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Rational Values of Powers of Trigonometric Functions

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Abstract

We extend the theorem by Olmsted (1945) and Carlitz-Thomas (1963) on rational values of trigonometric functions to powers of trigonometric functions.

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1 Introduction

Throughout the paper, we denote the ring of rational numbers by $\mathbb{Q}$, the ring of real numbers by $\mathbb{R}$, the set of positive rational numbers by $\mathbb{Q}_{>0}$ and a mth root of unity by $\zeta_m := e^{\frac{2\pi\sqrt{-1}}{m}}$. Olmsted [4] and Carlitz-Thomas [2] determined all rational values of trigonometric functions.

Theorem 1 (Olmsted (1945), Carlitz-Thomas (1963))**.** *If $\theta \in \mathbb{Q}$, then the only possible rational values of the trigonometric functions are:*

$$\sin(\pi\theta),\ \cos(\pi\theta) = 0,\ \pm\frac{1}{2},\ \pm 1\ ;\ \tan(\pi\theta) = 0,\ \pm 1.$$

By this Theorem 1 and well-known facts

$$\cos(\pi\theta)^2 = \frac{1+\cos(2\pi\theta)}{2}, \quad \tan(\pi\theta)^2 = \frac{1}{\cos(\pi\theta)^2} - 1,$$

we have the following result immediately.

Corollary 2. *If $\theta \in \mathbb{Q}$ and $\cos(\pi\theta)^2 \in \mathbb{Q}$, then the only possible values of the trigonometric functions are:*

$$\sin(\pi\theta),\ \cos(\pi\theta) = 0,\ \pm\frac{1}{2},\ \pm\frac{1}{\sqrt{2}},\ \pm\frac{\sqrt{3}}{2},\ \pm 1\ ;\ \tan(\pi\theta) = 0,\ \pm\frac{1}{\sqrt{3}},\ \pm 1,\ \pm\sqrt{3}.$$

In this note, we prove Theorem 3 and Theorem 4, which are generalizations of Theorem 1 and Corollary 2.

Theorem 3. *If* $N \geq 3$ *and* α *is a positive rational number such that* $\alpha^{\frac{1}{N}}, \ldots, \alpha^{\frac{N-1}{N}} \notin \mathbb{Q}$, *then for any positive integer* m, *we have*

$$\sqrt[N]{\alpha} \notin \mathbb{Q}(\zeta_m).$$

In particular, there is no $\theta \in \mathbb{Q}$ *such that* $\cos(\pi\theta), \cos(\pi\theta)^2, \ldots, \cos(\pi\theta)^{N-1} \notin \mathbb{Q}$ *and* $\cos(\pi\theta)^N \in \mathbb{Q}$ *(resp.* $\tan(\pi\theta), \tan(\pi\theta)^2, \ldots, \tan(\pi\theta)^{N-1} \notin \mathbb{Q}$ *and* $\tan(\pi\theta)^N \in \mathbb{Q}$*).*

Theorem 4. *If there exists a positive integer* n *and* $\theta \in \mathbb{Q}$ *such that* $\cos(\pi\theta)^n \in \mathbb{Q}$ *(resp.* $\tan(\pi\theta)^n \in \mathbb{Q}$*), then the only possible values of the trigonometric functions are:*

$$\sin(\pi\theta), \cos(\pi\theta) = \begin{cases} 0, \ \pm\frac{1}{2}, \ \pm 1 & (n : \textit{odd}) \\ 0, \ \pm\frac{1}{\sqrt{2}}, \ \pm\frac{1}{2}, \ \pm\frac{\sqrt{3}}{2}, \ \pm 1 & (n : \textit{even}) \end{cases}. \tag{1}$$

resp.

$$\tan(\pi\theta) = \begin{cases} 0, \ \pm 1 & (n : \textit{odd}) \\ 0, \ \pm\frac{1}{\sqrt{3}}, \ \pm 1, \ \pm\sqrt{3} & (n : \textit{even}) \end{cases}. \tag{2}$$

We note that one can easily verify the above theorems using the fundamental properties of Galois theory as in the following sections. But we could find no proof of these facts in print, and hence it will be of some interest to write down the proofs of these facts.

2 Preliminaries

To prove Theorem 3 and Theorem 4, we list some fundamental facts of the cyclotomic fields and Kummer extension in this section. First we mention the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ (see [3]).

Lemma 5. *The degree of the cyclotomic extension* $\mathbb{Q}(\zeta_n)$ *over* $\mathbb{Q}$ *is* $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) := |\{1 \leq a \leq n \mid \gcd(a, n) = 1\}|$ *and its Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ *is*

$$\begin{array}{ccccccc} (\mathbb{Z}/n\mathbb{Z})^{\times} & \simeq & \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) & \curvearrowright & \mathbb{Q}(\zeta_n) & \to & \mathbb{Q}(\zeta_n) \\ \cup & & \cup & & \cup & & \cup \\ c & \mapsto & \tau_c & \curvearrowright & \zeta_n & \mapsto & \tau_c(\zeta_n) := \zeta_n^c \end{array}.$$

In particular $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ *is an abelian extension, and its subfields* $L \supset \mathbb{Q}$ *are Galois and abelian extensions over* $\mathbb{Q}$.

Under the following, let α be a positive rational number such that

$$\alpha^{\frac{1}{n}}, \dots, \alpha^{\frac{n-1}{n}} \notin \mathbb{Q} \tag{3}$$

and $K := \mathbb{Q}(\sqrt[n]{\alpha}, \zeta_n)$.

Proposition 6. (1) *The binomial polynomial* $x^n - \alpha$ *is irreducible over* $\mathbb{Q}$ *and* $[\mathbb{Q}(\sqrt[n]{\alpha}) : \mathbb{Q}] = n$.
(2) *For any* $n \geq 2$, *we have* $\sqrt[n]{\alpha} \notin \mathbb{Q}(\zeta_n)$.

Proof. (1) We consider the following decomposition of the binomial polynomial:

$$x^n - \alpha = \prod_{i=1}^{n} (x - \sqrt[n]{\alpha}\zeta_n^i) = f_I(x) f_J(x) \tag{4}$$

where I, J are subsets of $[n] := \{1, 2, \dots, n\}$ such that

$$[n] = I \sqcup J, \quad I \neq \emptyset, \quad J \neq \emptyset, \quad I \cap J = \emptyset \tag{5}$$

and

$$f_I(x) := \prod_{i \in I} (x - \sqrt[n]{\alpha}\zeta_n^i) = x^{|I|} + \cdots + (-1)^{|I|} \alpha^{\frac{|I|}{n}} \prod_{i \in I} \zeta_n^i \in \mathbb{Q}[x],$$
$$f_J(x) := \prod_{j \in J} (x - \sqrt[n]{\alpha}\zeta_n^j) = x^{|J|} + \cdots + (-1)^{m} \alpha^{\frac{|J|}{n}} \prod_{j \in J} \zeta_n^j \in \mathbb{Q}[x].$$

If $x^n - \alpha$ is reducible over $\mathbb{Q}$, then there exist $I, J \subset [n]$ such that satisfy the condition (5) and $f_I(x), f_J(x)$ are rational coefficient polynomials. Since the constant term of $f_J(x)$ is real, the product $\prod_{j \in J} \zeta_n^j$ is real and $\left|\prod_{j \in J} \zeta_n^j\right| = 1$. Thus we have

$$\prod_{j \in J} \zeta_n^j = \pm 1.$$

From the assumption $\alpha^{\frac{|J|}{n}} \notin \mathbb{Q}$, the constant term of $f_J(x)$

$$(-1)^m \alpha^{\frac{m}{n}} \prod_{j \in J} \zeta_n^j = \pm(-1)^m \alpha^{\frac{m}{n}}$$

is irrational. It is a contradiction.
(2) Assume $\sqrt[n]{\alpha} \in \mathbb{Q}(\zeta_n)$. Then we have the contradiction

$$n = [\mathbb{Q}(\sqrt[n]{\alpha}) : \mathbb{Q}] \leq [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) < n.$$

□

Lemma 7. (1) *If* $n = p$ *is a odd prime, then the binomial type polynomial* $x^p - \alpha$ *is irreducible over* $\mathbb{Q}(\zeta_p)$ *and* $[K : \mathbb{Q}(\zeta_p)] = p$*. Its Galois group* $\mathrm{Gal}(K/\mathbb{Q})$ *is*

$$\begin{array}{ccccccccc}
\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^{\times} & \simeq & \mathrm{Gal}(K/\mathbb{Q}) & \curvearrowright & K & \to & K \\
\cup & & \cup & & \cup & & \cup \\
(1,1) & \mapsto & \sigma & \curvearrowright & \sqrt[p]{\alpha} & \mapsto & \sigma(\sqrt[p]{\alpha}) := \zeta_p \sqrt[p]{\alpha} \\
 & & & & \zeta_n & \mapsto & \sigma(\zeta_p) := \zeta_p \\
(0,c) & \mapsto & \tau_c & \curvearrowright & \sqrt[p]{\alpha} & \mapsto & \tau_c(\sqrt[p]{\alpha}) := \sqrt[p]{\alpha} \\
 & & & & \zeta_p & \mapsto & \tau_c(\zeta_p) := \zeta_p^c
\end{array}.$$

In particular, $\tau_c\sigma = \sigma^c\tau_c$ *and for* $n \geq 3$ *the Galois group* $\mathrm{Gal}(K/\mathbb{Q})$ *is non-abelian.*
(2) *For any* $n \geq 3$*, the Galois group* $\mathrm{Gal}(K/\mathbb{Q})$ *is non-abelian.*

Proof. (1) We consider the factorization (4) $x^p - \alpha = f_I(x)f_J(x)$ again. By $\gcd(|I|, p) = 1$ and $\gcd(|J|, p) = 1$, $\mathbb{Q}(\alpha^{\frac{|I|}{p}})$ and $\mathbb{Q}(\alpha^{\frac{|J|}{p}})$ contain $\sqrt[p]{\alpha}$. Hence, from Proposition 6 (2), $\alpha^{\frac{|I|}{p}}$ and $\alpha^{\frac{|J|}{p}}$ are not contained in $\mathbb{Q}(\zeta_p)$. Therefore the binomial polynomial $x^p - \alpha$ is irreducible over $\mathbb{Q}(\zeta_p)$ and $[K : \mathbb{Q}(\zeta_p)] = p$.
(2) When a odd prime p divides n, K contains a non-abelian Galois extension $\mathbb{Q}(\sqrt[p]{\alpha}, \zeta_p)$ over $\mathbb{Q}$, so the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is non-abelian. If $n = 2^m$ $(m \geq 2)$, then K contains a non-abelian Galois extension $\mathbb{Q}(\sqrt[4]{\alpha}, \zeta_4)$ over $\mathbb{Q}$ and $\mathrm{Gal}(K/\mathbb{Q})$ is also non-abelian. □

Remark 8. Lemma 7 (1) is not true in general. For example, when $n = 8$ and $\alpha = 2$ the polynomial $x^8 - 2$ is reducible over $\mathbb{Q}(\zeta_8)$ even though $2^{\frac{1}{8}}, \ldots, 2^{\frac{7}{8}}$ are irrational. In fact

$$x^8 - 2 = (x^4 - \sqrt{2})(x^4 + \sqrt{2}) = (x^4 - \zeta_8 - \zeta_8^{-1})(x^4 + \zeta_8 + \zeta_8^{-1}).$$

3 Proof of Theorem 3

Assume there exists $N \geq 3$, $\alpha \in \mathbb{Q}_{>0}$ and a positive integer m such that $\alpha^{\frac{1}{N}}, \ldots, \alpha^{\frac{N-1}{N}} \notin \mathbb{Q}$ and

$$\sqrt[N]{\alpha} \in \mathbb{Q}(\zeta_m).$$

Then

$$\mathbb{Q}(\sqrt[N]{\alpha}) \subset \mathbb{Q}(\zeta_m).$$

Although $\mathbb{Q}(\sqrt[N]{\alpha})$ is not a Galois extension over $\mathbb{Q}$, K is a Galois extension over $\mathbb{Q}$ and

$$K \subset \mathbb{Q}(\zeta_m, \zeta_N) \subset \mathbb{Q}(\zeta_{mN}).$$

Further the field K is a subfield of $\mathbb{Q}(\zeta_{mN})$ and the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is a normal subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_{mN})/\mathbb{Q})$:

$$\mathrm{Gal}(K/\mathbb{Q}) \lhd \mathrm{Gal}(\mathbb{Q}(\zeta_{mN})/\mathbb{Q}) \simeq (\mathbb{Z}/mN\mathbb{Z})^{\times}.$$

However the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is non-abelian. It is a contradiction. Then for any positive integer m,

$$\sqrt[N]{\alpha} \notin \mathbb{Q}(\zeta_m). \tag{6}$$

For the above $N \geq 3$ and positive rational number $\alpha \in \mathbb{Q}_{>0}$, assume there exists $\theta \in \mathbb{Q}$ such that

$$\cos(\pi\theta) = \sqrt[N]{\alpha}.$$

By $\theta \in \mathbb{Q}$, there exists a positive integer m such that

$$\sqrt[N]{\alpha} = \cos(\pi\theta) \in \mathbb{Q}(\zeta_m).$$

For $N \geq 3$, it is contrary to (6). For $\tan(\pi\theta)$, one can prove similarly.

Remark 9. From Theorem 3, there is no cyclotomic field over $\mathbb{Q}$ containing nth root of a positive rational number α with $\sqrt[n]{\alpha} \notin \mathbb{Q}$, for any $n \geq 3$. On the other hand, from Gauss sum's formulas [1]

$$\sum_{k=0}^{m-1} \zeta_m^{k^2} = \frac{1+\sqrt{-1}}{2}(1+(-\sqrt{-1})^m)\sqrt{m} = \begin{cases} (1+\sqrt{-1})\sqrt{m} & (m \equiv 0 \bmod 4) \\ \sqrt{m} & (m \equiv 1 \bmod 4) \\ 0 & (m \equiv 2 \bmod 4) \\ \sqrt{-1}\sqrt{m} & (m \equiv 3 \bmod 4) \end{cases},$$

$$\zeta_4 = \sqrt{-1}, \quad \zeta_8 + \zeta_8^{-1} = \sqrt{2},$$

there exists a positive integer m such that $\sqrt{\alpha} \in \mathbb{Q}(\zeta_m)$, for any $\alpha \in \mathbb{Q}$.

4 Proof of Theorem 4

Since the proof of (2) is similar to (1), we only prove (1). The cases of $n = 1$ and $n = 2$ are Theorem 1 and Corollary 2 respectively. For $n \geq 3$, the following three cases are possible:

1) $\cos(\pi\theta) \in \mathbb{Q}$ and $\cos(\pi\theta)^2 \in \mathbb{Q}$,
2) $\cos(\pi\theta) \notin \mathbb{Q}$ and $\cos(\pi\theta)^2 \in \mathbb{Q}$,
3) $\cos(\pi\theta) \notin \mathbb{Q}$ and $\cos(\pi\theta)^2 \notin \mathbb{Q}$.

In the case of 1), from Theorem 1 and Corollary 2, the possible values of $\cos(\pi\theta)$ (or $\sin(\pi\theta)$) are 0, $\pm\frac{1}{2}$, ± 1. Similarly, for the case of 2), the possible values of $\cos(\pi\theta)$ are $\pm\frac{1}{\sqrt{2}}$, $\pm\frac{\sqrt{3}}{2}$. Finally, the case of 3) is impossible from Theorem 3 (1). Then we obtain the conclusion (1).

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