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Fourier Transformation of $L^p_{\rm loc}$-functions

By

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Abstract

In this paper, we study the Fourier transformation of $L^p_{\rm loc}$-functions and L^q_c-functions. Here we assume that the condition $\dfrac{1}{p}+\dfrac{1}{q}=1,\ (1 \le p \le \infty,\ 1 \le q \le \infty)$ is satisfied. Thereby we prove the structure theorems of the image spaces $\mathcal{F}L^p_{\rm loc}$ and $\mathcal{F}L^q_c$. We study the convolution $f*g$ of a L^r_c-function f and a $L^p_{\rm loc}$-function g. Here assume $d \ge 1$. Further we assume that the condition $\dfrac{1}{q}=\dfrac{1}{p}+\dfrac{1}{r}-1,\ (1 \le p \le \infty,\ 1 \le q \le \infty,\ 1 \le r \le \infty)$ is satisfied. This is a generalization of the theory of Fourier transformations of $L^2_{\rm loc}$-functions.

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Introduction

In this paper, we study the Fourier transformation of $L^p_{\rm loc}$-functions and L^q_c-functions and some applications. Here we assume that the condition $\dfrac{1}{p}+\dfrac{1}{q}=1,\ (1 \le p \le \infty,\ 1 \le q \le \infty)$ is satisfied. In section 1, we define the Fourier transformation and the inverse Fourier transformation of $L^p_{\rm loc}$-functions. We show some examples of Fourier transformation of $L^p_{\rm loc}$-functions. We prove the inversion formulas of the Fourier transformation and the inverse Fourier transformation of L^p_{loc}-functions.

In section 2, we prove the structure theorems of the function spaces L^p_{loc} and L^q_c and the structure theorems of the Fourier images $\mathcal{F}L^p_{\text{loc}}$ and $\mathcal{F}L^q_c$.

In section 3, we study the convolution $f * g$ of a function f in $L^r_c = L^r_c(\boldsymbol{R}^d)$ and a function g in $L^p_{\text{loc}} = L^p_{\text{loc}}(\boldsymbol{R}^d)$. Here we assume that the condition $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{r} - 1$, $(1 \leq p \leq \infty,\ 1 \leq q \leq \infty,\ 1 \leq r \leq \infty)$ is satisfied.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Fourier transformation of L^p_{loc}-functions

In this section, at first we define the Fourier transformation of L^p_{loc}-functions and its fundamental properties. Here we assume $1 \leq p \leq \infty$. Let $\boldsymbol{R}^d$ be the d-dimensional Euclidean space. Here assume $d \geq 1$. Further we denote $L^p_{\text{loc}} = L^p_{\text{loc}}(\boldsymbol{R}^d)$ as usual. If we put $L^p = L^p(\boldsymbol{R}^d)$, we have the inclusion relation $L^p \subset L^p_{\text{loc}}$. For the points in $\boldsymbol{R}^d$

$$x = {}^t(x_1,\ x_2,\ \cdots,\ x_d),\ p = {}^t(p_1,\ p_2,\ \cdots,\ p_d),$$

we define the dual inner product by the formula

$$px = (p,\ x) = p_1x_1 + p_2x_2 + \cdots + p_dx_d.$$

Thereby the space $\boldsymbol{R}^d$ becomes a self-dual space. We define the norms of x and p by the formulas

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2},$$

$$|p| = \sqrt{p_1^2 + p_2^2 + \cdots + p_d^2}.$$

Let $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ be the space of all C^∞-functions with compact support in $\boldsymbol{R}^d$.

Here we define the Fourier transformation $\mathcal{F}\varphi$ of $\varphi \in \mathcal{D}$ by the relation

$$(\mathcal{F}\varphi)(p) = \frac{1}{(\sqrt{2\pi})^d} \int \varphi(x) e^{-ipx} dx,\ (p \in \boldsymbol{R}^d).$$

$\mathcal{F}\mathcal{D}$ denotes the space of the Fourier image of $\mathcal{D}$ by the Fourier transformation $\mathcal{F}$.

Here we define the symbol $e_d(x)$ by the formula

$$e_d(x) = \frac{1}{(\sqrt{2\pi})^d} e^x,\ (x \in \boldsymbol{R}^d).$$

Then we have the formula

$$(\mathcal{F}\varphi)(p) = \int \varphi(x)e_d(-ipx)dx,\ (p \in \boldsymbol{R}^d)$$

for the Fourier transformation $\mathcal{F}\varphi(p)$.

Further, let $\mathcal{D}' = \mathcal{D}'(\boldsymbol{R}^d)$ be the space of Schwartz distributions on $\boldsymbol{R}^d$.

Here, for the dual pair $\mathcal{D}'$ and $\mathcal{D}$ of two TVS's, we denote the dual inner product of $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ as $< T,\ \varphi >$ and, for the dual pair $(\mathcal{F}\mathcal{D})'$ and $\mathcal{F}\mathcal{D}$, we denote its dual inner product of $S \in (\mathcal{F}\mathcal{D})'$ and $\varphi \in \mathcal{F}\mathcal{D}$ as $< S,\ \varphi >$.

Now assume $T \in \mathcal{D}'$. Then, since we have $\mathcal{F}^{-1}\varphi \in \mathcal{D}$ for $\varphi \in \mathcal{F}\mathcal{D}$, we can define a continuous linear functional

$$S:\ \varphi \to < T,\ \mathcal{F}^{-1}\varphi >,\ (\varphi \in \mathcal{F}\mathcal{D})$$

and we have $S \in (\mathcal{F}\mathcal{D})'$. Namely, we have the equality

$$< S,\ \varphi > = < T,\ \mathcal{F}^{-1}\varphi > .$$

Then we define that S is a Fourier transform of T and denote it as $S = \mathcal{F}T$.

This is the new definition of the Fourier transformation of $\mathcal{D}'$. Since a Schwartz distribution is a generalized concept of functions, we had better to define the Fourier transformation of Schwartz distributions as in the same direction as the Fourier transformation of classical functions. Thus we define the new type of Fourier transformation of Schwartz distributions.

Therefore, for the Fourier transform $\mathcal{F}T \in \mathcal{F}\mathcal{D}'$ of $T \in \mathcal{D}'$, we have the relation

$$< \mathcal{F}T,\ \mathcal{F}\varphi > = < T,\ \varphi >,\ (\varphi \in \mathcal{D}).$$

This is a generalization of Parseval's formula for L^2-functions. Then the Fourier transformation $\mathcal{F}$ is a topological isomorphism from $\mathcal{D}'$ to $\mathcal{F}\mathcal{D}'$.

Thus we have the isomorphisms

$$\mathcal{D}' \cong \mathcal{F}\mathcal{D}' \cong (\mathcal{F}\mathcal{D})'.$$

Here we denote the dual mapping of the Fourier transformation $\mathcal{F}:\ \mathcal{D} \to \mathcal{F}\mathcal{D}$ as $\mathcal{F}^*:\ (\mathcal{F}\mathcal{D})' \to \mathcal{D}'$. Then we have the equality

$$\mathcal{F}^*\mathcal{F} = \text{the identity mapping of } \mathcal{D}'.$$

For $1 \leq p < p' \leq \infty$, we have the inclusion relations

$$L^{p'}_{\rm loc} \subset L^p_{\rm loc} \subset L^1_{\rm loc} \subset \mathcal{D}'.$$

We define the Fourier transformation of $f \in L^p_{\rm loc}$ considering it as an element of $\mathcal{D}'$.

We say that the limit in the sense of the topologies of $\mathcal{D}'$ or $\mathcal{F}\mathcal{D}'$ is **the limit in the sense of generalized functions**.

Then we give the following definition.

Definition 1.1 We define the Fourier transform $(\mathcal{F}f)(p)$ of $f \in L^p_{\mathrm{loc}}$ by the relation

$$(\mathcal{F}f)(p) = \lim_{R\to\infty}\int_{|x|\leq R} f(x)e_d(-ipx)dx$$

in the sense of generalized functions.

Then we denote $\mathcal{F}f(p)$ as

$$(\mathcal{F}f)(p) = \int f(x)e_d(-ipx)dx,\ (p \in \boldsymbol{R}^d).$$

Here, when the integration domain is equal to the entire space $\boldsymbol{R}^d$, we omit the symbol of the integration domain.

Let $\mathcal{C} = \mathcal{C}(\boldsymbol{R}^d)$ be the function space of all continuous functions on $\boldsymbol{R}^d$. Then we have the inclusion relation

$$\mathcal{C} \subset L^p_{\mathrm{loc}}.$$

In general, a continuous function is not necessarily a L^p-function. Then we can define the Fourier transformation of continuous functions considering them as L^p_{loc}-functions.

Example 1.1 We have the following equality:

$$(\mathcal{F}(-ix)^\alpha)(p) = \int (-ix)^\alpha e_d(-ipx)dx = (\sqrt{2\pi})^d\delta^{(\alpha)}(p).$$

Here $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ denotes a multi-index of natural numbers.

Especially, for $\alpha = 0 = (0,\ 0,\ \cdots,\ 0)$, we have the equality

$$(\mathcal{F}1)(p) = \int e_d(-ipx)dx = (\sqrt{2\pi})^d\delta(p).$$

Therefore, the Fourier transform of the constant function $\dfrac{1}{(\sqrt{2\pi})^d}$ is equal to the Dirac measure δ. Thereby, in general, the Fourier transform $\mathcal{F}f$ of a L^p_{loc}-function f is not necessarily a L^p_{loc}-function.

Now we give some examples of Fourier transforms of continuous functions.

Example 1.2 Assume $d \geq 1$ and $1 \leq p \leq \infty$. The constant function 1 belongs to $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\boldsymbol{R}^d)$. For $R > 0$, we put $\chi_R(x) = \chi_{|x|\leq R}(x)$. Then we have $\chi_R \in L^p_{\mathrm{loc}}$ and we have

$$\chi_R \to 1,\ (R \to \infty)$$

in the topology of $L^p_{\rm loc}$-convergence. Thus we have

$$\chi_R \to 1,\ (R \to \infty)$$

in the topology of $\mathcal{D}'$. Then we have

$$\hat{\chi}_R(p) = \int \chi_R(x) e_d(-ipx)dx \to \int e_d(-ipx)dx = \hat{1}(p) = (\sqrt{2\pi})^d \delta(p)$$

for $R \to \infty$ in the topology of $\mathcal{FD}'$.

Example 1.3 For $n \geq 1$, we put

$$\chi_n(x) = \chi_{[-n,\ n]}(x),\ (x \in \boldsymbol{R}).$$

Then, for $1 \leq p \leq \infty$, we have $\chi_n \in L^p_{\rm loc}$ and we have

$$\chi_n \to 1,\ (n \to \infty)$$

in the topology of $L^p_{\rm loc}$-convergence.

Thus we have

$$\chi_n \to 1, (n \to \infty)$$

in the topology of $\mathcal{D}'$. Then we have

$$\hat{\chi}_n((p) = \int \chi_n(x) e_1(-ipx)dx \to \int e_1(-ipx)dx = \hat{1}(p) = \sqrt{2\pi}\delta(p)$$

for $n \to \infty$ in the topology of $\mathcal{FD}'$.

Example 1.4 We have

$$\frac{1}{\pi}\frac{\sin pn}{p} \to \delta(p),\ (n \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof We have the equality

$$\int_{-n}^{n} e_1(-ipx)dx = \frac{1}{ip\sqrt{2\pi}}(e^{ipn} - e^{-ipn}) = \sqrt{\frac{2}{\pi}}\frac{\sin pn}{p}.$$

Thus we have the conclusion by virtue of Example 1.3.//

Example 1.5 Assume $d \geq 1$ and $1 \leq p \leq \infty$. Let $n = (n_1,\ n_2,\ \cdots,\ n_d)$ be a multi-index of positive natural numbers. We denote $|n| = n_1 + n_2 + \cdots + n_d$. By using the notation of Example 1.3, we denote

$$\chi_n(x) = \chi_{n_1}(x_1)\chi_{n_2}(x_2)\cdots\chi_{n_d}(x_d),\ (x \in \boldsymbol{R}^d),$$

$$\hat{\chi}_n(p) = \hat{\chi}_{n_1}(p_1)\hat{\chi}_{n_2}(p_2)\cdots\hat{\chi}_{n_d}(p_d),\ (p \in \boldsymbol{R}^d).$$

Then we have

$$\hat{\chi}_n(p) \to (\sqrt{2\pi})^d\delta(p),\ (|n| \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof By virtue of Example 1.3, because we have

$$\hat{\chi}_{n_j}(p_j) \to \sqrt{2\pi}\delta(p_j)$$

for $1 \leq j \leq d$, we have the conclusion. //

Theorem 1.1 *We use the same notation as Example* 1.5. *Then, for*

$$\chi_n(x) = \chi_{n_1}(x_1)\chi_{n_2}(x_2)\cdots\chi_{n_d}(x_d),\ (x \in \boldsymbol{R}^d),$$

we denote

$$\hat{\chi}_n(p) = \hat{\chi}_{n_1}(p_1)\hat{\chi}_{n_2}(p_2)\cdots\hat{\chi}_{n_d}(p_d),\ (p \in \boldsymbol{R}^d).$$

For $f(x) \in L^p_{\mathrm{loc}}$, we put $f_n(x) = \chi_n(x)f(x)$. Then we have $f_n(x) \in L^p$. Now, when we consider that f_n and f are elements of $\mathcal{D}'$, we denote their Fourier transformations as $\mathcal{F}f_n = \hat{f}_n$ and $\mathcal{F}f = \hat{f}$. Then we have

$$\hat{f}_n \to \hat{f},\ (|n| \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof When $|n| \to \infty$, we have

$$f_n(x) \to f(x),\ (x \in \boldsymbol{R}^d)$$

in the topology of L^p_{loc}. Therefore, when $|n| \to \infty$, we have

$$f_n \to f$$

in the topology of $\mathcal{D}'$.

Since we have $f_n = \chi_n f$, we have the equality

$$\hat{f}_n = (\chi_n f)^\wedge = \frac{1}{(\sqrt{2\pi})^d}\hat{\chi}_n * \hat{f}$$

in $\mathcal{FD}'$. Here the symbol $*$ denotes the convolution. By virtue of Example 1.5, we have

$$\hat{\chi}_n \to (\sqrt{2\pi})^d\delta,\ (|n| \to \infty).$$

Thus, when $|n| \to \infty$, we have

$$\hat{f}_n = \frac{1}{(\sqrt{2\pi})^d}\hat{\chi_n} * \hat{f} \to \delta * \hat{f} = \hat{f}$$

in the topology of $\mathcal{FD}'$. //

When we use the notation in Theorem 1.1, we have $f_n \in L^p$ and

$$\hat{f}_n(p) = \int f_n(x)e_d(-ipx)dx.$$

Therefore we have the equality

$$\lim_{|n|\to\infty} \int f_n(x)e_d(-ipx)dx = \hat{f}(p)$$

in $\mathcal{FD}'$. In this sense, we use the notation

$$\hat{f}(p) = \int f(x)e_d(-ipx)dx$$

for $\hat{f}(p) \in \mathcal{FD}'$. Here we consider this integral in the sense of convergence in the topology of $\mathcal{FD}'$.

In this case, we say that this integral converges in the sense of generalized functions.

Similarly, we define the Fourier inverse transformation as follows.

Definition 1.2(Fourier inverse transformation) Assume $1 \le p \le \infty$. We define the Fourier inverse transformation of $g(p) \in L^p_{\text{loc}}$ by the relation

$$(\mathcal{F}^{-1}g)(x) = \lim_{R\to\infty} \int_{|p|\le R} g(p)e_d(ipx)dp$$

in the sense of generalized functions.

We denote $(\mathcal{F}^{-1}g)(x)$ as

$$(\mathcal{F}^{-1}g)(x) = \int g(p)e_d(ipx)dp.$$

Theorem 1.2 *Let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ be a multi-index of natural numbers. Assume that $f(x) \in L^p_{\text{loc}}$ and $D^\alpha f(x) \in L^p_{\text{loc}}$ hold for $1 \le p \le \infty$. Then we have the following* (1) *and* (2):

(1) $D^\alpha(\mathcal{F}f)(p) = \mathcal{F}((-ix)^\alpha f)(p)$.

(2) $(ip)^\alpha(\mathcal{F}f)(p) = \mathcal{F}(D^\alpha f)(p)$.

In Theorem 1.2, the symbols x^α and D^α etc. are the same as usually used. Namely $D^\alpha f$ means a L^p_{loc}-derivatives, and $D^\alpha(\mathcal{F}f)$ means, in general, a partial derivative of $\mathcal{F}f$ in $\mathcal{FD}'$ and so on .

Next we prove the Fourier inversion formula.

Now we assume $f \in L^p_{\mathrm{loc}}$. Here we assume $1 \leq p \leq \infty$. Then, since we have

$$f_R(x) \in L^p, \ (0 < R < \infty),$$

we have

$$\mathcal{F}^{-1}\mathcal{F}f_R(x) = f_R(x), \ (0 < R < \infty)$$

in the sense of generalized functions.

Then, since we have

$$f_R(x) \to f(x), \ (R \to \infty)$$

is the sense of generalized functions, we have the equality

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Therefore we have the following inversion formula.

Theorem 1.3(Inversion formula) *Assume* $1 \leq p \leq \infty$, *For* $f(x) \in L^p_{\mathrm{loc}}$, *we have the following inversion formula*

$$f(x) = \lim_{R\to\infty} \int (\mathcal{F}f_R)(p)e_d(ipx)dp = \int e_d(ipx)dp \int f(y)e_d(-ipy)dy.$$

Here the integral converges in the sense of generalized functions. Namely we have the equality

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Similarly, for $g(p) \in L^p_{\mathrm{loc}}$, we denote the restriction of g to the closed ball $|p| \leq T$ as g_T. Then we have the equality

$$\mathcal{F}\mathcal{F}^{-1}g_T(p) = g_T(p), \ (0 < T < \infty)$$

in the sense of generalized functions. Then we have

$$g_T(p) \to g(p), \ (T \to \infty)$$

in the sense of generalized functions, Thus we have the equality

$$\mathcal{F}\mathcal{F}^{-1}g(p) = g(p)$$

in the sense of generalized functions.

Therefore we have the following inversion formula.

Theorem 1.4 (Inversion formula) *Assume* $1 \leq p \leq \infty$, *For* $g \in L^p_{\mathrm{loc}}$, *we have the following inversion formula*

$$g(p) = \int (\mathcal{F}^{-1}g)(x)e_d(-ipx)dx = \int e_d(-ipx)dx \int g(q)e_d(iqx)dq.$$

Here the integral converges in the sense of generalized functions. Namely we have the equality

$$\mathcal{F}\mathcal{F}^{-1}g = g.$$

Theorem 1.5 *For $f \in L^p_{\rm loc}$, we have the equalities*:

$$\mathcal{F}^2 f(x) = f(-x),\ \mathcal{F}^4 f(x) = f(x).$$

2 Structure theorems

In this section, we study the structure theorems of the function spaces $L^p_{\rm loc}$ and L^q_c and the structure theorems of the Fourier images $\mathcal{F}L^p_{\rm loc}$ and $\mathcal{F}L^q_c$. Here we assume that the real numbers p and q satisfy the condition

$$\frac{1}{p} + \frac{1}{q} = 1,\ (1 \le p \le \infty,\ 1 \le q \le \infty).$$

Now we choose an exhausting sequence $\{K_j\}$ of compact sets in $\boldsymbol{R}^d$ which satisfies the following conditions (i) and (ii):

(i) $K_1 \subset K_2 \subset \cdots \subset \boldsymbol{R}^d,\ \boldsymbol{R}^d = \bigcup_{j=1}^{\infty} K_j.$

(ii) $K_j = \mathrm{cl}(\mathrm{int}(K_j)),\ K_j \subset \mathrm{int}(K_{j+1}),\ (j = 1,\ 2,\ 3,\ \cdots).$

Then we denote the projective limit of projective system $\{L^p(K_j)\}$ of Banach spaces as

$$\varprojlim L^p(K_j).$$

Then we have the isomorphism

$$L^p_{\rm loc} \cong \varprojlim L^p(K_j)$$

as TVS's. Here, since, for each j, the restriction mapping $L^p(K_{j+1}) \to L^p(K_j)$ is a weakly compact mapping, $L^p_{\rm loc}$ is a FS*-space.

Further, because the system $\{L^q(K_j)\}$ of Banach spaces can be considered as an inductive system, we denote the inductive limit as

$$\varinjlim L^q(K_j).$$

Then we have the isomorphism

$$L^q_c \cong \varinjlim L^q(K_j)$$

as TVS's. Here L^q_c denotes the TVS of all L^q-functions with compact support. Then, since, for each j, the inclusion mapping $L^q(K_j) \to L^q(K_{j+1})$ is a weakly compact mapping, L^q_c is a DFS*-space.

Since $L^p(K_j)$ and $L^q(K_j)$ are the dual pair of Banach spaces, we have the isomorphism

$$L^p_{\mathrm{loc}} \cong (L^q_c)'$$

as TVS's. Here $(L^q_c)'$ denotes the dual space of L^q_c and we define the dual inner product of $f \in L^p_{\mathrm{loc}}$ and $g \in L^q_c$ by the equality

$$< f,\ g >= \int f(x)g(x)dx.$$

Here the dual inner product is a bilinear functional which defines the duality relation of the pair of two TVS's L^p_{loc} and L^q_c.

Because L^p_{loc} is a FS*-space and L^q_c is a DFS*-space, L^p_{loc} and L^q_c are reflexive. Thus we have the following theorem.

Theorem 2.1 *We use the the notation in the above. Assume that two real numbers satisfy the condition*

$$\frac{1}{p} + \frac{1}{q} = 1,\ (1 \leq p \leq \infty,\ 1 \leq q \leq \infty).$$

Then we have the following isomorphisms (1) *and* (2):

(1) $L^p_{\mathrm{loc}} \cong (L^q_c)' \cong (L^p_{\mathrm{loc}})''$.

(2) $L^q_c \cong (L^p_{\mathrm{loc}})' \cong (L^q_c)''$.

Theorem 2.2 *Assume* $1 \leq q \leq \infty$. *Then the function space* $\mathcal{D}$ *is dense in* L^q_c.

Proof Assume $1 \leq q \leq \infty$. Then we prove that $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ is dense in $L^q_c = L^q_c(\boldsymbol{R}^d)$. Now we choose a exhausting sequence $\{K_j\}$ of compact sets in $\boldsymbol{R}^d$. Here we define $\mathcal{D}_{K_j}$ is the subspace of $\mathcal{D}$ which is composed of the functions in $\mathcal{D}$ whose supports are included in K_j. Then we have the isomorphisms

$$\mathcal{D} \cong \varinjlim \mathcal{D}_{K_j} \cong \bigcup_{j=1}^{\infty} \mathcal{D}_{K_j}.$$

Further we have the isomorphisms

$$L^q_c \cong \varinjlim L^q(K_j) \cong \bigcup_{j=1}^{\infty} L^q(K_j).$$

Then, because $\mathcal{D}_{K_j}$ is dense in $L^q(K_j)$ for each $j \geq 1$, we have proved that $\mathcal{D}$ is dense in L^q_c. //

Corollary 2.1 *Assume* $1 \leq p \leq \infty$. *Let* V *be a complete TVS and let* T *be a linear mapping from* $L^p_{\rm loc}$ *into* V. *Then the following* (1) $\sim$ (3) *are equivalent*:

(1) *T is continuous with respect to the strong topology of* $L^p_{\rm loc}$.

(2) *T is continuous with respect to the weak topology of* $L^p_{\rm loc}$.

(3) *T is continuous with respect to the induced topology on* $L^p_{\rm loc}$ *from the topology of* $\mathcal{D}'$

Then, because we have the inclusion relation $L^q_c \subset L^q$, we define the Fourier transformation of a L^q_c-function $g(x)$ by using the Fourier transformation of a L^q-function

$$\mathcal{F}g(p) = \int g(x)e_d(-ipx)dx.$$

Further we define the Fourier transformation of a $L^p_{\rm loc}$-function f by the relation

$$\mathcal{F}f(p) = \lim_{j\to\infty}\int_{K_j} f(x)e_d(-ipx)dx$$

in the sense of generalized functions in $\mathcal{D}'$ and $\mathcal{F}\mathcal{D}'$.

By virtue of the definition of the Fourier transformation of $f \in L^p_{\rm loc}$, we have the equality

$$< \mathcal{F}f,\ \mathcal{F}g >=< f,\ g >$$

for any $g \in \mathcal{D}$.

Since a L^q_c-function g has the compact support, there exists some K_j such that ${\rm supp}(g) \subset K_j$ holds by the definition of $\{K_j\}$. Therefore, for an arbitrary $k \geq j$, we have the equalities

$$< f_{K_k},\ g >= \int_{K_k} f_{K_k}(x)g(x)dx = \int_{K_j} f(x)g(x)dx =< f,\ g > .$$

Here $f_{K_k}(x)$ denotes the image of $f(x) \in L^p_{\rm loc}$ by the restriction mapping $L^p_{\rm loc} \to L^p(K_k)$.

Since we have the equality

$$\int \mathcal{F}f_{K_k}(p)\mathcal{F}g(-p)dp = \int f_{K_k}(x)g(x)dx$$

by virtue of Parseval's formula, we have the equality

$$\lim_{k\to\infty}\int \mathcal{F}f_{K_k}(p)\mathcal{F}g(-p)dp = \lim_{k\to\infty}\int f_{K_k}(x)g(x)dx$$

$$= \int f_{K_j}(x)g(x)dx = \int \mathcal{F}f_{K_j}(p)\mathcal{F}g(-p)dp.$$

Especially, supposing that we have the relations

$$\mathcal{D}_{K_j} \subset L^p(K_j),\ g \in \mathcal{D}_{K_j},$$

we have the equality

$$\int \mathcal{F}f(p)\mathcal{F}g(-p)dp = \int f(x)g(x)dx.$$

We can choose a compact set K_j arbitrarily. Thus, if we consider an arbitrary $g \in \mathcal{D}$, we have $g \in \mathcal{D}_{K_j}$ for some $j \geq 1$. Thus we have the equality in the above for an arbitrary $g \in \mathcal{D}$.

Then, because the dual inner product

$$< f,\ g >= \int f(x)g(x)dx$$

is defined for an arbitrary $f \in L^p_{\mathrm{loc}}$ and $g \in L^q_c$, we have the equality

$$< \mathcal{F}f,\ \mathcal{F}g >= \int \mathcal{F}f(p)\mathcal{F}g(-p)dp = \int f(x)g(x)dx =< f,\ g >$$

for an arbitrary $f \in L^p_{\mathrm{loc}}$ and an arbitrary $g \in L^q_c$.

Now we choose one exhausting sequence $\{K_j\}$ of compact sets in $\boldsymbol{R}^d$ as in the above.

Then, for the sequence

$$L^r(K_1) \subset L^r(K_2) \subset \cdots,\ (1 \leq r \leq \infty,\ r = p \text{ or } q),$$

we have the isomorphisms

$$L^q_c \cong \varinjlim L^q(K_j),\ L^p_{\mathrm{loc}} \cong \varprojlim L^p(K_j).$$

Further we have the isomorphisms

$$L^q_c \cong \bigcup_{j=1}^{\infty} L^q(K_j),\ L^p_{\mathrm{loc}} \cong \bigcap_{j=1}^{\infty} L^p(K_j).$$

Then we have the isomorphisms

$$\mathcal{F}L^r(K_j) \cong L^r(K_j),\ (j = 1,\ 2,\ 3,\ \cdots)$$

for $1 \leq r \leq \infty$ and $r = p$ or q. Further, for the sequence

$$\mathcal{F}L^r(K_1) \subset \mathcal{F}L^r(K_2) \subset \cdots, (1 \leq r \leq \infty,\ r = p \text{ or } q),$$

we have the isomorphisms

$$\mathcal{F}L^q_c \cong \varinjlim \mathcal{F}L^q(K_j) \cong \varinjlim L^q(K_j) \cong L^q_c,$$

$$\mathcal{F}L^p_{\rm loc} \cong \varprojlim \mathcal{F}L^p(K_j) \cong \varprojlim L^p(K_j) \cong L^p_{\rm loc}.$$

Then we have the relations

$$\mathcal{F}L^p_{\rm loc} \subset \mathcal{F}\mathcal{D}',\ \mathcal{F}L^p_{\rm loc} \neq L^p_{\rm loc}.$$

Therefore we have the following theorem.

Theorem 2.3 *We use the notation in the above. Then we have the following isomorphisms* (1) $\sim$ (4):

(1) $L^q_c \cong \varinjlim L^q(K_j) \cong \bigcup_{j=1}^{\infty} L^q(K_j)$.

(2) $\mathcal{F}L^q_c \cong \varinjlim \mathcal{F}L^q(K_j)$.

(3) $\mathcal{F}L^q(K_j) \cong L^q(K_j),\ (j = 1,\ 2,\ 3,\ \cdots)$.

(4) $\mathcal{F}L^q_c \cong L^q_c,\ \mathcal{F}L^q_c \neq L^q_c$.

Further we have the following theorem.

Theorem 2.4 *We use the notation in the above. Then we have the following isomorphisms* (1) $\sim$ (3) *and the relation* (4):

(1) $L^p_{\rm loc} \cong \varprojlim L^p(K_j) \cong \bigcap_{j=1}^{\infty} L^p(K_j) \cong (L^q_c)'$.

(2) $\mathcal{F}L^p_{\rm loc} \cong \varprojlim \mathcal{F}L^p(K_j)$.

(3) $\mathcal{F}L^p_{\rm loc} \cong L^p_{\rm loc}$.

(4) $L^p_{\rm loc} \subset \mathcal{D}',\ \mathcal{F}L^p_{\rm loc} \subset \mathcal{F}\mathcal{D}',\ \mathcal{F}L^p_{\rm loc} \neq L^p_{\rm loc}$.

Theorem 2.5 *We use the notation in the above. If* $f \in L^p_{\rm loc}$ *and* $\mathcal{F}f \in L^p_{\rm loc}$ *are satisfied, we have the equality*

$$\mathcal{F}f(p) = \lim_{R\to\infty} \int_{|x|\le R} f(x)e_d(-ipx)dx,\ (p \in \boldsymbol{R}^d)$$

in the topology of $L^p_{\rm loc}$.

Theorem 2.6 *We use the notation in the above. If $f \in L^p_{\mathrm{loc}}$ and $\mathcal{F}^{-1}f \in L^p_{\mathrm{loc}}$ are satisfied, we have the equality*

$$\mathcal{F}^{-1}f(x) = \lim_{R\to\infty}\int_{|p|\le R} f(p)e_d(ipx)dp, \ (x \in \boldsymbol{R}^d)$$

in the topology of L^p_{loc}.

We remark that functions in $\mathcal{D}$ or $\mathcal{S}$ satisfy the conditions of Theorem 2.5 and Theorem 2.6.

3 Convolution

In this section, we study the convolution $f * g$ of a function f in $L^r_c = L^r_c(\boldsymbol{R}^d)$ and a function g in $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\boldsymbol{R}^d)$. Here assume $d \geq 1$. Further assume that three real numbers p, q and r satisfy the following condition

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \ (1 \le p \le \infty, \ 1 \le q \le \infty, \ 1 \le r \le \infty).$$

We define the convolution $f * g$ of $f \in L^r_c$ and $g \in L^p_{\mathrm{loc}}$ by the relation

$$(f * g)(x) = \int f(x-y)g(y)dy.$$

Then we have the equality

$$\int f(x-y)g(y)dy = \int g(x-y)f(y)dy.$$

Therefore we have the following theorem.

Theorem 3.1 *We use the notation in the above. For $f \in L^r_c$ and $g \in L^p_{\mathrm{loc}}$, we have $f * g \in L^q_{\mathrm{loc}}$. Further we have the equality*

$$f * g = g * f.$$

Theorem 3.2 *We use the notation in the above. Let $\alpha = (\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_d)$ be a multi-index of natural numbers. Then, for $f \in L^r_c$ and $g \in L^p_{\mathrm{loc}}$, we have the equality*

$$D^\alpha(f * g) = (D^\alpha f) * g = f * (D^\alpha g)$$

in L^q_{loc}. Here the partial derivatives are considered in the sense of topologies of L^r_c and L^p_{loc} and L^q_{loc}.

Corollary 3.1 *We use the notation in the above. Assume $f \in L^r_c$. Then the linear transformation of L^p_{loc} defined by the convolution*

$$T_f : \ g \to f * g, \ (g \in L^p_{\mathrm{loc}})$$

is a continuous linear mapping from L^p_{loc} into L^q_{loc}.

Now assume that $\{g_n\}$ is a sequence of L^p_{loc}-functions and it converges to $g \in L^p_{\mathrm{loc}}$ in the topology of L^p_{loc}. Namely, assume that $g_n \to g, \ (n \to \infty)$ in the topology of L^p_{loc}. Then we have

$$T_f(g_n) \to T_f(g), \ (n \to \infty).$$

in the topology of L^q_{loc}

Corollary 3.2 *We use the notation in the above. Assume $g \in L^p_{\mathrm{loc}}$. Then the linear mapping $T_g = f * g, \ (f \in L^r_c)$ defined by the convolution is a continuous linear mapping from L^r_c into L^q_{loc}.*

Therefore, if a sequence $\{f_n\}$ of L^r_c-functions converges to $f \in L^r_c$ in the topology of L^r_c, we have

$$T_g(f_n) \to T_g(f), \ (n \to \infty)$$

in the topology of L^q_{loc}. Here the convolution of a function f in L^r_c and a function g in L^p_{loc} is a separately continuous bilinear mapping $L^r_c \times L^p_{\mathrm{loc}} \to L^q_{\mathrm{loc}}$.

Theorem 3.3 *We use the notation in the above. Assume $f \in L^r_c$ and $g \in L^p_{\mathrm{loc}}$. Then we have the equality*

$$\mathcal{F}(f * g) = (\sqrt{2\pi})^d \mathcal{F}(f)\mathcal{F}(g).$$

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