(x)$ converges to $f(x)$ uniformly on F .

Corollary 1.4 *In Theorem 1.7, we can obtain a subset F so that it is a closed set.*

By using Egorov's Theorem and Corollary 1.4, we have the following theorem.

Theorem 1.8 (Luzin's Theorem) *We assume that we have three conditions (i) \sim (iii) in the following:*

- (i) $(\mathbf{R}^d, \mathcal{M}, \mu)$ is a positive LS-measurable space.
- (ii) E is a measurable set in \mathbf{R}^d .
- (iii) $f(x)$ is a finite valued measurable function almost everywhere in E .

Then, for an arbitrary $\varepsilon > 0$, there exists a certain closed set $F \subset E$ such that we have the following:

- (1) We have $\mu(E \setminus F) < \varepsilon$.
- (2) $f(x)$ is continuous on F .

2 Definition of the d -dimensional LS-integral

In this section, we define the concept of the LS-integral of a LS-measurable function.

Assume that the d -dimensional Euclidean space \mathbf{R}^d is a LS-measure space $(\mathbf{R}^d, \mathcal{M}, \mu)$.

Assume that a subset E of \mathbf{R}^d is a LS-measurable set. Then, by restricting $(\mathbf{R}^d, \mathcal{M}, \mu)$ to E , we have the d -dimensional LS-measurable space (E, \mathcal{M}, μ) on E .

Here we define the LS-integral of a LS-measurable function $f(x)$ on E and we denote this by the symbol

$$\int_E f(x) d\mu.$$

In the sequel, we define the LS-integral of $f(x)$ in the two steps in the following.

(1) In the case where $f(x)$ is a simple function

Assume that $f(x)$ is expressed as follows:

$$f(x) = \sum_{p=1}^{\infty} \alpha_p \chi_{E_p}(x), \quad (x \in E, \alpha_p \in \overline{\mathbf{R}}, (p \geq 1)), \quad (2.1)$$

$$E = E_1 + E_2 + \cdots, \quad (E_p \in \mathcal{M}, (p \geq 1)). \quad (2.2)$$

Then we define the **LS-integral** of $f(x)$ as the sum of the series in the right hand side of the equality

$$\int_E f(x) d\mu = \sum_{p=1}^{\infty} \alpha_p \mu(E_p). \quad (2.3)$$

Here we consider only the case where the series in the right hand side converges absolutely.

Then the sum of the absolutely convergent series in the right hand side of the formula has the fixed value independent of the choice of the expression in the formula (2.1) of a function $f(x)$.

Then we say that $f(x)$ is **LS-integrable** on E . We also say that $f(x)$ is **LS-sommable**.

Then, when we denote the total variation of μ as ν , $f(x)$ is LS-integrable on E with respect to μ if and only if $|f(x)|$ is LS-integrable on E with respect to ν . Here we can understand this equivalence as follows.

The absolute function of a function $f(x)$ in the formula (2.1) is expressed by the formula

$$|f(x)| = \sum_{p=1}^{\infty} |\alpha_p| \chi_{E_p}(x). \quad (2.4)$$

Therefore we have the equality

$$\begin{aligned} \int_E |f(x)| d\nu &= \sum_{p=1}^{\infty} |\alpha_p| \nu(E_p) \\ &= \sum_{p=1}^{\infty} |\alpha_p| \mu^+(E_p) + \sum_{p=1}^{\infty} |\alpha_p| \mu^-(E_p). \end{aligned} \quad (2.5)$$

Here μ^+ and μ^- are the positive variation and the negative variation of μ respectively.

Then the series in the right hand side of the formula (2.3) converges absolutely if and only if the series in the right hand side of the formula (2.5) converges.

Therefore we have the equalities

$$\int_E f(x) d\mu = \int_E f(x) d\mu^+ - \int_E f(x) d\mu^-,$$

$$\int_E f(x)d\nu = \int_E f(x)d\mu^+ + \int_E f(x)d\mu^-.$$

Remark 2.1 As for the convergence and the divergence of the series in the right hand side of the formula (2.3), we can consider the cases in the table 2.1.

Table 2.1 The convergence and the divergence of the series in the formula (2.3)

- (1) the convergence.
 - (1-i) the absolute convergence.
 - (1-ii) the conditional convergence.
- (2) the divergence.
 - (2-i) the divergence to either one of $\pm\infty$.
 - (2-ii) the case where it oscillates and it does not converges to a fixed value.

In the table 2.1, the case (1-i) is the case of the definition of the LS-integral and the case (1-ii) is the case of the conditional convergence of the LS-integral.

Here, the case of a LS-integrable simple function $f(x)$ is only the case (1-i) in the Table 2.1.

In general, as for the details of the convergence and the divergence of the LS-integral, we study it in the calculus of the LS-integral afterward.

(2) Case where $f(x)$ is a general measurable function

Here there exists a sequence of simple functions $\{f_n(x)\}$ such that it converges to $f(x)$ at each point on $E \setminus E(\infty)$.

Here, if each function $f_n(x)$ is LS-integrable and we have the limit

$$\lim_{n \rightarrow \infty} \int_E f_n(x)d\mu, \tag{2.6}$$

we say that this limit is the **LS-integral** of $f(x)$ on E and we denote it by the symbol

$$\int_E f(x)d\mu = \lim_{n \rightarrow \infty} \int_E f_n(x)d\mu. \tag{2.7}$$

Then we say that the LS-integral (2.7) **converges absolutely** if the limit (2.6) has the fined value independent of choice of a sequence of LS-integrable simple functions $\{f_n(x)\}$ which converges to $f(x)$ at each point in $E \setminus E(\infty)$.

Then we say that $f(x)$ is **LS-integral** on E or **LS-sommable** on E .

The usual LS-integral is the LS-integral in this case.

A function $f(x)$ is LS-integrable on E with respect to μ if and only if the absolute function $|f(x)|$ is LS-integrable with respect to ν .

Then we have the following theorem.

Theorem 2.1 *If $f(x)$ is LS-integrable on E , we choose a sequence of simple functions $\{f_n(x)\}$ in such a way as Corollary 1.3. Then the LS-integral of $f(x)$ on E is equal to the LS-integral*

$$\int_E f(x)d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{p=-\infty}^{\infty} p\mu \left(E \left(\frac{p}{n} \leq f < \frac{p+1}{n} \right) \right).$$

Theorem 2.2 *Assume that $f(x)$ is LS-integrable on E . When we put*

$$f^+(x) = \max\{f(x), 0\} \geq 0, \quad f^-(x) = -\min\{f(x), 0\} \geq 0,$$

we have the formulas

$$f(x) = f^+(x) - f^-(x), \quad |f(x)| = f^+(x) + f^-(x).$$

Then $f^+(x)$ and $f^-(x)$ are LS-integrable on E and we have the equality

$$\begin{aligned} \int_E f(x)d\mu &= \int_E f^+(x)d\mu - \int_E f^-(x)d\mu \\ &= \int_E f^+(x)d\mu^+ - \int_E f^+(x)d\mu^- \\ &\quad - \int_E f^-(x)d\mu^+ + \int_E f^-(x)d\mu^-. \end{aligned}$$

Further we have the equality

$$\begin{aligned} \int_E |f(x)|d\nu &= \int_E f^+(x)d\nu + \int_E f^-(x)d\nu \\ &= \int_E f^+(x)d\mu^+ + \int_E f^+(x)d\mu^- \\ &\quad + \int_E f^-(x)d\mu^+ + \int_E f^-(x)d\mu^-. \end{aligned}$$

Corollary 2.1 *Assume that $f(x)$ is LS-integrable on E and $g(x)$ is LS-measurable on E . Then, if we have the inequality $|g(x)| \leq |f(x)|$ on E , $g(x)$ is LS-integrable on E .*

Further we say that the LS-integral (2.7) is **conditionally convergent** if the limit (2.6) has a various value depending on the choice of a sequence of LS-integrable simple functions $\{f_n(x)\}$ which converges to $f(x)$ at each point on $E \setminus E(\infty)$.

We say that the LS-integral (2.7) **converges** if it converges absolutely or it converges conditionally. Then we say that LS-integral of $f(x)$ **exists**.

We say that the LS-integral of $f(x)$ **diverges** if the limit (2.6) does not exist. Then we say that the LS-integral of $f(x)$ **does not exist**.

Remark 2.2 In the case of the conditional convergence in the formula (2.7), we consider that the integral of a simple function is a LS-integral.

In general, as for the details of the conditions of the convergence and the divergence of a LS-integral, we study these in the section of the calculation of the LS-integrals afterward.

3 Fundamental properties of the d -dimensional LS-integral

In this section, we study the fundamental properties of the LS-integrals.

Assume that a subset E of \mathbf{R}^d is a LS-measurable set and we have the d -dimensional LS-measure space (E, \mathcal{M}, μ) on E . Here assume that $d \geq 1$ holds. We consider a general LS-measure space without any special permission.

We denote the total variation of μ as ν and we denote the positive variation and the negative variation of μ as μ^+ and μ^- respectively.

3.1 Fundamental properties of the d -dimensional LS-integral

In this subsection, we assume that a subset E of \mathbf{R}^d is a LS-measurable set and we have the d -dimensional LS-measure space (E, \mathcal{M}, μ) on E . Here we assume that $d \geq 1$ holds.

As for the several formulas in the several theorems in this section, we can easily prove them for the LS-integrable simple functions.

For the general LS-integrable functions, we can prove them by the limits of these several formulas corresponding to the LS-integrable simple functions by virtue of the definition of the LS-integral.

Therefore we omit the details of the proofs here.

Theorem 3.1.1 *Assume that (E, \mathcal{M}, μ) is a LS-measure space. Assume that a function $f(x)$ is LS-integrable on E . Then, if we have $\mu(E) = 0$, we have the equality*

$$\int_E f(x)d\mu = 0.$$

Theorem 3.1.2 *Assume that a function $f(x)$ is LS-integrable on E and a subset $F \subset E$ is a measurable subset. Then the restriction $f_F(x) = f(x)|_F$ of $f(x)$ to F is LS-integrable on F and we have the equality*

$$\int_F f_F(x)d\mu = \int_F f(x)d\mu.$$

Namely the function $f(x)$ is LS-integrable on F .

Theorem 3.1.3 *Assume that E is a LS-measurable set in \mathbf{R}^d and a function $f(x)$ is LS-integrable on E . Assume that $E = E_1 + E_2$ is a division of E and E_1 and E_2 are LS-measurable. Then we have the equality*

$$\int_E f(x)d\mu = \int_{E_1} f(x)d\mu + \int_{E_2} f(x)d\mu.$$

Corollary 3.1.1 *Assume that two functions $f(x)$ and $g(x)$ are LS-measurable on E and they are equal almost everywhere on E . Then, if $f(x)$ is LS-integrable on E , $g(x)$ is LS-integrable on E and we have the equality*

$$\int_E f(x)d\mu = \int_E g(x)d\mu.$$

By virtue of this Corollary 3.1.1, if two LS-integrable functions are equal almost everywhere, we need not distinguish their LS-integrals.

Theorem 3.1.4 *Assume that two functions $f(x)$ and $g(x)$ are LS-integrable on E . Then we have the results (1) and (2) in the following:*

- (1) $f(x) + g(x)$ is LS-integrable on E and we have the equality

$$\int_E \{f(x) + g(x)\}d\mu = \int_E f(x)d\mu + \int_E g(x)d\mu.$$

- (2) For an arbitrary real constant α , $\alpha f(x)$ is also LS-integrable on E and we have the equality

$$\int_E \{\alpha f(x)\}d\mu = \alpha \int_E f(x)d\mu.$$

Corollary 3.1.2 *We use the same notation as in Theorem 3.1.3. Assume that two functions $f(x)$ and $g(x)$ are LS-integrable on E . Then, for two arbitrary real constants α and β , $\alpha f(x) + \beta g(x)$ is also LS-integrable on E and we have the equality*

$$\int_E \{\alpha f(x) + \beta g(x)\} d\mu = \alpha \int_E f(x) d\mu + \beta \int_E g(x) d\mu.$$

Theorem 3.1.5 *Assume that (E, \mathcal{M}, μ) is a positive LS-measure space. If two functions $f(x)$ and $g(x)$ are LS-integrable on E , we have the results (1) \sim (3) in the following:*

(1) *If we have $f(x) \geq 0$, ($x \in E$), we have the inequality*

$$\int_E f(x) d\mu \geq 0.$$

(2) *If we have the inequality $f(x) \geq g(x)$, ($x \in E$), we have the inequality*

$$\int_E f(x) d\mu \geq \int_E g(x) d\mu.$$

(3) *We have the inequality*

$$\left| \int_E f(x) d\mu \right| \leq \int_E |f(x)| d\mu.$$

Corollary 3.1.3 *Assume that (E, \mathcal{M}, μ) is a LS-measure space. If $f(x)$ is LS-integrable on E , we have the inequality*

$$\left| \int_E f(x) d\mu \right| \leq \int_E |f(x)| d\mu.$$

Theorem 3.1.6 *Assume that (E, \mathcal{M}, μ) is a positive LS-measure space. Then, if a function $f(x)$ is LS-integrable on E , we have the equalities*

$$\mu(E(f = \infty)) = \mu(E(f = -\infty)) = 0.$$

Theorem 3.1.7 *If a function $f(x)$ is LS-integrable on E , $E(f \neq 0)$ is expressed as the sum of at most countable number of sets of the finite LS-measure.*

Theorem 3.1.8 (The first mean value Theorem of the integration)

Assume that (E, \mathcal{M}, μ) is a positive LS-measure space. Assume that a function $f(x)$ is a bounded measurable function on E and $g(x)$ is LS-integrable on E . Then, if we put

$$m = \inf_{x \in E} f(x), \quad M = \sup_{x \in E} f(x),$$

we have the following (1) and (2):

(1) $f(x)g(x)$ is LS-integrable on E .

(2) There exists a real constant α such that we have $m \leq \alpha \leq M$ and we have the equality

$$\int_E f(x)|g(x)|d\mu = \alpha \int_E |g(x)|d\mu.$$

Corollary 3.1.4 Assume that E is a bounded closed domain and (E, \mathcal{M}, μ) is a positive LS-measure space. Assume that a function $f(x)$ is continuous on E and $g(x)$ is LS-integrable on E . Further we have the inequality $g(x) \geq 0$, ($x \in E$). Then there exists a certain point $x_0 \in E$ such that we have the equality

$$\int_E f(x)g(x)d\mu = f(x_0) \int_E g(x)d\mu.$$

Theorem 3.1.9 Assume that E is a LS-measurable set in \mathbf{R}^d and a function $f(x)$ is LS-integrable on E . Then, for an arbitrary $\varepsilon > 0$, there exists a continuous function $f_\varepsilon(x)$ on \mathbf{R}^d which is identically equal to 0 outside a certain LS-measurable bounded closed set such that we have the inequality

$$\left| \int_E f(x)d\mu - \int_E f_\varepsilon(x)d\mu \right| \leq \int_E |f(x) - f_\varepsilon(x)|d\mu < \varepsilon.$$

3.2 d -dimensional LS-integral and limit

In this subsection, we study the d -dimensional LS-integral and its relations to limit.

In this section, we assume that a subset E of \mathbf{R}^d is a LS-measurable set and (E, \mathcal{M}, μ) is a d -dimensional LS-measure space.

Theorem 3.2.1 Assume that E is a LS-measurable set in \mathbf{R}^d and we have a division of E

$$E = E_1 + E_2 + \cdots$$

by using a countable number of mutually disjoint LS-measurable sets E_n , ($n \geq 1$). Then, if a function $f(x)$ is LS-integrable on E , we have the equality

$$\int_E f(x)d\mu = \int_{E_1} f(x)d\mu + \int_{E_2} f(x)d\mu + \cdots .$$

Further, if a function $f(x)$ is LS-integrable on each E_n and we have the condition

$$\sum_{n=1}^{\infty} \int_{E_n} |f(x)|d\nu < \infty,$$

we have the equality in the above.

Here ν is the total variation of μ .

Corollary 3.2.1 Assume that E is a LS-measurable set in \mathbf{R}^d and $\{E_n; n \geq 1\}$ is a monotone increasing sequence of LS-measurable sets and we have the condition

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Further assume that a function $f(x)$ is LS-integrable on E . Then, for an arbitrary $\varepsilon > 0$, there exists a certain natural number n_0 such that we have the condition

$$\int_{E \setminus E_n} |f(x)|d\nu < \varepsilon$$

for $n \geq n_0$. Especially we have the equality

$$\lim_{n \rightarrow \infty} \int_{E_n} f(x)d\mu = \int_E f(x)d\mu.$$

Remark 3.2.1 When the LS-integral

$$\int_E f(x)d\mu$$

of a function $f(x)$ converges conditionally, we have the limit such as Corollary 3.2.1 if we choose a special sequence of LS-measurable sets $\{E_n\}$ such as Corollary 3.2.1 in the above.

Corollary 3.2.2 Assume that E is a LS-measurable set in \mathbf{R}^d and a function $f(x)$ is LS-integrable on E .

Now we put

$$E_n = E(|f| < n), \quad (n \geq 1).$$

Then, for an arbitrary $\varepsilon > 0$, there exists a certain natural number n_0 such that we have the inequality

$$\int_{E \setminus E_n} |f(x)| d\nu < \varepsilon$$

for $n \geq n_0$.

Especially we have the equality

$$\lim_{n \rightarrow \infty} \int_{E_n} f(x) d\mu = \int_E f(x) d\mu.$$

Theorem 3.2.2 in the following shows the absolute continuity of the indefinite integral.

This is the application of Theorem 3.2.1 and Corollary 3.2.2.

Theorem 3.2.2 Assume that E is a LS-measurable set of \mathbf{R}^d and a function $f(x)$ is LS-integrable on E . Then, for an arbitrary $\varepsilon > 0$, there exists a certain $\delta > 0$ such that we have the inequality

$$\left| \int_e f(x) d\mu \right| < \varepsilon$$

if we have $\nu(e) < \delta$ for a LS-measurable set $e \subset E$.

Theorem 3.2.3 (Bounded convergence theorem) Assume that E is a bounded measurable set of \mathbf{R}^d . If a sequence $\{f_n(x); n \geq 1\}$ of uniformly bounded LS-measurable functions converges to $f(x)$ almost everywhere on E , we have the equality

$$\lim_{n \rightarrow \infty} \int_E f_n(x) d\mu = \int_E f(x) d\mu.$$

Theorem 3.2.4 (Lebesgue's convergence theorem) Assume that E is a LS-measurable set of \mathbf{R}^d and a sequence $\{f_n(x); n \geq 1\}$ of LS-measurable functions on E converges to a finite limit $f(x)$ almost everywhere on E . Further, if there exists a LS-integrable function $\Phi(x)$, (≥ 0) on E such that we have the inequality

$$|f_n(x)| \leq \Phi(x), \quad (x \in E, n \geq 1),$$

we have the equality

$$\lim_{n \rightarrow \infty} \int_E f_n(x) d\mu = \int_E f(x) d\mu.$$

By virtue of Lebesgue's convergence theorem, we have the termwise integration theorem.

Theorem 3.2.5 (Termwise integration theorem) *Assume that E is a LS-measurable set of \mathbf{R}^d and $\{f_n(x); n \geq 1\}$ is a sequence of LS-measurable functions on E . Now we put*

$$f(x) = f_1(x) + f_2(x) + \cdots .$$

Then, if the series in the right hand side converges almost everywhere on E and, further if there exists a LS-integrable function $\Phi(x)$, (≥ 0) on E such that we have the inequalities

$$\left| \sum_{p=1}^n f_p(x) \right| \leq \Phi(x), \quad (x \in E)$$

for an arbitrary $n \geq 1$, we can integrate $f(x)$ termwise. Namely we have the equality

$$\int_E f(x) d\mu = \int_E f_1(x) d\mu + \int_E f_2(x) d\mu + \cdots .$$

Corollary 3.2.3 *Let E , $\{f_n(x)\}$ and $f(x)$ be the same as in Theorem 3.2.5. Then we assume that we have either one of the conditions (i) and (ii) in the following:*

- (i) *There exists a LS-integrable function $\Phi(x)$, (≥ 0) on E such that we have the inequalities*

$$\sum_{p=1}^n |f_p(x)| \leq \Phi(x), \quad (x \in E, n \geq 1).$$

- (ii) *We have the condition*

$$\sum_{p=1}^{\infty} \int_E |f_p(x)| d\nu < \infty.$$

Then we have the termwise integration theorem.

Theorem 3.2.6 (Beppo Levi's Theorem) *Assume that E is a LS-measurable set of \mathbf{R}^d and (E, \mathcal{M}, μ) is a positive LS-measure space. Assume that $\{f_n(x); n \geq 1\}$ is a monotone increasing sequence of LS-integrable functions on E . Further assume that the monotone increasing sequence of numbers*

$$\left\{ \int_E f_n(x) d\mu \right\}$$

is bounded from the above.

Then, if we put

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad (x \in E),$$

the function $f(x)$ has a finite value almost everywhere on E and it is LS-integrable on E and we have the equality

$$\lim_{n \rightarrow \infty} \int_E f_n(x) d\mu = \int_E f(x) d\mu.$$

We give Corollary in the following for the fact used in the proof of Theorem 3.2.6 in the above.

Corollary 3.2.4 *We use the notation of Theorem 3.2.6. Assume that E is a LS-measurable set of \mathbf{R}^d and $\{E_n; n \geq 1\}$ is a monotone increasing sequence of LS-measurable sets on \mathbf{R}^d so that we have the equality*

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Further, if a LS-measurable function $f(x)$ on E is LS-integrable on each E_n and we have the inequality

$$\lim_{n \rightarrow \infty} \int_{E_n} |f(x)| d\mu < \infty,$$

$f(x)$ is LS-integrable on E and we have the equality

$$\lim_{n \rightarrow \infty} \int_{E_n} f(x) d\mu = \int_E f(x) d\mu.$$

Corollary 3.2.5 *We use the notation of Theorem 3.2.6. Assume that E is a LS-measurable set of \mathbf{R}^d and $\{f_n(x); n \geq 1\}$ is a monotone increasing sequence of LS-integrable functions on E . Then, if the limit function*

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad (x \in E)$$

has a finite value almost everywhere on E and it is LS-integrable on E , we have the equality

$$\lim_{n \rightarrow \infty} \int_E f_n(x) d\mu = \int_E f(x) d\mu.$$

,

Next we prove Fatou's Lemma as the Corollary of Beppo Levi's Theorem.

At first, we remark that Fatou's Lemma is used many times in the following form.

Assume that E is a LS-measurable set of \mathbf{R}^d and (E, \mathcal{M}, μ) is a positive LS-measurable space. Assume that, for the LS-integrable nonnegative functions $f_n(x)$, ($n \geq 1$) on E , we have

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad (x \in E).$$

If we have the inequality

$$\underline{\lim}_{n \rightarrow \infty} \int_E f_n(x) d\mu < \infty,$$

$f(x)$ is also LS-integrable on E and we have the inequality

$$\int_E f(x) d\mu \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n(x) d\mu.$$

Here we prove Fatou's Lemma in the fairly more generalized form.

Theorem 3.2.7 (Fatou's Lemma) *Assume that E is a LS-measurable set of \mathbf{R}^d and (E, \mathcal{M}, μ) is a positive LS-measure space. Assume that $\{f_n(x); n \geq 1\}$ is a sequence of LS-integrable nonnegative functions on E and we have the condition*

$$\underline{\lim}_{n \rightarrow \infty} \int_E f_n(x) d\mu < \infty.$$

Then the function

$$f(x) = \underline{\lim}_{n \rightarrow \infty} f_n(x), \quad (x \in E)$$

is LS-integrable on E and we have the inequality

$$\int_E f(x) d\mu = \int_E \left(\underline{\lim}_{n \rightarrow \infty} f_n(x) \right) d\mu \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n(x) d\mu.$$

The Theorem 3.2.8 in the following is the result concerning the differentiation under the integral symbol.

Theorem 3.2.8 *Assume that E is a LS-measurable set of \mathbf{R}^d and (a, b) is an interval of \mathbf{R} . Assume that a function $f(x, t)$ is defined on the set $E \times (a, b) = \{(x, t); x \in E, t \in (a, b)\}$ and we have the conditions (i) ~ (iii) in the following:*

- (i) *If we choose $t \in (a, b)$ arbitrarily and fix it, $f(x, t)$ is LS-integrable on E .*

- (ii) Assume that, for almost every x on E , $f(x, t)$ is differentiable with respect to t . Then we denote its partial derivative with respect to t as $f_t(x, t)$.
- (iii) Assume that there exists a LS-integrable function $\Phi(x)$, (≥ 0) on E such that we have the inequality

$$|f_t(x, t)| \leq \Phi(x), \quad (x \in E, t \in (a, b)).$$

Then, if we put

$$F(t) = \int_E f(x, t) d\mu,$$

$F(t)$ is differentiable on (a, b) and we have the equality

$$F'(t) = \int_E f_t(x, t) d\mu.$$

3.3 Method of calculation of the d -dimensional LS-integral

In this subsection, we study the one method of calculation of the d -dimensional LS-integral.

This method is the calculation of the LS-integral of a LS-measurable function $f(x)$ on E by approximating the integration domain E by an approximating direct family $\{E_\alpha\}$ of bounded closed sets in $E \setminus E(\infty)$. We assume that the integration domain E is a LS-measurable set of \mathbf{R}^d and (E, \mathcal{M}, μ) is a LS-measure space. Here ν is the total variation of μ and μ^+ and μ^- are the positive variation and the negative variation of μ respectively.

The integrand $f(x)$ is LS-measurable on E .

Assume that A is a direct set and $\{E_\alpha; \alpha \in A\}$ is a direct family of bounded closed sets included in E .

Now we say that the direct family $\{E_\alpha\}$ **converges** to E if, for an arbitrary bounded closed set K included in E , there exists a certain $\alpha_0 \in A$ such that we have $K \subset E_\alpha$ for an arbitrary α such as $\alpha \geq \alpha_0$.

Then the direct family $\{E_\alpha\}$ is an **approximating direct family** of E .

Especially, if we have $A = \{1, 2, 3, \dots\}$ and $E_1 \subset E_2 \subset \dots \subset E_n \subset \dots$ holds, the sequence $\{E_n\}$ **converges monotone** to E . In general, if a sequence $\{E_n\}$ converges to E and we put

$$E_1 \cup E_2 \cup \dots \cup E_n = H_n, \quad (n = 1, 2, 3, \dots),$$

the sequence $\{H_n\}$ converges monotone to E .

Assume that the set $E(\infty)$ of the singular points of $f(x)$ has the LS-measure 0. Then $E \setminus E(\infty)$ is also a LS-measurable set.

Further we assume that $f(x)$ is LS-integrable on an arbitrary bounded closed set included in $E \setminus E(\infty)$. In order to be so, a bounded closed set included in $E \setminus E(\infty)$ and the set $E(\infty)$ of the singular points are away from each other with a positive distance.

By virtue of this, we can construct an approximating direct family $\{E_\alpha; \alpha \in A\}$ of $E \setminus E(\infty)$ by using the bounded closed sets E_α .

Now we assume that a direct family $\{E_\alpha\}$ of the bounded closed sets is an **approximating direct family** of $E \setminus E(\infty)$. Thus this direct family $\{E_\alpha\}$ converges to $E \setminus E(\infty)$.

Then, if, for one approximating direct family $\{E_\alpha\}$ of $E \setminus E(\infty)$,

$$I(E_\alpha) = \int_{E_\alpha} f(x) d\mu \tag{3.3.1}$$

converges in the sense of Moore-Smith limit, the limit

$$I = \lim_{\alpha} I(E_\alpha)$$

is equal to the LS-integral

$$I = \int_E f(x) d\mu.$$

Here this LS-integral **converges absolutely** if and only if the value I of this LS-integral does not depend on the choice of a approximating direct family $\{E_\alpha\}$ of $E \setminus E(\infty)$.

Further this LS-integral **converges conditionally** if and only if the value I depends on the choice of the approximating direct family $\{E_\alpha\}$ of $E \setminus E(\infty)$.

Further, if the LS-integral exists, we say that $\int_E f(x) d\mu$ **converges**.

On other hand, if the LS-integral does not exist, we say that $\int_E f(x) d\mu$ **diverges**.

Remark 3.3.1 Assume that E is a LS-measurable set of \mathbf{R}^d and $f(x)$ is an extended real-valued LS-measurable function defined on E .

Then there exists a sequence of simple functions $\{f_n(x)\}$ which converges to $f(x)$ at each point in $E \setminus E(\infty)$ and there exists an approximating direct family $\{E_\alpha\}$ of bounded closed sets in $E \setminus E(\infty)$ such that we have the limits of (I) and (II) in the following:

$$(I) \quad \int_E f(x) d\mu = \lim_{n \rightarrow \infty} \int_E f_n(x) d\mu.$$

$$(II) \quad \int_E f(x) d\mu = \lim_{\alpha} \int_{E_\alpha} f(x) d\mu.$$

Here the limit of (II) denotes the Moore-Smith limit.

Then, the LS-integral of (I) converges or diverges if and only if the LS-integral of (II) converges or diverges respectively.

Further, in the case of convergence, (I) converges absolutely or converges conditionally if and only if (II) converges absolutely or converges conditionally respectively.

The LS-integral of (I) is the calculation by the method of approximation of a function $f(x)$ by the sequence of simple functions. The LS-integral of (II) is the calculation by the method of approximation of the integration domain E by using the approximating direct family of bounded closed sets in $E \setminus E(\infty)$.

Thus, as for the calculation of the LS-integral, we have two methods such as the method of calculation by the approximation by virtue of functions and the method of calculation by the approximation of the integration domain.

Here, for a function $f(x)$, we put

$$f^+(x) = \sup(f(x), 0), \quad f^-(x) = -\inf(f(x), 0). \quad (3.3.3)$$

Then we have the relations

$$|f(x)| \geq f^+(x) \geq 0, \quad |f(x)| \geq f^-(x) \geq 0, \quad (3.3.4)$$

$$f(x) = f^+(x) - f^-(x), \quad (3.3.5)$$

Table 3.3.1 Convergence and divergence of the LS-integral

$$(\mu(f) = \int_E f(x)d\mu, \quad \text{AC=absolute convergence,}$$

C=convergence, D=divergence)

$\mu(f)$	$\nu(f)$	$\mu^+(f^+)$	$\mu^-(f^-)$	$\mu^+(f^-)$	$\mu^-(f^+)$
AC	C	C	C	C	C
D	D	C	C	C	D
D	D	C	C	D	C
D	D	C	C	D	D
D	D	C	D	C	C
D	D	D	C	C	C
D	D	D	D	C	C

$$|f(x)| = f^+(x) + f^-(x). \quad (3.3.6)$$

Then we have the relations as in Table 3.3.1 in the above as for the convergence and divergence of the LS-integral of $f(x)$.

In the other cases than Table 3.3.1, $\nu(|f|)$ diverges always and $\mu(f)$ converges conditionally or diverges according to the choices of the approximating sequences.

Remark 3.3.2 In the case where the LS-integral $\int_E f(x)d\mu$ in Table 3.3.1 converges absolutely, the value of this LS-integral is determined as the fixed value independent of the choice of the approximating direct family $\{E_\alpha; \alpha \in A\}$ of E .

The LS-integral has the determined meaning only in the case of the absolute convergence.

In the case where the LS-integral $\int_E f(x)\mu$ diverges in Table 3.3.1, the LS-integral does not exist.

Nevertheless, in this case, for a LS-measurable set A included in E , the set function $m(A)$ on \mathcal{M}_E is defined by the equality

$$m(A) = \int_A f(x)d\mu.$$

Here \mathcal{M}_E is the family of all LS-measurable sets included in E .

Thereby the LS-measure space (E, \mathcal{M}_E, m) on E is defined. In these cases, the total mass is equal to $m(E) = -\infty$ or $m(E) = \infty$.

This measure space has the determined meaning as a σ -finite measure space.

Then, even though the LS-integral of $f(x)$ on E does not exist in itself, the indefinite integral of $f(x)$ on a LS-measurable set A included in E is defined by the formula

$$m(A) = \int_A f(x)d\mu$$

and its value is determined as a finite real value or $-\infty$ or ∞ .

On the other hand, in the case where the LS-integral $\int_E f(x)d\mu$ converges conditionally or diverges in Table 3.3.1, this LS-integral converges or diverges according to the choice of an approximating direct family $\{E_\alpha; \alpha \in A\}$ of E .

Then, in the case where the LS-integral diverges, we cannot give this LS-integral any meaning.

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