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Connection Formula of Basic Hypergeometric Series ${}_r\phi_{r-1}(\mathbf{0};\boldsymbol{b};q,x)$

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Abstract

We show a connection formula of a linear q-differential equation satisfied by ${}_r\phi_{r-1}(\mathbf{0};\boldsymbol{b};q,x)$ where any element of $\boldsymbol{b}$ are not zero. We use a q-Laplace transformation to obtain an integral representation of solutions of the q-differential equation.

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1 Introduction

We show a connection formula of a linear q-differential equation satisfied by ${}_r\phi_{r-1}(0,0,...,0;b_1,...,b_{r-1};q,x)$ in case that $b_1b_2\cdots b_{r-1}\neq 0$. The basic hypergeometric series ${}_r\phi_{r-1}(0,0,...,0;b_1,...,b_{r-1};q,x)$ satisfies a linear q-differential equation of the r-th order:

$$\left[x-(1-\sigma_q)\prod_{k=1}^{r-1}(1-\frac{b_k}{q}\sigma_q)\right]y(x)=0, \tag{1}$$

where $\sigma_q y(x)=y(xq)$. The condition $b_1b_2\cdots b_{r-1}\neq 0$ implies that the origin is a regular singular point of (1). Around the infinity (1) has r solutions which are represented by convergent power series on $x^{1/r}$. In this sense, (1) is the most degenerate case of hypergeometric equations.

Thomae [6, 7] showed a connection formula on ${}_2\phi_1(a_1, a_2; b_1)$ and ${}_3\phi_2(a_1, a_2, a_3; b_1, b_2)$. In [8] Watson gave connection formulae in more general cases. He showed a connection formula of ${}_r\phi_{r-1}(a_1, a_2, .., a_r; b_1, ..., b_s, \mathbf{0};\ q, z)$, where $s < r$. Watson also showed that an asymptotic expansion of ${}_{s+1}\phi_{r-1}(a'_1, a'_2, ..., a'_{s+1}; b'_1, ..., b'_{r-1}; q, z)$ (he used a notation ${}_{s+1}\mathfrak{P}_{r-1}$), but he did not give a resummation of divergent series. Later Slater [4, 5] also gave a more general form of a connection formula.

J.-P. Ramis, J. Sauloy and C. Zhang started modern study on divergent q-series and a q-analogue of the Stokes phenomenon [3]. Zhang studied the q-Stokes phenomenon of q-confluent hypergeometric function ${}_2\phi_0(a, b; 0; q, x)$ [10]. He has also shown a connection formula of Jackson's q-analogue of the Bessel function $J_\nu^{(1)}$ [11]. Since

$$J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu \ {}_2\phi_1\left(0, 0; q^{\nu+1}; q, -\frac{x^2}{4}\right),$$

the connection formula of $J_\nu^{(1)}(x; q)$ is essentially the case $r = 2$ of (1).

Since all of local solutions around the origin and the infinity are represented by convergent power series, we can determine the connection formula by a q-Laplace transformation [9]. We show a useful formula on p-Laplace transformation applied to q-difference equations ($p^m = q$) in section two.

We show a connection formula in section three. We study the q-differential equation

$$\left[x^r \prod_{k=1}^{r}(1 - a_k\sigma_p) - \left(-\frac{\sigma_p}{p^r}\right)^r\right] u(x) = 0.$$

Local solutions around the infinity are

$$u_{1,\infty}(x) = \frac{\theta_p(-a_1x)}{\theta_p(-x)}\, {}_r\phi_{r-1}\left(\begin{matrix} 0, 0, ..., 0 \\ p^r a_1/a_2, p^r a_2/a_3, ..., p^r a_1/a_r \end{matrix}; p^r, \frac{1}{a_1a_2\cdots a_r x^r}\right)$$

and $u_{2,\infty}(x), ..., u_{r,\infty}(x)$ are obtained by the cyclic transformation of $a_1, a_2, ..., a_r$.

We take a a primitive r-th root ω of unity. Local solutions around the origin are

$$u_{j,0}(x) = \frac{1}{\theta_p(-\omega^j p^{(1-r)/2}x)} v_j(x), \quad v_j(x) = \sum_{n=0}^{\infty} v_n^{(j)} x^n,$$

for $j = 0, 1, 2, ..., r-1$. We assume that $v_0^{(j)} = 1$. The connection formula between $(u_{0,0}, u_{1,0}, ..., u_{r-1,0})$ and $(u_{1,\infty}, ..., u_{r,\infty})$ is given by

$$v_j(x) = \frac{1}{(q, a_2/a_1, ..., a_r/a_1; q)_\infty} \frac{\theta_p(-\omega^j p^{(1-r)/2} a_1 x)\theta_p(-x)}{\theta_p(-\omega^j p^{(1-r)/2} x)\theta_p(-a_1 x)} u_{1,\infty}(x)$$
$$+ \operatorname{idem}(a_1; a_2, ..., a_r).$$

The symbol "idem $(a_1; a_2, ..., a_r)$" stands for the sum of the r expressions obtained from the preceding expression by interchanging a_1 with each $a_2, a_3, ..., a_r$.

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2 Notations

We denote the m-vetcor $(0, 0, ..., 0)$ by $\mathbf{0}_m$.

We assume that $0 < |q| < 1$. For $n = 0, 1, 2, ...$, we set the q-shifted factorial

$$(a; q)_n = \prod_{j=0}^{n-1}(1 - aq^j), \quad (a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

We set $(a_1, a_2, ..., a_m; q)_n = \prod_{j=1}^{m}(a_j; q)_n$ for $n = 0, 1, 2, ...$ or $n = \infty$.

We set the theta function

$$\theta_q(x) := \theta(x) = \sum_{k\in\mathbb{Z}} q^{k(k-1)/2}x^k = (q, -x, -q/x; q)_\infty .$$

The theta function satisfies

$$\theta_q(q^k x) = q^{-k(k-1)/2}x^{-k}\theta_q(x) \quad (k \in \mathbb{Z}),$$
$$x\theta_q(1/x) = \theta_q(x), \quad \theta_q(1/x) = \theta_q(qx).$$

The basic hypergeometric series [1] is defined by

$$\begin{aligned}&{}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)\\ &:= \sum_{n\geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n(q; q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} x^n.\end{aligned}$$

Let σ_q be a q-shift operator $\sigma_q[f(x)] = f(xq)$. When $1 + s \geq r$, ${}_r\phi_s$ is convergent and satisfies a q-difference equation with $(s + 1)$-th order

$$\left[x\prod_{j=1}^{r}(1 - a_j\sigma_q) - (1 - \sigma_q)\prod_{k=1}^{r-1}(1 - \frac{b_k}{q}\sigma_q)\right] y(x) = 0.$$

3 q-Borel transformation and q-Laplace transformation

We review a q-Borel transformation and a q-Laplace transformation. See [9, 3] for detail.

The q-Borel transformation $\mathcal{B}_q^-:\mathbb{C}[[x]]\to\mathbb{C}[[\tau]]$ is defined by

$$\mathcal{B}_q^-\left[\sum_{n=0}^{\infty}a_nx^n\right]:=\sum_{n=0}^{\infty}a_nq^{-n(n-1)/2}\tau^n.$$

We identify a germ of holomorphic functions at the origin $\mathcal{O}_{\mathbb{C},0}$ as a subset of $\mathbb{C}[[x]]$. As a linear operator on $\mathbb{C}[[x]]$, the following lemma is useful to study q-difference equations.

Lemma 1. *(1) The q-Borel transformation $\mathcal{B}_q^-$ shifts the power of σ_q:*

$$\mathcal{B}_q^-(x^m\sigma_q^nf)=q^{-m(m-1)/2}\tau^m\sigma_q^{n-m}\mathcal{B}_q^-(f).$$

(2) Multiplication by the theta function shifts the power of x:

$$x^m\sigma_q^n\left[\frac{1}{\theta_q(cx)}f(x)\right]=\frac{q^{n(n-1)/2}c^n}{\theta_q(cx)}x^{m+n}\sigma_q^nf(x).$$

The inverse transformation of $\mathcal{B}_q^-$ is given by a q-Laplace transform $\mathcal{L}_q^-$. Assume that $\varphi(\tau)$ is holomorphic on $|\tau|\leqq\varepsilon$. We define

$$\mathcal{L}_q^-\varphi(x)=\frac{1}{2\pi i}\int_{|\tau|=\varepsilon}\varphi(\tau)\theta_q(x/\tau)\frac{d\tau}{\tau}.$$

Under a suitable condition, we have $\mathcal{L}_q^-\circ\mathcal{B}_q^-f=f$.

We consider the p-Laplace transform of a ratio of p^m-products.

Proposition 2. *Let m be a positive integers. We set $p^m=q$. We assume that $s+m\leqq r$. When $s+m=r$, we need $|q^{(1+m)/2}b_1\cdots b_s/a_1a_2\cdots a_rx^m|<1$. We consider the contour integral*

$$I=\frac{1}{2\pi i}\int_{|\tau|=\varepsilon}\frac{\prod_{j=1}^{s}(b_j\tau;q)_\infty}{\prod_{k=1}^{r}(a_k\tau;q)_\infty}\theta_p(x/\tau)\,\frac{d\tau}{\tau},$$

where $\prod_{k=1}^{r}(a_k\tau;q)_\infty$ does not have any zero on $|\tau|\leqq\varepsilon$. Then we obtain

$$\begin{aligned}I=&\frac{(b_1/a_1,\ldots,b_s/a_1;q)_\infty}{(q,a_2/a_1,\ldots,a_r/a_1;q)_\infty}\theta_p(a_1x)\\&\times{}_{s+m}\phi_{r-1}\left(\begin{matrix}qa_1/b_1,\ldots,qa_1/b_s,\mathbf{0}_m\\qa_1/a_2,\ldots,qa_1/a_r\end{matrix};q,\frac{(-1)^rq^{r-s+(1-m)/2}b_1\cdots b_s}{a_1^{m+s-r+1}a_2\cdots a_rx^m}\right)\\&+\mathrm{idem}(a_1;a_2,\ldots,a_r).\end{aligned}\tag{2}$$

Proof. The following relations are directly proved :

$$\mathrm{Res}_{\tau=1/aq^n}\frac{1}{(a\tau;q)_\infty}\frac{d\tau}{\tau}=-\frac{(-1)^nq^{n(n+1)/2}}{(q;q)_\infty(q;q)_n},$$

$$\theta_p(x/\tau)|_{\tau\to 1/aq^n} = (ax)^{-mn}p^{-nm(nm-1)/2}\theta_p(ax),$$

$$(b\tau;q)_\infty|_{\tau\to 1/aq^n} = (-b/a)^n q^{-n(n+1)/2}(b/a;q)_\infty(aq/b;q)_n.$$

By using the above relations we can show Proposition. □

4 Connection formula

We consider the equation

$$\left[z\prod_{j=k}^{r}(1-a_k\sigma_q) - \left(-\frac{\sigma_q}{q}\right)^r\right]y(z) = 0. \tag{3}$$

Local solutions of (3) around the infinity are

$$y_{1,\infty}(z) = \frac{\theta_q(-a_1 z)}{\theta_q(-z)}\,{}_r\phi_{r-1}\left(\begin{matrix}0,0,...,0\\ qa_1/a_2, qa_j/a_3, ..., qa_1/a_r\end{matrix};q,\frac{1}{a_1\cdots a_r z}\right)$$

and $y_{2,\infty}(z), ..., y_{r,\infty}(z)$ are obtained by the cyclic transform $a_1 \to a_2 \to \cdots \to a_r \to a_1$.

Since (3) has ramified solutions around the origin, we take a covering transformation $z = x^r$. We set $p^r = q$.

$$\left[x^r\prod_{k=1}^{r}(1-a_k\sigma_p) - \left(-\frac{\sigma_p}{q}\right)^r\right]u(x) = 0. \tag{4}$$

We give a connection formula of (4). Local solutions of (4) around the infinity are

$$u_{1,\infty}(x) = \frac{\theta_p(-a_1 x)}{\theta_p(-x)}\,{}_r\phi_{r-1}\left(\begin{matrix}0,0,...,0\\ qa_1/a_2, qa_2/a_3, ..., qa_1/a_r\end{matrix};q,\frac{1}{a_1 a_2\cdots a_r x^r}\right)$$

and $u_{2,\infty}(x), ..., u_{r,\infty}(x)$ are obtained by the cyclic transform.

We take a complex number ω, which is a primitive r-th root of unity: $\omega^r = 1$. Local solutions of (4) around the origin are

$$u_{j,0}(x) = \frac{1}{\theta_p(-\omega^j p^{(1-r)/2}x)}v_j(x), \quad v_j(x) = \sum_{n=0}^{\infty} v_n^{(j)}x^n,$$

for $j = 0, 1, 2, ..., r-1$. We assume that $v_0^{(j)} = 1$. We show a connection formula between $(u_{0,0}, u_{1,0}, ..., u_{r-1,0})$ and $(u_{1,\infty}, ..., u_{r,\infty})$.

We set elementary symmetric polynomials $s_1, s_2, ..., s_r$ so that

$$\prod_{k=1}^{r}(1-a_k x) = \sum_{k=1}^{r}(-1)^k s_k x^k.$$

We set $c_j = -\omega^j p^{(1-r)/2}$. Then $v_j(x)$ satisfies a q-difference equation

$$\left[\sum_{k=1}^{r}(-1)^k c_j^k p^{k(k-1)/2} s_k x^k \sigma_p^k - \sigma_p^r\right] v_j(x) = 0.$$

We remark that $\sigma_p^r = \sigma_q$. Since $w_j(\tau) = (\mathcal{B}_p^- v_j)(\tau)$ satisfies

$$\left[\prod_{k=1}^{r}(1 - c_j a_k \tau) - \sigma_p^r\right] w_j(\tau) = 0,$$

we have

$$w_j(\tau) = \frac{1}{(c_j a_1 \tau, c_j a_2 \tau, ..., c_j a_r \tau; q)_\infty}.$$

By (2) we obtain

$$\begin{aligned} v_j(x) = \mathcal{L}_p^- w_j(x) &= \frac{1}{2\pi i}\int_{|\tau|=\varepsilon} \frac{1}{(c_j a_1 \tau, c_j a_2 \tau, ..., c_j a_r \tau; q)_\infty} \theta_p(x/\tau)\frac{d\tau}{\tau} \\ &= \frac{\theta_p(c_j a_1 x)}{(q, a_2/a_1, ..., a_r/a_1; q)_\infty} {}_r\phi_{r-1}\left(\begin{matrix} 0, 0, ..., 0 \\ q a_1/a_2, ..., q a_1/a_r \end{matrix}; q, \frac{(-1)^r q^{(1-r)/2}}{c_j^r a_1 a_2 \cdots a_r x^r}\right) \\ &\qquad + \mathrm{idem}(a_1; a_2, ..., a_r) \\ &= \frac{\theta_p(c_j a_1 x)}{(q, a_2/a_1, ..., a_r/a_1; q)_\infty} {}_r\phi_{r-1}\left(\begin{matrix} 0, 0, ..., 0 \\ q a_1/a_2, ..., q a_1/a_r \end{matrix}; q, \frac{1}{a_1 a_2 \cdots a_r x^r}\right) \\ &\qquad + \mathrm{idem}(a_1; a_2, ..., a_r). \end{aligned}$$

We remark that $c_j^r = (-1)^r q^{(1-r)/2}$.

The main result is as follows:

Theorem 3. *We take a primitive r-th root ω of unity. A connection formula of* (4) *is given by*

$$\begin{aligned} u_{j,0}(x) = \frac{1}{(q, a_2/a_1, ..., a_r/a_1; q)_\infty} \frac{\theta_p(-\omega^j p^{(1-r)/2} a_1 x)\theta_p(-x)}{\theta_p(-\omega^j p^{(1-r)/2} x)\theta_p(-a_1 x)} u_{1,\infty}(x) \\ + \mathrm{idem}(a_1; a_2, ..., a_r), \end{aligned}$$

for $j = 0, 1, ..., r-1$.

The case $r = 2$:
We set $p^2 = q$. We take a p-difference equation

$$\left[p^2 x^2 (1 - a_1 \sigma_p)(1 - a_2 \sigma_p) - \sigma_p^2\right] u(x) = 0. \tag{5}$$

We give a connection formula of (5). Local solutions of (5) around the infinity are

$$u_{1,\infty}(x)=\frac{\theta_p(-a_1x)}{\theta_p(-x)}{}_2\phi_1\left(0,0;qa_1/a_2;q,\frac{1}{a_1a_2x^2}\right),$$
$$u_{2,\infty}(x)=\frac{\theta_p(-a_2x)}{\theta_p(-x)}{}_2\phi_1\left(0,0;qa_2/a_1;q,\frac{1}{a_1a_2x^2}\right).$$

Local solutions of (5) around the origin are

$$u_{0,1}(x)=\frac{1}{\theta_p(-p^{-1/2}x)}v_1(x),\qquad v_1(x)=\sum_{n=0}^{\infty}v_n^{(1)}x^n,$$
$$u_{0,2}(x)=\frac{1}{\theta_p(p^{-1/2}x)}v_2(x),\qquad v_2(x)=\sum_{n=0}^{\infty}v_n^{(2)}x^n.$$

We assume that $v_0^{(j)}=1$ for $j=1,2$. By Theorem 3 we obtain

$$u_{0,1}(x)=\frac{1}{(q,a_2/a_1;q)_\infty}\frac{\theta_p(-p^{1/2}a_1x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_1x)}u_{1,\infty}(x)+\frac{1}{(q,a_1/a_2;q)_\infty}\frac{\theta_p(-p^{1/2}a_2x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_2x)}u_{2,\infty}(x),$$

$$u_{0,2}(x)=\frac{1}{(q,a_2/a_1;q)_\infty}\frac{\theta_p(p^{1/2}a_1x)\theta_p(-x)}{\theta_p(p^{1/2}x)\theta_p(-a_1x)}u_{1,\infty}(x)+\frac{1}{(q,a_1/a_2;q)_\infty}\frac{\theta_p(-p^{1/2}a_2x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_2x)}u_{2,\infty}(x).$$

This connection formula is essentially equivalent to the connection formula of Jackson's first q-Bessel functions in [11].

5 Conclusion

We show a connection formula of (4), which is a generalization of Jackson's first q-analogue of the Bessel functions [2]. We can obtain a connection formula of solutions represented by a convergent (non-hypergeometric) series of $x^{1/m}$ by applying the p-Laplace transformation (2) to a product of p^m-shifted factorials p for other q-hypergeometric equations.

We should study the q-Stokes phenomenon [3] for divergent series solutions. By using the other q-Borel transformation $\mathcal{B}_q^+$, we can give a resummation for divergent hypergeometric series.

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