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Axiomatic Method of Measure and Integration (XIV). The Measure and the Integration on a Riemanniann Manifold

By

Yoshifumi Ito

Professor Emeritus, Tokushima University
209-15 Kamifukuman Hachiman-cho
Tokushima 770-8073, JAPAN
e-mail address : *itoyoshifumi@fd5.so-net.ne.jp*

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Abstract

In this paper, we define the Lebesgue type measure and the Lebesgue type integral on a Riemannian manifold and study their fundamental properties. These results are the new results.

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Introduction

This paper is the part XIV of the series of the papers on the axiomatic method of measure and integration. As for the details, we refer to Ito [5]. Further we refer to Ito [1] $\sim$ [4], [6] and [7].

In this paper, we define the Lebesgue type measure and the Lebesgue type integral on a Riemannian manifold and study their fundamental properties.

We assume that a n-dimensional Riemannian manifold M is a Lebesgue type measure space $(M,\ \mathcal{M},\ \mu)$. Here we assume $n \geq 1$.

Here we define that an extended real-valued function $f(p)$ defined on a measurable set E in M is measurable if it is a limit of a sequence of simple functions in a sense of pointwise convergence outside the set of the singular points of $f(p)$. Here we assume that we have $E(\infty) = \{p \in E;\ |f(p)| = \infty\} \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.

Then we define the class of Lebesgue type measurable functions adapting to this Lebesgue type measure and we define the Lebesgue type integral for these Lebesgue type measurable functions and we study their fundamental properties.

Here I express my heartfelt gratitude to my wife Mutuko for her help of typesetting of the TEX-file of this manuscript.

1 Riemannian manifold

In this section, we give an outline of the fundamental concepts concerning a Riemannian manifold.

We assume that $(M,\ g)$ is a **n-dimensional Riemannian manifold**. Here we assume that $n \geq 1$ holds and M is a n-dimensional C^∞-manifold and g is the Riemannian metric on M. We say that g is the **metric tensor** or the **fundamental tensor** on M. M is assumed to be a paracompact C^∞-manifold. M is assumed to be oriented with the positive orientation. We denote the system of the positive C^∞-coordinate neighborhoods as

$$S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha \in A}.$$

For a point $p \in U_\alpha$, we denote the local coordinate of the point p as

$$\psi_\alpha(p) = (x_\alpha^1(p),\ x_\alpha^2(p),\ \cdots,\ x_\alpha^n(p)).$$

Then we define that, for each point $p \in U_\alpha \subset M$, the value of a real-valued function $f(p)$ defined on M is the value of the real-valued function of the n-variables on the right hand side of the formula

$$f(p) = f \circ \psi_\alpha^{-1}(x_\alpha^1(p),\ x_\alpha^2(p),\ \cdots,\ x_\alpha^n(p)).$$

Then we define that the real-valued function $f(p)$ defined on M is of class C^r if, for each point $p \in U_\alpha \subset M$, the function $f \circ \psi_\alpha^{-1}$ is a C^r-function of a local coordinate $(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)$ in a neighborhood of the point $\psi_\alpha(p)$ in $\boldsymbol{R}^n$. Here we assume that $0 \leq r \leq \infty$ holds.

The fact that the function $f(p)$ is of class C^r does not depend on the choice of the local coordinate system of M. Similarly we define that a real-valued function $f(p)$ defined on a subset E of M is of class C^r. Here we assume that $0 \leq r \leq \infty$ holds.

We assume that

$$S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha\in A}$$

is the system of positive coordinate neighborhoods, each $\overline{U_\alpha}$ is compact and $\{U_\alpha;\ \alpha \in A\}$ is a locally finite open covering of M. We denote the partition of unity associated with the open covering $\{U_\alpha\}_{\alpha\in A}$ of M as $\{\chi_\alpha\}_{\alpha\in A}$. Namely, for each α, χ_α is a C^∞-function on M and it satisfies the conditions (i) $\sim$ (iii) in the following:

(i) $0 \leq \chi_\alpha \leq 1$.

(ii) The support of χ_α is included in U_α.

(iii) $\displaystyle\sum_{\alpha\in A} \chi_\alpha(p) = 1,\ (p \in M)$.

Let $(U_\alpha,\ \psi_\alpha)$ be a C^∞-coodinate neighborhood. $\psi_\alpha(U_\alpha)$ is an open set in $\boldsymbol{R}^n$. Let $(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)$ be a system of local coordinates on U_α.

For $p \in U_\alpha$, this determines the coordinate $(x_\alpha^1(p),\ x_\alpha^2(p),\ \cdots,\ x_\alpha^n(p))$ of the point $\psi_\alpha(p)$ in $\boldsymbol{R}^n$. Let g_{ij}^α be the components of g with respect to the system of positive local coordinates and we express

$$G_\alpha = \det(g_{ij}^\alpha).$$

Then the volume element $d\mu_\alpha$ on U_α is given by the formula

$$d\mu_\alpha = \sqrt{G_\alpha}dx_\alpha^1 dx_\alpha^2 \cdots dx_\alpha^n.$$

Then assume that $(U_\beta,\ \psi_\beta)$ is one another C^∞-coordinate neighborhood. Assume that $(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n)$ is the local coordinate system on U_β.

Now, if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ holds, $\psi_\beta \circ \psi_\alpha^{-1}$ is the diffeomorphism from $\psi_\alpha(U_\alpha \cap U_\beta)$ to $\psi_\beta(U_\alpha \cap U_\beta)$. By expressing with local coordinates, the diffeomorphism $\psi_\beta \circ \psi_\alpha^{-1}$ is expressed by the formula

$$(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n) = (\psi_\beta \circ \psi_\alpha^{-1})(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)$$

$$= \Big(f_{\beta\alpha}^1(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n),\ f_{\beta\alpha}^2(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n),$$

$$\cdots,\ f_{\beta\alpha}^n(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n) \Big).$$

Here the functions $f_{\beta\alpha}^1,\ f_{\beta\alpha}^2,\ \cdots,\ f_{\beta\alpha}^n$ are the C^∞-functions of the real variables $(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)$. Then, on $U_\alpha \cap U_\beta$, the volume elements $d\mu_\alpha$ and $d\mu_\beta$

satisfy the formula

$$\begin{aligned} d\mu_\beta &= \sqrt{G_\beta}dx_\beta^1 dx_\beta^2 \cdots dx_\beta^n \\ &= \sqrt{G_\beta}\frac{\partial(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n)}{\partial(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)}dx_\alpha^1 dx_\alpha^2 \cdots dx_\alpha^n \\ &= \sqrt{G_\alpha}dx_\alpha^1 dx_\alpha^2 \cdots dx_\alpha^n. \end{aligned}$$

Here we use the formula

$$\sqrt{G_\alpha} = \sqrt{G_\beta}\frac{\partial(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n)}{\partial(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)}.$$

2 Lebesgue type measure space on the Riemannian manifold

In this section, we define the Lebesgue type measure space $(M,\ \mathcal{M},\ \mu)$ on a Riemannian manifold M.

2.1 Lebesgue type measure space on a coordinate neighborhood $(U,\ \psi)$

In this paragraph, we study the Lebesgue type measure space on a coordinate neighborhood $(U,\ \psi)$.

Assume that $(U,\ \psi)$ is a C^∞-coordinate neighborhood on M. Since $\psi(U)$ is an open set in $\boldsymbol{R}^n$, it is a Lebesgue measurable set. Therefore we have the Lebesgue measure space $(\psi(U),\ \mathcal{N}_\psi,\ \nu_\psi)$. Namely we have the following theorem.

Theorem 2.1 *Assume that $(U,\ \psi)$ is a C^∞-coordinate neighborhood on M. Then $(\psi(U),\ \mathcal{N}_\psi,\ \nu_\psi)$ is the Lebesgue measure space. Namely we have the following* (1)$\sim$(5):

(1) *$\mathcal{N}_\psi$ is a σ-ring composed of the all Lebesgue measurable sets on $\psi(U)$.*

(2) *If we have $A \in \mathcal{N}_\psi$, we have $0 \leq \nu_\psi(A) \leq \infty$.*

(3) *If $A_p \in \mathcal{N}_\psi,\ (p \geq 1)$ are mutually disjoint, we have the equality*

$$\nu_\psi(\sum_p A_p) = \sum_p \nu_\psi(A_p).$$

(4) *If, for a certain* $a > 0$, *we have* $A \subset \psi(U)$ *which is congruent to* $[0,\ a]^n$, *we have the equality*

$$\nu_\psi(A) = a^n.$$

(5) *If* $A_1,\ A_2 \in \mathcal{N}_\psi$ *are congruent, we have the equality* $\nu_\psi(A_1) = \nu_\psi(A_2)$.

Definition 2.1 Assume that $(U,\ \psi)$ is a C^∞-coordinate neighborhood. Then we define the Lebesgue type measure space $(U,\ \mathcal{M}_\psi,\ \mu_\psi)$ is a measure space which satisfies the conditions (i)∼(ii) in the following:

(i) We define $B \in \mathcal{M}_\psi$ if we have $\psi(B) \in \mathcal{N}_\psi$.

(ii) For $B \in \mathcal{M}_\psi$, we define $\mu_\psi(B)$ by the formula

$$\mu_\psi(B) = \int_{\psi(B)} \sqrt{G_\psi} d\nu_\psi$$

Her we define G_ψ is equal to

$$G_\psi = \det(g_{ij}^\psi)$$

by using the metric tensor g_{ij}^ψ concerning the system of positive local coordinates on U. Here the integral on the right hand side of (ii) is the Lebesgue integral.

Theorem 2.2 *We use the notation in Definition* 2.1. *Then* $(U,\ \mathcal{M}_\psi,\ \mu_\psi)$ *is the Lebesgue type measure space. Namely we have the following* (1)∼(3):

(1) $\mathcal{M}_\psi$ *is a* σ*-ring on* U.

(2) *If we have* $B \in \mathcal{M}_\psi$, *we have* $0 \leq \mu_\psi(B) \leq \infty$.

(3) *If* $B_p \in \mathcal{M}_\psi$, $(p \geq 1)$ *are mutually disjoint, we have the equality*

$$\mu_\psi(\sum_p B_p) = \sum_p \mu_\psi(B_p).$$

Theorem 2.3 *Assume that* $(U_\alpha,\ \psi_\alpha)$ *and* $(U_\beta,\ \psi_\beta)$ *are two* C^∞*-coordinate neighborhoods on* M. *Then, if we denote* $\mathcal{M}_\alpha = \mathcal{M}_{\psi_\alpha}$, $\mu_\alpha = \mu_{\psi_\alpha}$, $\mathcal{M}_\beta = \mathcal{M}_{\psi_\beta}$ *and* $\mu_\beta = \mu_{\mu_\beta}$, $(U_\alpha,\ \mathcal{M}_\alpha,\ \mu_\alpha)$ *and* $(U_\beta,\ \mathcal{M}_\beta,\ \mu_\beta)$ *are two Lebesgue type measure spaces defined in Definition* 2.1. *Assume that* $(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)$ *is the system of local coordinates on* U_α *and* $(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n)$ *is the system of local coordinates on* U_β. *Assume that* g_{ij}^α *and* g_{ij}^β *are the component of the Riemannian metric tensors* g^α *and* g^β *concerning the systems of local coordinates of*

U_α and U_β respectively. Put $G_\alpha = \det(g_{ij}^\alpha)$ and $G_\beta = \det(g_{ij}^\beta)$. Then we have the formula

$$\sqrt{G_\alpha} = \sqrt{G_\beta}\frac{\partial(x_\beta^1,\ x_\beta^2,\ \cdots,\ x_\beta^n)}{\partial(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)}.$$

Then, if, putting $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have $B \subset U_{\alpha\beta}$ and $B \in \mathcal{M}_\alpha \cap \mathcal{M}_\beta$, we have the equality

$$\begin{aligned}\mu_\beta(B) &= \int_{\psi_\beta(B)} \sqrt{G_\beta} dx_\beta^1 dx_\beta^2 \cdots dx_\beta^n \\ &= \int_{\psi_\alpha(B)} \sqrt{G_\alpha} dx_\alpha^1 dx_\alpha^2 \cdots dx_\alpha^n = \mu_\alpha(B).\end{aligned}$$

2.2 Lebesgue type measure space on a Riemannian manifold M

In this paragraph, we study the Lebesgue type measure space on a Riemannian manifold M.

Definition 2.2 Assume that $(M,\ g)$ is a n-dimensional Riemannian manifold and a paracompact C^∞-manifold and it is oriented with the positive direction.

Assume that $S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha\in A}$ is a system of C^∞-coordinate neighborhoods on M and $\{\chi_\alpha\}_{\alpha\in A}$ is the partition of unity associated with the open covering $\{U_\alpha\}_{\alpha\in A}$ of M. When the family of sets $\mathcal{M}$ on M and the set function μ on $\mathcal{M}$ satisfy the axioms (I), (II) in the following, we say that the triplet $(M,\ \mathcal{M},\ \mu)$ is the **Lebesgue type measure space** on M. Then we say that an element of $\mathcal{M}$ is a **Lebesgue type measurable set** and μ is a **Lebesgue type measure**.

(I) We have $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$. Here we put $\mathcal{M}_\alpha = \mathcal{M}_{\psi_\alpha}$. Namely $\mathcal{M}$ is a σ-algebra on M generated by the family of σ-rings $\{\mathcal{M}_\alpha\}_{\alpha\in A}$.

(II) We have the conditions (i)∼(iii) in the following:

(i) For $B \in \mathcal{M}$, we have $0 \le \mu(B) \le \infty$.

(ii) If $B_p \in \mathcal{M},\ (p \ge 1)$ are mutually disjoint, the direct sum

$$B = \sum_{p=1}^\infty B_p$$

is an element of $\mathcal{M}$ and we have the equality

$$\mu(B) = \sum_{p=1}^{\infty} \mu(B_p).$$

(iii) For $B \in \mathcal{M}$, we have the equality

$$\begin{aligned}\mu(B) &= \int_B d\mu = \sum_{\alpha} \int_B \chi_\alpha d\mu \\ &= \sum_{\alpha} \int_{B\cap U_\alpha} \chi_\alpha d\mu_\alpha = \sum_{\alpha} \int_{\psi_\alpha(B\cap U_\alpha)} \chi_\alpha \sqrt{G_\alpha} d\nu_\alpha.\end{aligned}$$

Here we denote $B = \bigcup_\alpha (B \cap U_\alpha)$.

In Definition 2.2, we remark that the definition of the Lebesgue type measure space $(M,\ \mathcal{M},\ \mu)$ does not depend on the choice of the system of C^∞-coordinate neighborhoods

$$S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha\in A}.$$

Corollary 2.1 *We use the notation in* Definition 2.2. *Assume that B is a subset in M. Then the conditions* (1)$\sim$(3) *in the following are equivalent*:

(1) *We have $B \in \mathcal{M}$.*

(2) *For an arbitrary $\alpha \in A$, we have $B \cap U_\alpha \in \mathcal{M}_\alpha$ and the equality*

$$B = \bigcup_{\alpha} (B \cap U_\alpha).$$

(3) *For an arbitrary $\alpha \in A$, there exists $B_\alpha \in \mathcal{M}_\alpha$ such that we have the equality*

$$B = \bigcup_{\alpha} B_\alpha.$$

Theorem 2.4 *Assume that M is a n-dimensional Riemannian manifold and a paracompact C^∞-manifold and it is oriented with the positive direction. Assume that $S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha\in A}$ is a system of C^∞-coordinate neighborhoods and $\{\chi_\alpha\}_{\alpha\in A}$ is the partition of unity associated with an open covering $\{U_\alpha\}_{\alpha\in A}$ of M. We express the family of the Lebesgue type measure spaces on the coordinate neighborhoods as $\{(U_\alpha,\ \mathcal{M}_\alpha,\ \mu_\alpha)\}_{\alpha\in A}$. Now we define $\mathcal{M}$ and μ by the conditions* (i) *and* (ii) *in the following*:

(i) *We have* $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$. *Here we put* $\mathcal{M}_\alpha = \mathcal{M}_{\psi_\alpha}$.

(ii) *For* $B \in \mathcal{M}$, *we define*

$$\mu(B) = \int_B d\mu = \sum_\alpha \int_B \chi_\alpha d\mu = \sum_\alpha \int_{B \cap U_\alpha} \chi_\alpha d\mu_\alpha$$
$$= \sum_\alpha \int_{\psi_\alpha(B \cap U_\alpha)} \chi_\alpha(x_\alpha^1,\ x_\alpha^2,\ \cdots,\ x_\alpha^n)\sqrt{G_\alpha}dx_\alpha^1 dx_\alpha^2 \cdots dx_\alpha^n.$$

Here we put $B = \bigcup_\alpha (B \cap U_\alpha)$. *Then the triplet* $(M,\ \mathcal{M},\ \mu)$ *is the Lebesgue type measure space on* M *such that we have the conditions* (1) *and* (2) *in the following*:

(1) *For* $B \in \mathcal{M}$, *we have* $0 \leq \mu(B) \leq \infty$.

(2) *If* $B_p \in \mathcal{M},\ (p \geq 1)$ *are mutually disjoint, the direct sum*

$$B = \sum_{p=1}^{\infty} B_p$$

is an element of $\mathcal{M}$ *and we have the equality*

$$\mu(B) = \sum_{p=1}^{\infty} \mu(B_p).$$

Here $(M,\ \mathcal{M},\ \mu)$ *does not depend on the choice of the system of* C^∞*-coordinate neighborhoods*

$$S = \{(U_\alpha,\ \psi_\alpha)\}_{\alpha \in A}.$$

Corollary 2.2 *We use the notation in* Theorem 2.4. *If, for* $B \in \mathcal{M}$, *we have the conditions* $B \subset U_\alpha$ *and* $B \in \mathcal{M}_\alpha$, *we have the equality*

$$\mu(B) = \int_B d\mu = \int_{\psi_\alpha(B)} \sqrt{G_\alpha} d\nu_\alpha.$$

3 Measurable functions

In this section, we define the concept of the Lebesgue type measurable functions and study their fundamental properties.

Assume that $(M,\ g)$ is a n-dimensional Riemannian manifold and $(M,\ \mathcal{M},\ \mu)$ is a Lebesgue type measure space on M. Here we assume that $n \geq 1$ holds.

Assume that a subset E of M is a measurable set.

Here we assume that every considered function $f(p)$ is a extended real-valued function defined on E. Namely we assume that the range of the function $f(p)$ is included in the extended real number space $\overline{\boldsymbol{R}} = [-\infty,\ \infty]$.

We denote the σ-ring of all measurable sets included in E as $\mathcal{M}_E$ and we denote the restricted measure on $\mathcal{M}_E$ of the Lebesgue type measure μ on M as the same symbol μ.

Then the measure space $(E,\ \mathcal{M}_E,\ \mu)$ is the **Lebesgue type measure space** on E.

In the sequel, we denote $\mathcal{M}_E$ as $\mathcal{M}$ as the abbreviation.

When we put $E(\infty) = \{p \in E;\ |f(p)| = \infty\}$, we say that a point p of $E(\infty)$ is a **singular point** of $f(p)$.

At first, we define a simple function.

Definition 3.1 We define that a function $f(p)$ defined on a measurable set E of M is a **simple function** if we have the expression

$$f(p) = \sum_{i=1}^{\infty} \alpha_j \chi_{E_j}(p) \tag{3.1}$$

for a countable division Δ of E:

$$(\Delta); E = \sum_{j=1}^{\infty} E_j = E_1 + E_2 + \cdots . \tag{3.2}$$

Here α_j is a real number or $\pm\infty$, $(j \geq 1)$ and they need not be different each other. $\chi_{E_j}(p)$ is the defining function of the set E_j, $(i \geq 1)$.

Then we denote the simple function $f(p)$ as $f_\Delta(p)$.

Here we assume that all the subsets $E_1,\ E_2,\ \cdots$ of E are measurable subsets and they are mutually disjoint.

Further we denote $E(\infty) = \{p \in E;\ |f(p)| = \infty\}$ and we assume that we have $E(\infty) \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.

In Definition 3.1, we define the defining function $\chi_A(p)$ of a set A in the following:

$$\chi_A(p) = \begin{cases} 1, & (p \in A), \\ 0, & (p \notin A). \end{cases}$$

Nevertheless the expression of a simple function $f(p)$ in the formula (3.1) has an infinitely many varieties because the form of the division Δ of E in the formula (3.2) has an infinitely many varieties.

Even if the range of a simple function $f(p)$ is fixed as above, we often use the symbol $f_\Delta(p)$ in order to distinguish the simple function of the different expression in the form of the formula (3.1).

Then we define a measurable function in the following.

Definition 3.2 We define that an extended real-valued function $f(p)$ defined on a measurable set E of M is a **measurable function** if we have the conditions (i) and (ii) in the following:

(i) If we put $E(\infty) = \{p \in E;\ |f(p)| = \infty\}$, we have $E(p) \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.

(ii) There exists a sequence of simple functions $\{f_m(p);\ m \geq 1\}$ such that we have the limit

$$\lim_{m\to\infty} f_m(p) = f(p) \tag{3.3}$$

in the sense of point wise convergence on $E \backslash E(\infty)$. Here we assume that we have

$$E_m(\infty) = \{p;\ |f_m(p)| = \infty\} \subset E(\infty),\ (m \geq 1).$$

The condition (ii) of Definition 3.2 is equivalent to the condition (iii) in the following:

(iii) At an arbitrary point p in $E \backslash E(\infty)$ and for an arbitrary positive number $\varepsilon > 0$, there exists a certain natural number m_0 such that, for an arbitrary natural number m such as $m \geq m_0$, we have the inequality

$$|f_m(p) - f(p)| < \varepsilon.$$

Example 3.1 Assume that E is a measurable set on M. A simple function $f(p)$ and a continuous function $f(p)$ defined on E are measurable.

Theorem 3.1 *Assume that E is a measurable set on M. Assume that two functions f and g are some measurable functions defined on E. Then the following functions* (1)$\sim$(10) *are also the measurable functions defined on E:*

(1) $f + g$. (2) $f - g$. (3) fg.

(4) f/g. *Here we assume that we have* $g(p) \neq 0,\ (p \in E)$.

(5) αf, *Here we assume that α is a real constant.*

(6) $|f|^p$, *Here we assume that $p \neq 0$ is a real number.*

(7) $\sup(f,\ g)$. (8) $\inf(f,\ g)$.

(9) $f^+ = \sup(f,\ 0)$. (10) $f^- = -\inf(f,\ 0)$.

The functions $\sup(f,\ g)$ and $\inf(f,\ g)$ in Theorem 3.1 are defined in the following:

$$\sup(f,\ g)(p) = \sup(f(p),\ g(p)),\ (p \in E),$$

$$\inf(f,\ g)(p) = \inf(f(p),\ g(p)),\ (p \in E).$$

Further we have the formulas

$$f = f^{+} - f^{-},\ |f| = f^{+} + f^{-}.$$

Theorem 3.2 *If a function $f(p)$ is measurable on E and we have $F \subset E$ and $F \in \mathcal{M}_E$, the restriction $f_F(p) = f(p)|_F$ of $f(p)$ on F is measurable on F.*

Now we have the notation in the following. If α and β are tow arbitrary real numbers or $\pm\infty$, we put in the following:

$$E(f > \alpha) = \{p \in E;\ f(p) > \alpha\},$$

$$E(f \leq \alpha) = \{p \in E;\ f(p) \leq \alpha\},$$

$$E(f = \alpha) = \{p \in E;\ f(p) = \alpha\},$$

$$E(\alpha < f \leq \beta) = \{p \in E;\ \alpha < f(p) \leq \beta\},\ (\alpha < \beta).$$

Theorem 3.3 *Assume that $f(p)$ is a function defined on E. Then the following four statements are equivalent*:

(1) *For an arbitrary real number α, we have $E(f > \alpha) \in \mathcal{M}_E$.*

(2) *For an arbitrary real number α, we have $E(f \leq \alpha) \in \mathcal{M}_E$.*

(3) *For an arbitrary real number α, we have $E(f \geq \alpha) \in \mathcal{M}_E$.*

(4) *For an arbitrary real number α, we have $E(f < \alpha) \in \mathcal{M}_E$.*

Corollary 3.1 *For a function $f(p)$ defined on E, the following statements* (1) *and* (2) *are equivalent*:

(1) *For an arbitrary real number α, we have $E(f > \alpha) \in \mathcal{M}_E$.*

(2) *For an arbitrary rational number r, we have $E(f > r) \in \mathcal{M}_E$.*

Corollary 3.2 *Assume that a function $f(p)$ defined on E satisfies the condition in* Theorem 3.3. *Then each set in the following belongs to $\mathcal{M}_E$*:

(1) $E(f=\alpha)$, *Here* α *is an arbitrary real number.*

(2) $E(f<\infty)$. (3) $E(f=\infty)$.

(4) $E(f>-\infty)$. (5) $E(f=-\infty)$.

Theorem 3.4 *For a function* $f(p)$ *defined on* E, *the following statements* (1) *and* (2) *are equivalent*:

(1) $f(p)$ *is measurable on* E. *Namely there exists a sequence of simple functions* $\{f_m(p)\}$ *which converges to* $f(p)$ *at each point in* $E\backslash E(\infty)$. *Here we assume that we have*

$$E_m(\infty)=\{p\in E;\ |f_m(p)|=\infty\}\subset E(\infty),\ (m\geq 1).$$

(2) *For an arbitrary real number* α, *we have* $E(f>\alpha)\in\mathcal{M}_E$.

If $f(p)$ is measurable, there exists a sequence of simple functions which converges to $f(p)$ at each point in $E\backslash E(\infty)$ by virtue of the definition. Then, by virtue of this theorem, we have the method of constructing such a sequence of simple functions concretely.

We give this result in the Corollary 3.3 in the following.

Corollary 3.3 *Assume that a function* $f(p)$ *is measurable on* E. *Then we put*

$$E_m^j=E\ (\ \frac{j}{m}\leq f<\frac{j+1}{m}\ ),(j=0,\ \pm 1,\ \pm 2,\ \cdots)$$

for each natural number $m\geq 1$ *and we express the defining function of* E_m^j *as*

$$C_m^j(p)=\chi_{E_m^j}(p).$$

Then, if we define the simple function $f_m(p)$ *by the formula*

$$f_m(p)=\sum_{j=-\infty}^{\infty}\frac{j}{m}C_m^j(p),\ (p\in E),$$

the sequence of simple functions $\{f_m(p)\}$ *converges to* $f(p)$ *at each point on* $E\backslash E(\infty)$.

Theorem 3.5 *If a function* $f(p)$ *is measurable on* E *and we have* $f(p)\geq 0$, *there exists a sequence of simple functions* $\{f_m(p)\}$ *which converges to* $f(p)$ *at each point on* $E\backslash E(\infty)$ *and satisfies the conditions* $f_m(p)\geq 0,\ (m\geq 1)$.

Theorem 3.6 *If the functions* $f_m(p),\ (m\geq 1)$ *defined on* E *are measurable, the functions* $(1)\sim(5)$ *in following are measurable on* E:

(1) $\sup_{m\geq 1} f_m(p)$. (2) $\inf_{m\geq 1} f_m(p)$.

(3) $\overline{\lim_{m\to\infty}} f_m(p)$. (4) $\underline{\lim_{m\to\infty}} f_m(p)$.

(5) *If we have the limit* $f(p) = \lim_{m\to\infty} f_m(p)$ *at almost every* $p \in E$, $f(p)$ *is measurable on* E.

If we have the certain property (P) for a measurable function $f(p)$ or a sequence of measurable functions $\{f_m(p)\}$ on E excluding a null set e, we say that we have the property (P) for the function $f(p)$ and the sequence of simple functions $\{f_m(p)\}$ **almost everywhere**.

For an example, if we have the equality

$$f(p) = 0,\ (p \in E\backslash e,\ \mu(e) = 0),$$

$f(p)$ is equal to 0 almost everywhere on E. We express this as

$$f(p) = 0,\ (\text{a.e. } p \in E).$$

Further, if we have the limit

$$\lim_{m\to\infty} f_m(p) = f(p),\ (p \in E\backslash e,\ \mu(e) = 0),$$

$f_m(p)$ converges to $f(p)$ almost everywhere on E.

We express this as

$$\lim_{m\to\infty} f_m(p) = f(p),\ (\text{a.e. } p \in E).$$

Then the values of the limit function $f(p)$ on the null set e happen to be undetermined. But we give the function $f(p)$ one value at each point of such a null set e and fix its value. Thereby the domain of the function is determined constantly. Namely, when the domains of some functions are different, we cannot state a certain determined statement concerning these functions.

In this case, because the value of the Lebesgue type integral of $f(p)$ is not influenced even if we give any value of $f(p)$ on the null set e, this is the idea for the clear statement of the proposition.

Theorem 3.7 (Egorov's Theorem) *Assume that* E *is a measurable set in* M *and we have* $\mu(E) < \infty$.

Assume that $f_m(p)$, $(m \geq 1)$ *are the measurable functions which take the finite values almost everywhere on* E. *Further, we assume that there exists the finite limit* $f(p) = \lim_{m\to\infty} f_m(p)$ *almost everywhere on* E.

Then, for an arbitrary positive number $\varepsilon > 0$, *there exists a set* $F \in \mathcal{M}_E$ *such that we have the statements* (1) *and* (2) *in the following*:

(1) *We have* $F \subset E$ *and* $\mu(E \backslash F) < \varepsilon$.

(2) $f_m(p)$ *converges to* $f(p)$ *uniformly on* F.

Corollary 3.4 *In* Theorem 3.7, *we can take a closed set as* F.

By virtue of Egorov's Theorem and Corollary 3.4, we have the theorem in the following.

Theorem 3.8 (Luzin's Theorem) *Assume that* E *is a measurable set in* M. *Assume that* $f(p)$ *is a measurable function which takes a finite value almost everywhere in* E.

Then, for an arbitrary positive number $\varepsilon > 0$, *there exists a certain closed set* $F \subset E$ *such that we have the statements* (1) *and* (2) *in the following*:

(1) *We have* $\mu(E \backslash F) < \varepsilon$.

(2) $f(p)$ *is continuous on* F.

4 Definition of the Lebesgue type integral

In this section, we define the concept of the Lebesgue type integral for the Lebesgue type measurable functions.

Assume that $(M,\ g)$ is a n-dimensional Riemannian manifold and the n-dimensional Riemannian manifold M is a Lebesgue type measure space $(M,\ \mathcal{M},\ \mu)$. Here we assume $n \geq 1$. Assume that a subset E of M is a Lebesgue type measurable set.

Then, by restricting $(M,\ \mathcal{M},\ \mu)$ on E, we have the n-dimensional Lebesgue type measure space $(E,\ \mathcal{M},\ \mu)$ on E.

Then we define the Lebesgue type integral on E of a measurable function $f(p)$ defined on E and denote it by the symbol

$$\int_E f(p) d\mu.$$

In the following, we define the Lebesgue type integral of $f(p)$ in the two steps in the following.

(1) The case where $f(p)$ is a simple function

Assume that a function $f(p)$ is expressed as follows:

$$f(p) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}(p), \ (\alpha_j \in \overline{\boldsymbol{R}}, \ j \geq 1), \tag{4.1}$$

$$E = E_1 + E_2 + \cdots, \ (E_j \in \mathcal{M}_E, \ j \geq 1). \tag{4.2}$$

Then we define the Lebesgue type integral of $f(p)$ as the sum of the series on the right hand side of the equality

$$\int_E f(p) d\mu = \sum_{j=1}^{\infty} \alpha_j \mu(E_j) \tag{4.3}$$

and denote it by the symbol on the left hand side. Here we consider only the case of the absolute convergence of the series on the right hand side.

The sum of the absolutely convergent series on the right hand side of the formula (4.3) has the decided value independent of the expression of the function $f(p)$ such as the formula (4.1). Then we say that $f(p)$ is **integrable** on E in the sense of Lebesgue type integral. We also say that it is **integrable** on E.

$f(p)$ is integrable on E if and only if $|f(p)|$ is integrable on E.

This equivalence is understood by the following reasoning.

For the absolute value function of the function $f(p)$ in the formula (4.1), we have the equality

$$|f(p)| = \sum_{j=1}^{\infty} |\alpha_j| \chi_{E_j}(p). \tag{4.4}$$

Therefor, we have the equality

$$\int_E |f(p)| d\mu = \sum_{j=1}^{\infty} |\alpha_j| \mu(E_j). \tag{4.5}$$

Then the series on the right hand side of the formula (4.3) is absolutely convergent if and only if the series on the right hand side of the formula (4.5) is convergent.

Remark 4.1 As for the convergence and divergence of this series on the right hand side of the formula (4.3), we can consider the two cases: (1) convergence and (2) divergence.

By considering precisely, we can consider the cases in the following:

(1) On the case of convergence.

(1-i) The case of absolute convergence.

(1-ii) The case of conditional convergence.

(2) In the case of divergence.

(2-i) The case where it converges to either one of $\pm\infty$.

(2-ii) The case where it oscillates and it does not converge to a fixed value.

The case (1-i) is the case of the definition of the Lebesgue type integral and the case (1-ii) is the case where the integral converges conditionally.

Here, because we consider only the case where a simple function $f(p)$ is integrable, this means that we consider only the case (1-i) of Remark 4.1.

In general, we study the precise of the situations of convergence and divergence of the Lebesgue type integral afterward in the section of the calculation of the Lebesgue type integral.

(2) The case where $f(p)$ is a general measurable function

In this case, these exists a sequence of simple functions $\{f_m(p)\}$ which converges to $f(p)$ at each point on $E\backslash E(\infty)$.

Here, if each $f_m(p)$ is integrable and we have the limit

$$\lim_{m\to\infty}\int_E f_m(p)d\mu, \tag{4.6}$$

we say that this limit is a **Lebesgue type integral** of $f(p)$ on E and we denote it as

$$\int_E f(p)d\mu = \lim_{m\to\infty}\int_E f_m(p)d\mu. \tag{4.7}$$

Further we say that the Lebesgue type integral (4.7) **converges absolutely** if the limit (4.6) has the fixed value independent to the choice of a sequence of integrable simple functions $\{f_m(p)\}$ which converges to $f(p)$ at each point on $E\backslash E(\infty)$.

Then we say that $f(p)$ is **integrable** on E. The Lebesgue type integral is usually the Lebesgue type integral in this case.

A function $f(p)$ defined on E is integrable if and only if the absolute value of the function $|f(p)|$ is integrable on E.

Theorem 4.1 *When $f(p)$ is integrable on E, we choose the sequence of simple functions $\{f_m(p)\}$ as* Corollary 3.3. *Then the Lebesgue type integral of $f(p)$ on E is given by the equality*

$$\int_E f(p)d\mu = \lim_{m\to\infty}\frac{1}{m}\sum_{j=-\infty}^{\infty} j\mu\left(E(\frac{j}{m}\le f<\frac{j+1}{m})\right).$$

Theorem 4.2 *Assume that $f(p)$ is integrable on E. Now, if we put*

$$f^+(p) = \sup\{f(p),\ 0\},\ f^-(p) = -\inf\{f(p),\ 0\},$$

we have the formulas

$$|f(p)| \geq f^+(p) \geq 0,\ |f(p)| \geq f^-(p) \geq 0,$$

$$f(p) = f^+(p) - f^-(p),\ |f(p)| = f^+(p) + f^-(p).$$

Then both $f^+(p)$ and $f^-(p)$ are integrable on E and we have the equality

$$\int_E f(p)d\mu = \int_E f^+(p)d\mu - \int_E f^-(p)d\mu.$$

Further we have the equality

$$\int_E |f(p)|d\mu = \int_E f^+(p)d\mu + \int_E f^-(p)d\mu.$$

Corollary 4.1 *Assume that $f(p)$ is integrable on E and $g(p)$ is measurable on E. Then, if we have the inequality $|g(p)| \leq |f(p)|$, $g(p)$ is integrable on E.*

Further we say that the Lebesgue type integral (4.7) **converges conditionally** if the limit (4.6) has the various value depending on the choice of a sequence of integrable simple functions $\{f_m(p)\}$ which converges to $f(p)$ at each point on $E\backslash E(\infty)$.

If the limit (4.6) does not exist, we say that the Lebesgue type integral **diverges**.

Then the Lebesgue type integral does not exist.

Remark 4.2 The case of the conditional convergence of (1-ii) in Remark 4.1 means the case of the conditional convergence of the integral of this simple function.

In general, we study the precise of the situations of the convergence and divergence of the Lebesgue type integral afterward in the section of the calculation of the Lebesgue type integral.

5 Fundamental properties of the Lebesgue type integral

In this section, we study the fundamental properties of the Lebesgue type integral on the Riemannian manifold.

Assume that a subset E of M is a measurable set and E is a n-dimensional Lebesgue type measure space $(E,\ \mathcal{M},\ \mu)$. Here we assume $n \geq 1$.

5.1 Fundamental properties of the Lebesgue type integral

In this paragraph, we study the fundamental properties of the Lebesgue type integral.

As for the various formulas in the various theorems in this paragraph, we can prove simply that they are true for the integrable simple functions. For the general integrable functions, we can prove them by taking the limits of the various formulas for the integrable simple functions by virtue of the definition of the Lebesgue type integral. Therefore we omit the precises of the proofs here.

In the sequel, we assume that E is a measurable set in M.

Theorem 5.1.1 *Assume that a function $f(p)$ is integrable on E and a subset $F \subset E$ is measurable. Then the restrictions $f_F(p) = f(p)|_F$ of $f(p)$ on F integrable on F and we have the equality*

$$\int_F f_F(p)d\mu = \int_F f(p)d\mu.$$

Namely the function $f(p)$ is integrable on F.

Theorem 5.1.2 *Assume that E and $f(x)$ are the same as in* Theorem 5.1.1. *Further, assume that $E = E_1 + E_2$ is a division of E. Here E_1 and E_2 are measurable. Then we have the equality*

$$\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu.$$

Theorem 5.1.3 *Assume that two functions $f(p)$ and $g(p)$ are integrable on E. Then we have the statements* (1) *and* (2) *in the following*:

(1) *$f(p) + g(p)$ is integrable on E and we have the equality*

$$\int_E \{f(p) + g(p)\}d\mu = \int_E f(p)d\mu + \int_E g(p)d\mu.$$

(2) *For an arbitrary real number α, $\alpha f(p)$ is also integrable on E and we have the equality*

$$\int_E \{\alpha f(p)\}d\mu = \alpha \int_E f(p)d\mu.$$

Corollary 5.1.1 *Assume that two functions $f(p)$ and $g(p)$ are integrable on E. Then, for two arbitrary real numbers α and β, $\alpha f(p)+\beta g(p)$ is integrable on E and we have the equality*

$$\int_E \{\alpha f(p) + \beta g(p)\} d\mu = \alpha \int_E f(p) d\mu + \beta \int_E g(p) d\mu.$$

Theorem 5.1.4 *Assume that two functions $f(p)$ and $g(p)$ are integrable on E. Then we have the results* (1) $\sim$ (3) *in the following*:

(1) *If we have $f(p) \geq 0$, $(p \in E)$, we have the inequality*

$$\int_E f(p) d\mu \geq 0.$$

(2) *If we have $f(p) \geq g(p)$, $(p \in E)$, we have the inequality*

$$\int_E f(p) d\mu \geq \int_E g(p) d\mu.$$

(3) *We have the inequality*

$$|\int_E f(p) d\mu| \leq \int_E |f(p)| d\mu.$$

Theorem 5.1.5 *Assume that a function $f(p)$ is integrable on E. Then we have the results* (1) *and* (2) *in the following*:

(1) *If we have $\mu(E) = 0$, we have $\int_E f(p) d\mu = 0$.*

(2) *We have $\mu(E(f = \infty)) = \mu(E(f = -\infty)) = 0$.*

Corollary 5.1.2 *Assume that two functions $f(p)$ and $g(p)$ are measurable on E and they are equal almost everywhere on E. Then, if $f(p)$ is integrable on E, $g(p)$ is also integrable on E and we have the equality*

$$\int_E f(p) d\mu = \int_E g(p) d\mu.$$

By virtue of Corollary 5.1.2, if two integrable functions are equal almost everywhere, we need not distinguish their Lebesgue type integrals.

Theorem 5.1.6 *If a function $f(p)$ is integrable on E, $E(f \neq 0)$ is equal to the sum of at most countable subsets of the finite Lebesgue type measures.*

Theorem 5.1.7(The first mean value theorem of the integration) *Assume that a function $f(p)$ is a bounded measurable function on E and $g(p)$ is integrable on E. Then, if we put*

$$m = \inf_{p \in E} f(p), \ M = \sup_{p \in E} f(p),$$

we have the statements (1) *and* (2) *in the following*:

(1) *$f(p)g(p)$ is integrable on E.*

(2) *There exists a real constant α such that we have $m \leq \alpha \leq M$ and we have the equality*

$$\int_E f(p)|g(p)|d\mu = \alpha \int_E |g(p)|d\mu.$$

Corollary 5.1.3 *Assume that a function $f(p)$ is continuous on a bounded closed domain and $g(p)$ is integrable on E and we have $g(p) \geq 0$, $(p \in E)$. Then there exists a certain point $p_0 \in E$ such that we have the equality*

$$\int_E f(p)g(p)d\mu = f(p_0) \int_E g(p)d\mu.$$

Theorem 5.1.8 *Assume that E is a measurable set on M and a function $f(p)$ is integrable on E. Then, for an arbitrary real number $\varepsilon > 0$, there exists a continuous function $f_\varepsilon(p)$ on M which is identically* 0 *outside of a certain bounded measurable set such that we have the inequality*

$$|\int_E f(p)d\mu - \int_E f_\varepsilon(p)d\mu| \leq \int_E |f(p) - f_\varepsilon(p)|d\mu < \varepsilon.$$

5.2 Lebesgue type integral and the limiting processes

In this paragraph, we study the Lebesgue type integral on M and the limiting processes.

Theorem 5.2.1 *Assume that E is a measurable set in M and E_m, $(m \geq 1)$ are mutually disjoint measurable subsets of E and we have a division*

$$E = E_1 + E_2 + \cdots .$$

Then, if a function $f(p)$ is integrable on E, we have the equality

$$\int_E f(p)d\mu = \int_{E_1} f(p)d\mu + \int_{E_2} f(p)d\mu + \cdots . \tag{5.1}$$

Further, if a function $f(p)$ is integrable on each E_m, $(m \geq 1)$ and we have the condition

$$\sum_{m=1}^{\infty} \int_{E_m} |f(p)|d\mu < \infty,$$

the function $f(p)$ is integrable on E and we have the equality in the formula (5.1).

Corollary 5.2.1 *Assume that E is a measurable set in M and E_m, $(m \geq 1)$ are a monotone increasing sequence of measurable sets and we have the equality*

$$E = \bigcup_{m=1}^{\infty} E_m.$$

Assume that a function $f(p)$ is integrable on E. Then, for an arbitrary positive number $\varepsilon > 0$, there exists a natural number m_0 such that, for any $m \geq m_0$, we have the inequality

$$\int_{E \setminus E_m} |f(p)|d\mu < \varepsilon.$$

Especially we have the equality

$$\lim_{m \to \infty} \int_{E_m} f(p)d\mu = \int_E f(p)d\mu.$$

Remark 5.2.1 If the Lebesgue type integral of a function $f(p)$

$$\int_E f(p)d\mu$$

converges conditionally and if we take a special choice of a sequence of measurable sets $\{E_m\}$ such as in Corollary 5.2.1 in the above, we have the limit as in Corollary 5.2.1.

Corollary 5.2.2 *Assume that E is a measurable set in M and a function $f(p)$ is integrable on E. Now we put*

$$E_m = E(|f| < m), \ (m \geq 1).$$

Then, for an arbitrary positive number $\varepsilon > 0$, there exists a certain natural number m_0 such that, for any $m \geq m_0$, we have the inequality

$$\int_{E \setminus E_m} |f(p)|d\mu < \varepsilon.$$

Especially we have the equality

$$\lim_{m\to\infty}\int_{E_m} f(p)d\mu = \int_E f(p)d\mu.$$

The Theorem 5.2.2 in the following gives the absolute continuity of the indefinite integral.

This is the application of Theorem 5.2.1 and Corollary 5.2.2.

Theorem 5.2.2 *Assume that E is a measurable set in M and a function $f(p)$ is integrable on E. Then, for an arbitrary positive number $\varepsilon > 0$, there exists a certain positive number $\delta > 0$ such that, for any measurable sets e in E which satisfies the condition $\mu(e) < \delta$, we have the inequality*

$$|\int_e f(p)d\mu| < \varepsilon.$$

Theorem 5.2.3(Lebesgue bounded convergence theorem) *Assume that E is a compact set in M. If a sequence of uniformly bounded measurable functions $\{f_m(p);\ m \geq 1\}$ converges to $f(p)$ almost everywhere in E, we have the equality*

$$\lim_{m\to\infty}\int_E f_m(p)d\mu = \int_E f(p)d\mu.$$

Theorem 5.2.4(Lebesgue convergence theorem) *Assume that E is a measurable set in M. Here we assume the conditions (1) and (2) in the following:*

(1) *A sequence of measurable functions $\{f_m(p);\ m \geq 1\}$ on E converges to a finite limit $f(p)$ almost everywhere on E.*

(2) *There exists an integrable function $\Phi(p)$ such that we have the condition $\Phi(p) \geq 0,\ (p \in E)$ and we have the inequalities*

$$|f_m(p)| \leq \Phi(p),\ (p \in E,\ m \geq 1).$$

Then we have the equality

$$\lim_{m\to\infty}\int_E f_m(p)d\mu = \int_E f(p)d\mu.$$

By virtue of the Lebesgue convergence theorem, we have the theorem of termwise integration.

Theorem 5.2.5 (Theorem of termwise integration) *Assume that* E *is a measurable set in* M *and* $\{f_m(p);\ m \geq 1\}$ *is a sequence of measurable functions on* E*. Here we put*

$$f(p) = f_1(p) + f_2(p) + \cdots .$$

Then, if the series on the right hand side converges almost everywhere on E *and there exists an integrable function* $\Phi(p)$ *on* E *such that we have the condition* $\Phi(p) \geq 0,\ (p \in E)$ *and, for an arbitrary natural number* $m \geq 1$*, we have the inequality*

$$|\sum_{j=1}^{m} f_j(p)| \leq \Phi(p),\ (p \in E),$$

we have the termwise integration.

Namely we have the equality

$$\int_E f(p)d\mu = \int_E f_1(p)d\mu + \int_E f_2(p)d\mu + \cdots .$$

Corollary 5.2.3 *Assume that* E, $\{f_m(p);\ m \geq 1\}$ *and* $f(p)$ *are the same as in* Theorem 5.2.5. *Here we assume that we have either one of the conditions* (i) *and* (ii) *in the following*:

(i) *There exists a Lebesgue type integrable function* $\Phi(p)$ *on* E *such that we have the condition* $\Phi(p) \geq 0,\ (p \in E)$ *and we have the inequalities*

$$\sum_{j=1}^{m} |f_j(p)| \leq \Phi(p),\ (p \in E,\ m \geq 1).$$

(ii) *We have the condition*

$$\sum_{j=1}^{\infty} \int_E |f_j(p)|d\mu < \infty.$$

Then we have the theorem of termwise integration.

Theorem 5.2.6 (Beppo Levi Theorem) *Assume that* E *is a measurable set in* M *and* $\{f_m(p);\ m \geq 1\}$ *is a monotone increasing sequence of integrable functions on* E*. Further we assume that a monotone increasing sequence*

$$\{\int_E f_m(p)d\mu\}$$

is bounded from above. Then, if we put

$$\lim_{m\to\infty} f_m(p) = f(p),\ (p \in E),$$

the function $f(p)$ has the finite value almost everywhere on E, it is integrable on E and we have the equality

$$\lim_{m\to\infty} \int_E f_m(p)d\mu = \int_E f(p)d\mu.$$

We give the results used in the proof of the theorem in the above as two Corollaries in the following.

Corollary 5.2.4 *Assume that, for a measurable set E in M and for a monotone increasing sequence $\{E_m;\ m \geq 1\}$ of measurable sets in E, we have the equality*

$$E = \bigcup_{m=1}^{\infty} E_m.$$

Further, if a measurable function $f(p)$ on E is integrable on each E_m, $(m \geq 1)$ and we have the condition

$$\lim_{m\to\infty} \int_{E_m} |f(p)|d\mu < \infty,$$

then $f(p)$ is integrable on E and we have the equality

$$\lim_{m\to\infty} \int_{E_m} f(p)d\mu = \int_E f(p)d\mu.$$

Corollary 5.2.5 *Assume that E is a measurable set in M and $\{f_m(p);\ m \geq 1\}$ is a monotone sequence of integrable functions on E. Then, if*

$$\lim_{m\to\infty} f_m(p) = f(p),\ (p \in E)$$

has the finite value almost everywhere on E and it is integrable on E, we have the equality

$$\lim_{m\to\infty} \int_E f_m(p)d\mu = \int_E f(p)d\mu.$$

Next, as the Corollary of Beppo Levi Theorem, we prove Fatou's Lemma.

At first, we remark that Fatou's Lemma is used many times in the following form.

Assume that, for the integrable nonnegative functions $f_m(p),\ (m \geq 1)$ on a measurable set E, we have the equality

$$\lim_{m\to\infty} f_m(p) = f(p),\ (p \in E).$$

If we have the condition

$$\underline{\lim}_{m\to\infty} \int_E f_m(p)d\mu < \infty,$$

$f(p)$ is integrable on E and we have the inequality

$$\int_E f(p)d\mu \leq \underline{\lim}_{m\to\infty} \int_E f_m(p)d\mu.$$

Here we prove Fatou's Lemma in a nealy more generalized form.

Theorem 5.2.7 (Fatou's Lemma) *Assume that E is a measurable set in M and assume that $\{f_m(p);\ m \geq 1\}$ is a sequence of integrable nonnegative functions on E and we have the condition*

$$\underline{\lim}_{m\to\infty} \int_E f_m(p)d\mu < \infty.$$

Then the function

$$f(p) = \underline{\lim}_{m\to\infty} f_m(p)$$

is integrable on E and we have the inequality

$$\int_E f(p)d\mu = \int_E \Big(\underline{\lim}_{m\to\infty} f_m(p) \Big) d\mu$$
$$\leq \underline{\lim}_{m\to\infty} \int_E f_m(p)d\mu.$$

Theorem 5.2.8 in the following is the result on the differentiation under the integral symbol

Theorem 5.2.8 *Assume that E is a measurable set in M and $(a,\ b)$ is an interval in $\boldsymbol{R}$. Assume that a function $f(p,\ t)$ is defined on a set $E \times (a,\ b) = \{(p,\ t);\ p \in E,\ t \in (a,\ b)\}$ and it satisfies the conditions* (i) $\sim$ (iii) *in the following*:

(i) *For an arbitrary $t \in (a,\ b)$, $f(p,\ t)$ is integrable on E.*

(ii) *For almost every p on E, $f(p,\ t)$ is differentiable with respect to t. Then we denote the partial derivative of $f(p,\ t)$ with respect to t as $f_t(p,\ t)$.*

(iii) *There exists an integrable function* $\Phi(p)$ *on* E *such that we have the condition* $\Phi(p) \geq 0,\ (p \in E)$ *and we have the inequality*

$$|f_t(p,\ t)| \leq \Phi(p),\ (p \in E,\ t \in (a,\ b)).$$

Then, if we put

$$F(t) = \int_E f(p,\ t) d\mu,$$

$F(t)$ *is differentiable on* $(a,\ b)$ *and we have the equality*

$$F'(t) = \int_E f_t(p,\ t) d\mu.$$

6 Calculation of the Lebesgue type integral

In this section, we study the calculation of the Lebesgue type integral on M by the method of approximation of the integral domain by using an approximating direct family of compact sets in the integral domain.

Assume that an integral domain E is a measurable set in M and an integrand function $f(p)$ is measurable on E.

We consider a direct set A and a direct family $\{E_\alpha;\ \alpha \in A\}$ of compact sets included in E.

Here we say that a direct family $\{E_\alpha\}$ **converges** to E if, for an arbitrary compact set K included in E, there exists a certain $\alpha_0 \in A$ such that, for an arbitrary α such as $\alpha \geq \alpha_0$, we have $K \subset E_\alpha$. Then we say that the direct family $\{E_\alpha\}$ is an **approximating direct family** of E.

Especially, if we have $A = \{1,\ 2,\ 3,\ \cdots\}$ and $E_1 \subset E_2 \subset \cdots \subset E_m \subset \cdots$, we say that a sequence $\{E_m\}$ **converges to** E **monotonously**. In general, assume that a sequence $\{E_m\}$ converges to E and we put $E_1 \cup E_2 \cup \cdots \cup E_m = H_m,\ (m = 1,\ 2,\ 3 \cdots)$, the sequence $\{H_m\}$ converges to E monotonously.

Assume that the set $E(\infty)$ of singular points of $f(p)$ has the measure 0. Then $E \backslash E(\infty)$ is also a measurable set.

Further assume that $f(p)$ is integrable on any compact set included in $E \backslash E(\infty)$. In order that such a condition is satisfied, a compact set included in $E \backslash E(\infty)$ and the set E_0 of singular points of $f(p)$ must not contact each other. Namely they are in a certain positive distance away from each other. Thereby we can construct an approximating direct family $\{E_\alpha;\ \alpha \in A\}$ of $E \backslash E(\infty)$ by using compact sets $E_\alpha,\ (\alpha \in A)$ included in $E \backslash E(\infty)$.

Here we say that a direct family $\{E_\alpha\}$ of compact sets converging to $E \backslash E(\infty)$ is an **approximating direct family** of $E \backslash E(\infty)$. Then, if, for an approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$, we have the limit of

$$I(E_\alpha) = \int_{E_\alpha} f(p) d\mu$$

in the sense of the Moore-Smith limit, the limit

$$I = \lim_{\alpha} I(E_\alpha)$$

is equal to the **Lebesgue type integral** of $f(p)$ on E

$$I = \int_E f(p)d\mu.$$

Here we say that this Lebesgue type integral **converges absolutely** if the value I of the Lebesgue type integral does not depend on the choice of an approximating direct family $\{E_\alpha\}$ of $E\backslash E(\infty)$.

On the other hand, we say that this Lebesgue type integral **converges conditionally** if the value I depends on the choice of an approximating direct family $\{E_\alpha\}$ of $E\backslash E(\infty)$.

Further, if the Lebesgue type integral converges absolutely or conditionally, we say that the Lebesgue type integral exists. If the Lebesgue type integral of $f(p)$ converges absolutely, we say that $f(p)$ is integrable.

If the Lebesgue type integral does not exist, we say that the Lebesgue type integral **diverges**.

A function $f(p)$ is integrable on E if and only if $|f(p)|$ is integrable on E.

Thereby the extended Lebesgue type integral called as usual is well considered to be the calculation of the Lebesgue type integral by approximating the integral domain by using an approximating direct family composed of compact sets.

The Lebesgue type integral is usually considered to be the case of the absolute convergence. We call this as the Lebesgue type integral in a narrow sense. Against this, if the Lebesgue type integral converges including the case of the conditional convergence, we call this as the extended Lebesgue type integral. We remark that the Lebesgue type integral defined in section 2 of this paper is the definition of the unified Lebesgue type integral of the Lebesgue type integral in a narrow sense and the extended Lebesgue type integral.

Remark 6.1 Assume that E is a measurable set in M and $f(p)$ is an extended real-valued measurable function defined on E.

Then there exist a direct family of simple functions $\{f_\Delta(p)\}$ which converges to $f(p)$ at each point in $E\backslash E(\infty)$ and an approximating direct family $\{E_\alpha\}$ composed of compact sets in $E\backslash E(\infty)$ such that we have the limits of (I) and (II) in the following in the sense of Moore-Smith limits:

$$\text{(I)} \quad \int_E f(p)d\mu = \lim_{\Delta} \int_E f_\Delta(p)d\mu,$$

$$\text{(II)} \quad \int_E f(p)d\mu = \lim_{\alpha} \int_{E_\alpha} f(p)d\mu.$$

Then the Lebesgue type integrals in (I) converges or diverges if and only if the Lebesgue type integral (II) converges or diverges respectively.

Further, in the case of convergence, the Lebesgue type integral in (I) converges absolutely or converges conditionally if and only if the Lebesgue type integral in (II) converges absolutely or converges conditionally respectively.

We calculate the Lebesgue type integral in (I) by using the approximation of $f(p)$ by a direct family of simple functions and the Lebesgue type integral in (II) by using the approximation of E by an approximating direct family composed of compact sets in $E \backslash E(\infty)$.

Thus, as for the calculation of the Lebesgue type integral, there is either one of the calculation by using an approximation by functions and the calculation by using an approximation by integration domains.

In the case of (I), we remark that we may use the approximation by a sequence of simple functions $\{f_m(p)\}$ instead of the approximation by a direct family $\{f_\Delta(p)\}$ of simple functions.

Here, for a function $f(p)$, we put

$$f^+(p) = \sup(f(p),\ 0),\ f^-(p) = -\inf(f(p),\ 0).$$

Then we have the formulas in the following:

$$|f(p)| \geq f^+(p) \geq 0,\ |f(p)| \geq f^-(p) \geq 0,$$

$$f(p) = f^+(p) - f^-(p),\ |f(p)| = f^+(p) + f^-(p).$$

Then we have the relations in the Table 6.1 in the following as for the convergence and divergence of the Lebesgue type integral of $f(p)$ on E.

Table 6.1 Convergence and divergence of the Lebesgue type integral

$$\left(\begin{array}{l} \text{conv. = convergence, div. = divergence,} \\ \text{ab.conv. = absolute convergenve,} \\ \text{cond.conv. = conditional convergence} \end{array}\right)$$

$\int_E f(p)d\mu$	$\int_E \lvert f(p)\rvert d\mu$	$\int_E f^+(p)d\mu$	$\int_E f^-(p)d\mu$
ab. conv.	conv.	conv.	conv.
div.	div.	conv.	div.
div.	div.	div.	conv.
cond.conv. or div.	div.	div.	div.

Remark 6.2 In the case where the Lebesgue type integral converges absolutely in Table 6.1, the value of this Lebesgue type integral is determined as a fixed value independently on the choice of an approximating direct family $\{E_\alpha;\ \alpha \in A\}$ of E. It is only in this case of absolute convergence that the Lebesgue type integral has the fixed meaning. The cases where the Lebesgue type integral $\displaystyle\int_E f(p)d\mu$ diverges to $\pm\infty$ in Table 6.1 are the cases (1) and (2) in the following:

(1) $\displaystyle\int_E f^+(p)d\mu < \infty,\ \int_E f^-(p)d\mu = \infty.$

(2) $\displaystyle\int_E f^+(p)d\mu = \infty,\ \int_E f^-(p)d\mu < \infty.$

In these cases, the Lebesgue type integrals do not exist.

Nevertheless, in these cases, for a measurable set A in E, we can define the set function $\nu(A)$ on $\mathcal{M}_E$ by the equality

$$\nu(A) = \int_A f(p)d\mu.$$

Here we assume that $\mathcal{M}_E$ is the family of all measurable sets in E.

Thereby we can define a Lebesgue-Stieltjes type measure space $(E,\ \mathcal{M}_E,\ \nu)$ on E. In the case (1), the total measure is equal to $\nu(E) = -\infty$, and, in the case (2), the total measure is equal to $\nu(E) = \infty$.

This measure space has the fixed meaning as a σ-finite measure space.

Then, even if the Lebesgue type integral of $f(p)$ on E does not exist itself, an indefinite integral of $f(p)$ on a measurable set A in E is defined by the formula

$$\nu(A) = \int_A f(p)d\mu$$

and its value is determined as a finite real value or $-\infty$ or ∞.

Against this, in the case where the Lebesgue type integral converges conditionally or diverges in Table 6.1, this integral converges or diverges according as the choice of an approximating direct family $\{E_\alpha;\ \alpha \in A\}$ of E.

Then, in the case where the Lebesgue type integral diverges, we cannot give any meaning to this.

On the other hand, in the case where the Lebesgue type integral converges conditionally, we can define its value as the mathematically meaningful value.

But, it is difficult to make a general theory in this case and it is necessary to devise a way to give its meaning according to each one of functions with singular points

Theorem 6.1 *Assume that E is a measurable set in M and a function $f(p)$ is an extended real-valued nonnegative measurable function on E. Then, if,* *for an approximating direct family* $\{E_\alpha\}$ *composed of compact sets in* $E\backslash E(\infty)$, *we have the* Moor-Smith *limit*

$$\lim_\alpha I(E_\alpha) = \lim_\alpha \int_{E_\alpha} f(p)d\mu,$$

the Lebesgue type integral of $f(p)$ *on* E *converges absolutely.*

Theorem 6.2 *Assume that* E *and* $f(p)$ *are as in* Theorem 6.1. *Then the Lebesgue type integral* $\displaystyle\int_E f(p)d\mu$ *converges if and only if, for any compact set* H *included in* $E\backslash E(\infty)$,

$$I(H) = \int_H f(p)d\mu$$

is bounded.

Theorem 6.3 *Assume that* E *is a measurable set in* M *and a function* $f(p)$ *is integrable in the sense of Lebesgue type integral on* E.

Further, we assume $f(p) \geq 0$ *and a sequence* $\{E_m\}$ *of measurable subsets of* E *satisfies the conditions* (i) *and* (ii) *in the following*:

(i) $f(p)$ *is integrable on* E_m, $(m \geq 1)$.

(ii) *We have* $\mu(E\backslash E_m) \to 0$, $(m \to \infty)$.

Then we have the equality

$$\lim_{m\to\infty} \int_{E_m} f(p)d\mu = \int_E f(p)d\mu.$$

Remark 6.3 If we have not the Lebesgue type integral of a function $f(p)$ on E in Theorem 6.3, we have

$$\int_E f(p)d\mu = \infty.$$

Then, if the conditions are the same as Theorem 6.3, we have the equality

$$\lim_{m\to\infty} \int_{E_m} f(p)d\mu = \infty = \int_E f(p)d\mu.$$

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