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Global Solvability for Mildly Degenerate Kirchhoff Type Dissipative Wave Equations in Bounded Domains

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Abstract

Consider the initial boundary value problem for degenerate dissipative wave equations of Kirchhoff type. When the wave coefficient $\rho > 0$ or the initial energy $E(0)$ is small, we show the global existence theorem.

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1 Introduction

In this paper, we study on the existence of global solutions to the initial boundary value problem for the following degenerate dissipative wave equations of Kirchhoff type :

$$\begin{cases} \rho u'' + \|A^{1/2}u(t)\|^{2\gamma} Au + u' = 0 \quad \text{in} \quad \Omega \times [0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega\,, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty)\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $L^2(\Omega)$, and $\rho > 0$ and $\gamma > 0$ are positive constants.

It is well known that Equation (1.1) describes the damped small amplitude vibrations of an elastic, stretched string when the dimension N is one or membrane when the dimension N is two (see Kirchhoff [6] and Carrier [2]).

The unique global solvability has been considered for the initial data $[u_0, u_1]$ belonging to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $\|A^{1/2}u_0\| \neq 0$. When $\gamma \geq 1$, under the assumption that the initial data $[u_0, u_1]$ are small Nishihara and Yamada [9] have shown global existence theorems.

Under the assumption that the coefficient $\rho > 0$ is small, Ghisi and Gobbino [4] have derived some decay estimates such that

$$C'(1+t)^{-\frac{1}{\gamma}} \le \|A^{m/2}u(t)\|^2 \le C(1+t)^{-\frac{1}{\gamma}} \qquad \text{for} \quad m = 1, 2$$

(see Ghisi [3] for weak dissipative cases, and Nishihara [8], Ono [11] for lower decay estimates, also [5], [7], [12] for upper decay estimates).

In this paper, we discuss to another smallness condition on the coefficient $\rho > 0$ or the initial energy $E(0)$, related to the unique global existence theorem.

We introduce an energy $E(t)$ as

$$E(t) \equiv \rho\|u'(t)\|^2 + \frac{1}{\gamma+1}M(t)^{\gamma+1} \qquad \text{with} \quad M(t) \equiv \|A^{1/2}u(t)\|^2 \,. \tag{1.2}$$

By simple calculation, we see that the energy $E(t)$ has the so-called energy identity such that

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \tag{1.3}$$

or

$$E(t) + 2\int_0^t \|u'(s)\|^2\, ds = E(0) \tag{1.4}$$

where

$$E(0) = \rho\|u_1\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u_0\|^{2(\gamma+1)} \,. \tag{1.5}$$

Our main result is as follows.

Theorem 1.1 *Let the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ *and* $\|A^{1/2}u_0\| \ne 0$. *Suppose that the coefficient* $\rho > 0$ *or the initial energy* $E(0)$ *is small in the following sense*

$$2(\gamma+1)^{\frac{2\gamma+1}{\gamma+1}} G(0)^{\frac{1}{2}} B(0)^{\frac{1}{2}} \rho E(0)^{\frac{\gamma}{\gamma+1}} < 1 \tag{1.6}$$

(*equivalent to* (3.2)) *with* $G(0)$ *and* $B(0)$ *given by* (2.3) *and* (2.8), *respectively. Then, the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); \mathcal{D}(A)) \cap C^1([0,\infty); \mathcal{D}(A^{1/2})) \cap C^2([0,\infty); L^2(\Omega))$.

Theorem 1.1 follows from Theorem 3.1 in the continuing sections.

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2(\Omega)$ or sometimes duality between the space X and its dual X'. Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

We obtain the following local existence theorem by standard arguments and we omit the proof here (see [1], [10], [13], [14], and the references cited therein).

Proposition 2.1 *Suppose that the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$ *and* $\|A^{1/2}u_0\| \neq 0$. *Then the problem* (1.1) *admits a unique local solution* $u(t)$ *in the class* $C^0([0,T);\mathcal{D}(A))\cap C^1([0,T);\mathcal{D}(A^{1/2}))\cap C^2([0,T);L^2(\Omega))$ *for some* $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$.

Moreover, if $\|A^{1/2}u(t)\| \neq 0$ *and* $\|Au(t)\| + \|A^{1/2}u'(t)\| < \infty$ *for* $t \geq 0$, *then we can take that* $T = \infty$.

In what follows in this section, we assume that $M(0) > 0$ and the function $u = u(t)$ is a solution of (1.1) and satisfies

$$\rho\frac{|M'(t)|}{M(t)} \leq \frac{1}{\gamma+1}\,. \tag{2.1}$$

Proposition 2.2 *Under the assumption* (2.1), *it holds that*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0) \tag{2.2}$$

where

$$G(t) \equiv \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\,, \tag{2.3}$$

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+1}}\left(M(t)\|A^{1/2}u'(t)\|^2 - \frac{1}{4}|M'(t)|^2\right)\,. \tag{2.4}$$

Proof. From Equation (1.1), we observe

$$\begin{aligned}
&\frac{d}{dt}\frac{\|Au(t)\|^2}{M(t)}\\
&= \frac{1}{M(t)^{\gamma+2}}\big(2(M(t)^\gamma Au(t), Au'(t))M(t) - (M(t)^\gamma Au(t), Au(t))M'(t)\big)\\
&= \frac{-1}{M(t)^{\gamma+1}}\Bigg(2\left(\|A^{1/2}u'(t)\|^2 + \rho(A^{1/2}u''(t), A^{1/2}u'(t))\right)M(t)\\
&\qquad - \left(\frac{1}{2}M'(t) + \rho\left(\frac{1}{2}M''(t) - \|A^{1/2}u'(t)\|^2\right)\right)M'(t)\Bigg)\\
&= -2Q(t) - \rho R(t)
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
R(t) \equiv &\frac{1}{M(t)^{\gamma+2}}\Bigg(2M(t)(A^{1/2}u''(t), A^{1/2}u'(t))\\
&+ M'(t)\left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)\Bigg).
\end{aligned}$$

Moreover, we observe

$$\frac{d}{dt}Q(t) = -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) + R(t) \quad \text{and} \quad Q(t) \geq 0\,. \tag{2.6}$$

From (2.1), (2.5), and (2.6), we have

$$\frac{d}{dt}G(t) + 2\left(1 + \frac{\gamma+2}{2}\rho\frac{M'(t)}{M(t)}\right)Q(t) \leq 0\,,$$

and hence, we obtain the desired estimate (2.2). □

Proposition 2.3 *Under the assumption* (2.1)*, it holds that*

$$\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \leq B(0) \tag{2.7}$$

where

$$B(0) \equiv \max\left\{\frac{\|u_1\|^2}{M(0)^{2\gamma+1}}\,,\ (2(\gamma+1))^2G(0)\right\}\,. \tag{2.8}$$

Proof. Multiplying (1.1) by $2M(t)^{-2\gamma-1}u'$ and integrating it over Ω, we have from the Young inequality that

$$\begin{aligned}\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + 2\left(1+\frac{2\gamma+1}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} &= -\frac{M'(t)}{M(t)^{\gamma+1}} \\ \leq \frac{1}{2(\gamma+1)}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + 2(\gamma+1)\frac{\|Au(t)\|^2}{M(t)}\,. &\end{aligned} \tag{2.9}$$

Since it follows from (2.1) that

$$1 + \frac{2\gamma+1}{2}\rho\frac{M'(t)}{M(t)} \geq \frac{1}{2(\gamma+1)}\,, \tag{2.10}$$

we observe from (2.2) and (2.9) that

$$\begin{aligned}\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{2(\gamma+1)}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} &\leq 2(\gamma+1)\frac{\|Au(t)\|^2}{M(t)} \\ &\leq 2(\gamma+1)G(0)\,.\end{aligned}$$

Thus, by standard calculation for ODE, we obtain the desired estimate (2.7). □

Proposition 2.4 *Under the assumption* (2.1)*, it holds that*

$$M(t) \geq C'(1+t)^{-\frac{1}{\gamma}} \tag{2.11}$$

with some positive constant C'.

Proof. Multiplying (1.1) by $2M(t)^{-2\gamma-1}u'$ and integrating it over Ω, we have from the Young inequality that

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\right)+2\left(1+\frac{2\gamma+1}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}$$
$$=-(\gamma+1)\frac{M'(t)}{M(t)^{\gamma+1}}$$
$$\le\frac{1}{\gamma+1}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}+(\gamma+1)^3\frac{\|Au(t)\|^2}{M(t)}\,,$$

and from (2.10) that

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\right)\le(\gamma+1)^3\frac{\|Au(t)\|^2}{M(t)}\le(\gamma+1)^3G(0)$$

Thus, immediately we obtain

$$\rho\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\le C(1+t)\qquad\text{or}\qquad M(t)^\gamma\ge C^{-1}(1+t)^{-1}$$

which implies the desired estimate (2.11). □

3 Global Solvability

We introduce the function $H(t)$ (a second order energy) as

$$H(t)\equiv\rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma}+\|Au(t)\|^2\,. \tag{3.1}$$

Theorem 3.1 *Let the initial data* $[u_0,u_1]$ *belong to* $\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$ *and* $M(0)>0$*. Suppose that*

$$2\rho B(0)^{\frac12}G(0)^{\frac12}\left((\gamma+1)E(0)\right)^{\frac{\gamma}{\gamma+1}}<\frac{1}{\gamma+1}\,. \tag{3.2}$$

Then, the problem (1.1) *admits a unique global solution* $u(t)$ *in the class*

$$C^0([0,\infty);\mathcal{D}(A))\cap C^1([0,\infty);\mathcal{D}(A^{1/2}))\cap C^2([0,\infty);L^2(\Omega))$$

and this solution $u(t)$ *satisfies*

$$\rho\frac{|M'(t)|}{M(t)}<\frac{1}{\gamma+1}\quad\textit{and}\quad H(t)\le H(0)\,, \tag{3.3}$$

$$\frac{\|Au(t)\|^2}{M(t)}\le G(0)\quad\textit{and}\quad\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}\le B(0)\,, \tag{3.4}$$

$$C'(1+t)^{-\frac{1}{\gamma}}\le M(t)\le\left((\gamma+1)E(0)\right)^{\frac{1}{\gamma+1}}\quad\textit{for}\quad t\ge0 \tag{3.5}$$

with $E(0)$, $G(0)$, *and* $B(0)$ *given by* (1.5), (2.3), *and* (2.8), *respectively, where* C' *is some positive constant.*

Proof. Let $u(t)$ be a solution on $[0,T]$. Since $M(0)>0$, putting

$$T_1 \equiv \sup\left\{t \in [0,\infty) \;\middle|\; M(s)>0 \text{ for } 0 \le s < t\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$M(t) > 0 \quad \text{for} \quad 0 \le t < T_1 \quad \text{and} \quad M(T_1) = 0\,. \tag{3.6}$$

Since it follows from (1.4) and (3.2) that

$$\begin{aligned}\rho\frac{|M'(0)|}{M(0)} &\le 2\rho\left(\frac{\|u_1\|^2}{M(0)^{2\gamma+1}}\right)^{\frac{1}{2}}\left(\frac{\|Au_0\|^2}{M(0)}\right)^{\frac{1}{2}} M(0)^{\gamma} \\ &\le 2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}}\left((\gamma+1)E(0)\right)^{\frac{1}{\gamma+1}} < \frac{1}{\gamma+1}\,,\end{aligned}$$

putting

$$T_2 \equiv \sup\left\{t \in [0,\infty) \;\middle|\; \rho\frac{|M'(s)|}{M(s)} < \frac{1}{\gamma+1} \text{ for } 0 \le s < t\right\},$$

we see that $T_2 > 0$. If $T_2 < T_1$, then

$$\rho\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \text{for} \quad 0 \le t < T_2 \quad \text{and} \quad \rho\frac{|M'(T_2)|}{M(T_2)} = \frac{1}{\gamma+1}\,. \tag{3.7}$$

From Proposition 2.2 and Proposition 2.3 we observe

$$\begin{aligned}\rho\frac{|M'(t)|}{M(t)} &\le 2\rho\left(\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}}\right)^{\frac{1}{2}}\left(\frac{\|Au(t)\|^2}{M(t)}\right)^{\frac{1}{2}} M(t)^{\gamma} \\ &\le 2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}}\left((\gamma+1)E(0)\right)^{\frac{1}{\gamma+1}} < \frac{1}{\gamma+1}\end{aligned} \tag{3.8}$$

for $0 \le t \le T_2$, which is a contradiction to (3.7), and hence, we have that $T_2 \ge T_1$. Moreover, from Proposition 2.4 we observe

$$M(t) \ge C'(1+t)^{-\frac{1}{\gamma}} > 0 \quad \text{for} \quad 0 \le t \le T_1\,,$$

which is a contradiction to (3.6), and hence, we have that $T_1 \ge T$.

Multiplying (1.1) by $2M(t)^{-\gamma}Au'$ and integrating it over Ω we have

$$\frac{d}{dt}H(t) + 2\left(1+\frac{\gamma}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)^{\gamma}} = 0\,.$$

Since it follows from (3.8) that

$$1+\frac{\gamma}{2}\rho\frac{M'(t)}{M(t)} \ge 0\,,$$

we observe

$$\frac{d}{dt}H(t) \leq 0 \quad \text{and} \quad H(t) \leq H(0) \tag{3.9}$$

for $0 \leq t \leq T$. Thus, from above argument we see that $M(t) > 0$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| \leq C$ for $t \geq 0$. Therefore, by the second argument of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from Propositions 2.2–2.4, we obtain the desired estimates (3.3)–(3.5). □

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