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Lower Decay Estimates for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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Abstract

We consider the Cauchy problem for non-degenerate Kirchhoff type dissipative wave equations $\rho u'' + a\left(\|A^{1/2}u(t)\|^2\right)Au + u' = 0$ and $(u(0), u'(0)) = (u_0, u_1)$, where $u_0 \neq 0$. We derive the lower decay estimate $\|u(t)\|^2 \geq Ce^{-\beta t}$ for $t \geq 0$ with $\beta > 0$ for the solution $u(t)$.

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1 Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let A be a linear operator on H with dense domain $\mathcal{D}(A)$. We assume that the operator A is self-adjoint and nonnegative such that $(Av, v) \geq 0$ for $v \in \mathcal{D}(A)$. The α-th power of A with dense domain $\mathcal{D}(A^\alpha)$ is denoted by A^α for $\alpha > 0$, and the graph-norm of A^α is denoted by $\|v\|_\alpha = \left(\|v\|^2 + \|A^\alpha v\|^2\right)^{\frac{1}{2}}$ for $v \in \mathcal{D}(A^\alpha)$. We use that $\|A^{1/2}v\|^2 = (Av, v)$ for $v \in \mathcal{D}(A^{1/2})$.

We study on the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a\left(\|A^{1/2}u(t)\|^2\right)Au + u' = 0\,, \quad t \geq 0 \\ (u(0), u'(0)) = (u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})\,, \end{cases} \tag{1.1}$$

where $u = u(t)$ is an unknown real value function, $' = d/dt$, and ρ is a positive constant.

For the non-local nonlinear term $a(M) \in C^0([0,\infty)) \cap C^1((0,\infty))$, we assume that as follows :

Hyp.1 $\quad K_1 \le a(M) \le K_2 + K_3 M^\gamma \quad$ for $M \ge 0$

Hyp.2 $\quad 0 \le a'(M)M \le K_4 a(M) \quad$ for $M > 0$

with $\gamma > 0$ and $K_j > 0$ $(j = 1, 2, 3, 4)$.

From Hyp.1, we see that

$$K_1 M \le \int_0^M a(\mu)\,d\mu \le \left(K_2 + \frac{K_3}{\gamma+1} M^\gamma\right) M\,. \tag{1.2}$$

For typical examples, we have that

$$a(M) = 1 + M^\gamma\,, \quad (1+M)^\gamma\,, \quad \log(2 + M^\gamma)\,.$$

In the case of one dimension, (1.1) describes small amplitude vibrations of an elastic string (see [3], [4], [6]).

We obtain the following global existence theorem (see Theorem 4.1 and Proposition 5.1).

Theorem 1.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data* (u_0, u_1) *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ *and satisfy* $u_0 \ne 0$, *and moreover, the coefficient* ρ *and the initial data* (u_0, u_1) *satisfy the smallness condition* (4.1), *then the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class*

$$C^0([0,\infty); \mathcal{D}(A)) \cap C^1([0,\infty); \mathcal{D}(A^{1/2})) \cap C^2([0,\infty); H)\,.$$

Moreover, the solution $u(t)$ *satisfies*

$$\|A^{1/2}u(t)\|^2 \ge Ce^{-\alpha t} \qquad \textit{for} \quad t \ge 0 \tag{1.3}$$

with some $\alpha > 0$.

In previous paper [10], we have derived the upper decay estimates of the solution $u(t)$ of (1.1) in the case of $a(M) = (1+M)^\gamma$ with $\gamma > 0$ and $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N)$:

$$\begin{aligned} &\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1}\,, \quad \|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-2}\,, \\ &\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-3} \qquad \text{for} \quad t \ge 0 \end{aligned}$$

(see [5], [8] for $a(M) = 1 + M^\gamma$ with $\gamma \ge 1$, that is, $a(\cdot) \in C^1([0,\infty))$).

On the other hand, Ghisi and Gobbino [5] have derived the lower decay estimate (1.3) for (1.1) (see [9] for bounded domains).

The purpose of this paper is to derive the lower decay estimate for $\|u(t)\|^2$.

For the non-local nonlinear term $a(M) \in C([0,\infty)) \cap C^2((0,\infty))$, we assume that as follows :

Hyp.3 $\quad |a''(M)|M^2 \le K_5 a(M) \quad$ for $M > 0$

with $K_5 > 0$.

We obtain the following lower decay estimate of the solution $u(t)$ of (1.1) (see Theorem 5.4). Our main result is as follows.

Theorem 1.2 *Suppose that the assumption of Theorem 1.1 and Hyp.3 are fulfilled. Then, the solution $u(t)$ satisfies*

$$\|u(t)\|^2 \ge Ce^{-\beta t} \qquad \text{for} \quad t \ge 0 \tag{1.4}$$

with some $\beta > 0$.

The notations we use in this paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Local Existence and Energy

We have the following local existence theorem by standard arguments (see [1], [2], [7], [11] and the references cited therein).

Proposition 2.1 *Suppose that Hyp.1 and Hyp.2 are satisfied. If the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, then the problem (1.1) admits a unique local solution $u(t)$ in the class $C^0([0,T);\mathcal{D}(A)) \cap C^1([0,T);\mathcal{D}(A^{1/2})) \cap C^2([0,T); H)$ for some $T = T(\|u_0\|_2, \|u_1\|_1) > 0$.*

Moreover, $\|u(t)\|_2 + \|u'(t)\|_1 < \infty$ for $t \ge 0$, then we can take $T = \infty$.

In what follows, let $u(t)$ be a solution of (1.1) under the assumption of Proposition 2.1.

We set that

$$M(t) = \|A^{1/2}u(t)\|^2 \tag{2.1}$$

and

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu)\, d\mu \tag{2.2}$$

for simplicity of the notations.

Proposition 2.2 *Under the assumption of Proposition 2.1, the solution $u(t)$ of* (1.1) *satisfies that*

$$E(t)+2\int_0^t \|u'(s)\|^2 ds = E(0)\,, \tag{2.3}$$

$$M(t) \le K_1^{-1}E(0)\,, \tag{2.4}$$

$$a(M(t)) \le K_2 + K_3(K_1^{-1}E(0))^\gamma \quad (\equiv I(0)\,)\,, \tag{2.5}$$

$$\|u(t)\|^2 \le 6(\|u_0\|^2 + \rho E(0))\,. \tag{2.6}$$

for $t \ge 0$.

Proof. Taking the inner product of (1.1) with $2u'(t)$, we have

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0\,, \tag{2.7}$$

and integrating (2.7) in time t, we obtain (2.3).

Moreover, it follows from (5.1) and (2.2) that

$$K_1 M(t) \le E(t) \le E(0)\,,$$

and from Hyp.2 that

$$a(M(t)) \le K_2 + K_3 M(t)^\gamma \le I(0)\,.$$

Taking the inner product of (1.1) with $u(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + a(M(t))M(t) = \rho\left(\|u'(t)\|^2 - \frac{d}{dt}(u'(t),u(t))\right)\,,$$

and we observe from the Young inequality that

$$\begin{aligned}&\|u(t)\|^2 + 2\int_0^t a(M(s))M(s)\,ds \\ &\le \|u_0\|^2 + 2\rho\int_0^t \|u'(s)\|^2 ds + \|u_0\|^2 + \rho\|u_1\|^2 + \frac{1}{2}\|u(t)\|^2 + 2\rho^2\|u'(t)\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\|u(t)\|^2 + 2\int_0^t a(M(s))M(s)\,ds \\ &\le 2\|u_0\|^2 + \rho\left(\rho\|u_1\|^2 + 2\rho\|u'(t)\|^2 + 2\int_0^t \|u'(s)\|^2 ds\right) \\ &\le 2\|u_0\|^2 + 3\rho E(0)\end{aligned}$$

which implies the desired estimate (2.6). □

3 Several Estimates

In order to obtain a-priori estimates of the solution $u(t)$, we assume that

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{K_4 + 1} \tag{3.1}$$

where $M(t)$ is defined by (5.1).

Proposition 3.1 *Under the assumption* (3.1), *the solution* $u(t)$ *satisfies*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0)\,, \tag{3.2}$$

where

$$G(t) = \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\,, \tag{3.3}$$

$$Q(t) = \frac{\|A^{1/2}u'(t)\|^2 \|A^{1/2}u(t)\|^2 - |(A^{1/2}u'(t), A^{1/2}u(t))|^2}{a(M(t))M(t)^2} \quad (\geq 0)\,. \tag{3.4}$$

Proof. We have from (1.1) that

$$\begin{aligned}
&\frac{d}{dt}\frac{\|Au(t)\|^2}{M(t)} \\
&= \frac{1}{a(M(t))M(t)^2}\left(2(a(M(t))Au, Au')M(t) - (a(M(t))Au, Au)M'(t)\right) \\
&= \frac{1}{a(M(t))M(t)^2}\Big(2(\|A^{1/2}u'\|^2 + \rho(A^{1/2}u'', A^{1/2}u'))M(t) \\
&\quad - \left(\tfrac{1}{2}|M'(t)|^2 + \rho\left(\|A^{1/2}u'(t)\|^2 - \tfrac{1}{2}M''(t)\right)M'(t)\right)\Big) \\
&= -2Q(t) + \rho R(t)
\end{aligned}$$

where we set

$$R(t) = \frac{2(A^{1/2}u'', A^{1/2}u')M(t) + \left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)}{a(M(t))M(t)^2}\,.$$

Since we observe

$$\begin{aligned}
&\frac{d}{dt}Q(t) \\
&= -\frac{a'(M(t))M'(t)M(t)^2 + 2a(M(t))M(t)M'(t)}{(a(M(t))M(t)^2)^2} \\
&\quad \times \left(\|A^{1/2}u'\|^2 M(t) - \frac{1}{4}|M'(t)|^2\right) \\
&\quad + \frac{2(A^{1/2}u'', A^{1/2}u')M(t) + \|A^{1/2}u'\|^2 M'(t) - \frac{1}{2}M'(t)M''(t)}{a(M(t))M(t)^2} \\
&= -\frac{M'(t)}{M(t)}\frac{a'(M(t))M(t) + 2a(M(t))}{a(M(t))^2 M(t)^2}\left(\|A^{1/2}u'\|^2 M(t) - \frac{1}{4}|M'(t)|^2\right) + R(t) \\
&= -\frac{M'(t)}{M(t)}\left(2 + \frac{a'(M(t))M(t)}{a(M(t))}\right)Q(t) + R(t)\,,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\right) \\
&\quad + 2\left(1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\left(2 + \frac{a'(M(t))M(t)}{a(M(t))}\right)\right)Q(t) = 0\,.
\end{aligned}$$

Moreover, we observe

$$1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\left(2 + \frac{a'(M(t))M(t)}{a(M(t))}\right) \geq 1 - \frac{1}{2}\frac{1}{K_4 + 1}(2 + K_4) \geq 0$$

and $Q(t) \geq 0$, we have

$$\frac{d}{dt}G(t) = \frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\right) \leq 0$$

which implies the desired estimate (3.2). □

Proposition 3.2 *Under the assumption* (3.1), *the solution* $u(t)$ *satisfies*

$$\frac{\|u'(t)\|^2}{M(t)} \leq B(0)\,, \tag{3.5}$$

where

$$B(0) = \max\left\{\frac{\|u_1\|^2}{M(0)}\,,\ \frac{K_4 + 1}{K_4}I(0)^2 G(0)\right\}\,. \tag{3.6}$$

Proof. Taking the inner product of (1.1) with $2u'(t)/M(t)$, we have

$$\begin{aligned}\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)} + \left(2+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)} &= -a(M(t))\frac{M'(t)}{M(t)} \\ &\le 2a(M(t))\frac{\|Au(t)\|\|u'(t)\|}{M(t)} \\ &\le a(M(t))^2\frac{\|Au(t)\|^2}{M(t)} + \frac{\|u'(t)\|^2}{M(t)}\end{aligned} \tag{3.7}$$

where we used the Young inequality.

Since

$$1+\rho\frac{M'(t)}{M(t)} \ge \frac{K_4}{K_4+1} \quad \text{and} \quad a(M(t))^2\frac{\|Au(t)\|^2}{M(t)} \le I(0)^2G(0)\,,$$

we have

$$\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)} + \frac{K_4}{K_4+1}\frac{\|u'(t)\|^2}{M(t)} \le I(0)^2G(0)$$

and hence, we obtain (3.6). □

Remark. If the nonnegative function $f(t)$ satisfies

$$f'(t) + af(t) \le b\,, \quad t \ge 0$$

with positive constants a and b, then

$$f(t) \le \max\{f(0)\,,\, b/a\}\,, \quad t \ge 0\,.$$

Indeed, taking

$$g(t) = \max\{f(0)\,,\, b/a\}\,, \quad t \ge 0\,,$$

we see that $-ag(t) + b \le 0$ and $g'(t) = 0$, and hence,

$$g'(t) + ag(t) \ge b \quad \text{and} \quad f(0) \le g(0)\,.$$

Thus, by the comparison principle, we conclude.

4 Global Existence

Theorem 4.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data* (u_0, u_1) *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ *are satisfies* $u_0 \ne 0$ *and*

$$2\rho G(0)^{\frac{1}{2}}B(0)^{\frac{1}{2}} < \frac{1}{K_4+1}\,, \tag{4.1}$$

then the problem (1.1) *admits a unique global solution* $u(t)$ *in the class*

$$C^0([0,\infty);\mathcal{D}(A))\cap C^1([0,\infty);\mathcal{D}(A^{1/2}))\cap C^2([0,\infty);H)$$

and the solution $u(t)$ *satisfies*

$$\|u(t)\|^2 \le 6(\|u_0\|^2+\rho E(0))\,, \tag{4.2}$$

$$E(t)\le E(0)\,,\quad a(M(t))\le I(0)\,, \tag{4.3}$$

$$\rho\frac{|M'(t)|}{M(t)}\le\frac{1}{K_4+1}\,, \tag{4.4}$$

$$\frac{\|Au(t)\|^2}{M(t)}\le G(0)\,,\quad \frac{\|u'(t)\|^2}{M(t)}\le B(0)\,. \tag{4.5}$$

Proof. Let $u(t)$ be a solution on $[0,T]$. Since we observe from (3.2), (3.5), and (4.1) that

$$\rho\frac{|M'(0)|}{M(0)}\le 2\rho\frac{\|u_1\|}{M(0)^{\frac12}}\frac{\|Au_0\|}{M(0)^{\frac12}}\le 2\rho B(0)^{\frac12}G(0)<\frac{1}{K_4+1}\,,$$

putting

$$T=\sup\{t\in[0,\infty)\mid \rho\frac{|M'(s)|}{M(s)}<\frac{1}{K_4+1}\quad\text{for}\quad 0\le s<t\}\,,$$

we see that $T_1>0$. If $T_1<T$, we have

$$\rho\frac{|M'(t)|}{M(t)}<\frac{1}{K_4+1}\quad\text{for}\quad 0\le t<T_1\,,\quad\text{and}\quad \rho\frac{|M'(T_1)|}{M(T_1)}=\frac{1}{K_4+1}\,.$$

Again, from (3.2), (3.5), and (4.1) it follows that

$$\rho\frac{|M'(t)|}{M(t)}\le 2\rho\frac{\|u'(t)\|}{M(t)^{\frac12}}\frac{\|Au(t)\|}{M(t)^{\frac12}}\le 2\rho B(0)^{\frac12}G(0)<\frac{1}{K_4+1}$$

for $0\le t\le T$, and hence, we obtain $T_1\ge T$, and we see that the solution $u(t)$ satisfies the estimates (2.3)–(2.6), (3.2), and (3.5), which implies (4.2)–(4.5).

Taking the inner product of (1.1) with $2Au'(t)/a(M(t))$, we have

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{a(M(t))}+\|Au(t)\|^2\right)$$
$$+2\left(1+\frac{\rho}{2}\frac{a'(M(t))M(t)}{a(M(t))}\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{a(M(t))}=0\,.$$

Since

$$1+\frac{\rho}{2}\frac{a'(M(t))M(t)}{a(M(t))}\frac{M'(t)}{M(t)}\ge 1-\frac{K_4}{2}\rho\frac{|M'(t)|}{M(t)}$$
$$\ge 1-\frac{K_4}{2}\frac{1}{K_4+1}\ge 0\,,$$

we have

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{a(M(t))} + \|Au(t)\|^2\right) \le 0\,,$$

and hence,

$$\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \le C \quad \text{for} \quad 0 \le t \le T\,.$$

Thus, we observe that $\|u(t)\|_2 + \|u'(t)\|_1 \le C$, and by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. □

5 Lower Decay Estimates

Proposition 5.1 *Under the assumption of Theorem 4.1, it holds that*

$$M(t) \ge Ce^{-\alpha t} \qquad \textit{for} \quad t \ge 0 \tag{5.1}$$

with some $\alpha > 0$.

Proof. Taking the inner product of (1.1) with $2u'(t)/M(t)^2$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)}\right) + 2\left(1 + \rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^2} \\
&= \frac{-2a(M(t)) + a'(M(t))M(t)}{M(t)}\frac{M'(t)}{M(t)} \\
&\le C\frac{a(M(t))}{M(t)}\frac{M'(t)}{M(t)} \le \alpha\frac{a(M(t))}{M(t)}
\end{aligned}$$

with some $\alpha > 0$, where we used Hyp.2 and (4.4).

Since $1 + \rho M'(t)/M(t) \ge 0$, we have

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)}\right) \le \alpha\left(\rho\frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)}\right)$$

and hence, we obtain

$$\rho\frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)} \le Ce^{\alpha t} \quad \text{or} \quad M(t) \ge Ce^{-\alpha t}$$

where we used the assumption that $a(M(t)) \ge K_1 > 0$. □

Proposition 5.2 *Under the assumption of Theorem 4.1, it holds that*

$$\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \le C \qquad \textit{for} \quad t \ge 0\,. \tag{5.2}$$

Proof. Taking the inner product of (1.1) with $(2Au'(t)+\rho^{-1}Au(t))/M(t)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)}+a(M(t))\frac{\|Au(t)\|^2}{M(t)}+\frac{(Au(t),u'(t))}{M(t)}\right)\\
&+\left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)}+\frac{a(M(t))}{\rho}\frac{\|Au(t)\|^2}{M(t)}+\frac{1}{2}\frac{|M'(t)|^2}{M(t)^2}\\
&=-\left(a(M(t))+a'(M(t))M(t)\right)\frac{M'(t)}{M(t)}\frac{\|Au(t)\|^2}{M(t)}-\frac{1}{2\rho}\frac{M'(t)}{M(t)}\,. \qquad (5.3)
\end{aligned}
$$

Moreover, taking (5.3) + (3.7) $\times\ \rho^{-1}K_1^{-1}$, we have

$$
\begin{aligned}
&\frac{d}{dt}F(t)+\left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)}+\frac{a(M(t))}{\rho}\frac{\|Au(t)\|^2}{M(t)}+\frac{1}{2}\frac{|M'(t)|^2}{M(t)^2}\\
&+\frac{1}{\rho K_1}\left(2+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)}=R(t)
\end{aligned}
$$

where

$$
\begin{aligned}
F(t)=&H(t)+\frac{(Au(t),u'(t))}{M(t)}\,,\\
H(t)=&\rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)}+a(M(t))\frac{\|Au(t)\|^2}{M(t)}+\frac{1}{K_1}\frac{\|u'(t)\|^2}{M(t)}\quad(\geq 0)\,,\\
R(t)=&-\left(a(M(t))+a'(M(t))M(t)\right)\frac{M'(t)}{M(t)}\frac{\|Au(t)\|^2}{M(t)}-\frac{1}{2\rho}\frac{M'(t)}{M(t)}\\
&-\frac{a(M(t))}{\rho K_1}\frac{M'(t)}{M(t)}\,.
\end{aligned}
$$

Since we observe from the Young inequality and Hyp.1 that

$$
\begin{aligned}
\frac{|(Au(t),u'(t))|}{M(t)}&\leq\frac{K_1}{2}\frac{\|Au(t)\|^2}{M(t)}+\frac{1}{2K_1}\frac{\|u'(t)\|^2}{M(t)}\\
&\leq\frac{a(M(t))}{2}\frac{\|Au(t)\|^2}{M(t)}+\frac{1}{2K_1}\frac{\|u'(t)\|^2}{M(t)}\,,
\end{aligned}
$$

and from (4.4) that

$$
1+\rho\frac{M'(t)}{M(t)}\geq\frac{K_4}{K_4+1}\quad(>0)
$$

and from (4.3)–(4.5) that

$$
|R(t)|\leq C\frac{|M'(t)|}{M(t)}\leq C\,,
$$

we have

$$\frac{d}{dt}F(t) + \nu F(t) \le C$$

with some $\nu > 0$, and hence,

$$F(t) \le C \quad \text{or} \quad H(t) \le C$$

which implies the desired estimate (5.2). □

Proposition 5.3 *Under the assumption of Theorem 4.1 and Hyp.3, it holds that*

$$\frac{\|u''(t)\|^2}{M(t)} \le C \qquad \textit{for} \quad t \ge 0\,. \tag{5.4}$$

Proof. Taking the inner product of (1.1) with $(2u''(t) + \rho^{-1}u'(t))/M(t)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|u''(t)\|^2}{M(t)} + a(M(t))\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2}\frac{|M'(t)|^2}{M(t)^2}\right. \\
&\quad\left. + \frac{1}{2\rho}\frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t),u'(t))}{M(t)}\right) + \left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u''(t)\|^2}{M(t)} \\
&\quad + \frac{a(M(t))}{\rho}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2\rho}\frac{|M'(t)|^2}{M(t)^2} \\
&= (-a(M(t)) + 3a'(M(t))M(t))\frac{M'(t)}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\
&\quad + \frac{1}{2}\left(-a'(M(t))M(t) + a''(M(t))M(t)^2\right)\left(\frac{M'(t)}{M(t)}\right)^3 \\
&\quad - \frac{M'(t)}{M(t)}\left(\frac{1}{2\rho}\frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t),u'(t))}{M(t)}\right).
\end{aligned} \tag{5.5}$$

Moreover, taking (5.5)+(3.7), we have

$$\begin{aligned}
&\frac{d}{dt}G(t) + \left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u''(t)\|^2}{M(t)} + \frac{a(M(t))}{\rho}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\
&\quad + \frac{a'(M(t))M(t)}{2\rho}\frac{|M'(t)|^2}{M(t)^2} + \left(2+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)} = S(t)
\end{aligned}$$

where

$$
\begin{aligned}
G(t) =& K(t) + \frac{(u''(t), u'(t))}{M(t)}, \\
K(t) =& \rho \frac{\|u''(t)\|^2}{M(t)} + a(M(t)) \frac{\|A^{1/2} u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2} \frac{|M'(t)|^2}{M(t)^2} \\
& + \left(\frac{1}{2\rho} + \rho \right) \frac{\|u'(t)\|^2}{M(t)}, \\
S(t) =& (-a(M(t)) + 3a'(M(t))M(t)) \frac{M'(t)}{M(t)} \frac{\|A^{1/2} u'(t)\|^2}{M(t)} \\
& + \frac{1}{2} \left(-a'(M(t))M(t) + a''(M(t))M(t)^2 \right) \left(\frac{M'(t)}{M(t)} \right)^3 \\
& - \frac{M'(t)}{M(t)} \left(\frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)} \right) - a(M(t)) \frac{M'(t)}{M(t)}.
\end{aligned}
$$

Since we observe from the Young inequality that

$$
\frac{|(u''(t), u'(t))|}{M(t)} \leq \frac{\rho}{2} \frac{\|u''(t)\|^2}{M(t)} + \frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)}
$$

and from (4.4) that

$$
1 + \rho \frac{M'(t)}{M(t)} \geq \frac{K_4}{K_4 + 1} \quad (> 0)
$$

and from (4.3)–(4.5), (5.2) that

$$
|S(t)| \leq C + \frac{K_4}{2(K_4 + 1)} \frac{\|u'(t)\|^2}{M(t)},
$$

we have

$$
\frac{d}{dt} G(t) + \nu G(t) \leq C
$$

with some $\nu > 0$, and hence,

$$
G(t) \leq 0 \quad \text{or} \quad K(t) \leq 0
$$

which implies the desired estimate (5.4). □

Theorem 5.4 *Suppose that the assumption of Theorem 4.1 and Hyp.3 are fulfilled. Then, the solution $u(t)$ satisfies*

$$
\|u(t)\|^2 \geq Ce^{-\beta t} \qquad \text{for} \quad t \geq 0 \tag{5.6}
$$

with some $\beta \geq \alpha > 0$.

Proof. Using (1.1), we observe that

$$\begin{aligned}\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2} &= \frac{1}{\|u(t)\|^4}\left(2(Au(t),u'(t))\|u(t)\|^2 - 2M(t)(u(t),u'(t))\right)\\ &= \frac{-2}{\|u(t)\|^2}\Bigg(\rho(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), u''(t))\\ &\quad + a(M(t))((Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), Au(t))\Bigg)\end{aligned}$$

and

$$(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), Au(t)) = \|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2 .$$

Thus, we have

$$\begin{aligned}&\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2} + \frac{2a(M(t))}{\|u(t)\|^2}\|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2\\ &= \frac{-2\rho}{\|u(t)\|^2}\rho(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), u''(t))\\ &\le 2\rho\frac{1}{\|u(t)\|}\|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|\frac{\|u''(t)\|}{\|u(t)\|}\\ &\le \frac{2K_1}{\|u(t)\|^2}\|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2 + \frac{\rho^2}{2K_1}\frac{\|u''(t)\|^2}{\|u(t)\|^2},\end{aligned}$$

and moreover, by $a(M(t)) \ge K_1 > 0$,

$$\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2} \le C\frac{\|u''(t)\|^2}{\|u(t)\|^2} = C\frac{\|u''(t)\|}{M(t)}\frac{M(t)}{\|u(t)\|^2} \le \nu\frac{M(t)}{\|u(t)\|^2}$$

with some $\nu \ge 0$, where we used (5.4). Therefore, we obtain

$$\frac{M(t)}{\|u(t)\|^2} \le Ce^{\nu t}$$

and hence

$$\|u(t)\|^2 \ge Ce^{-\nu t}M(t) \ge Ce^{-\nu t}e^{-\alpha t} = Ce^{-\beta t}$$

with some $\beta \ge \alpha > 0$, where we used (5.1). □

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