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Uniqueness in π-Regular Unital Rings

By

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Abstract

We establish that a ring is uniquely π-regular if, and only if, it is a division ring. This somewhat improves on our result proved for uniquely von Neumann regular rings in Palestine J. Math. (2018).

1 Introduction and Background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which differs from the zero element 0. Our terminology and notations are mainly in agreement with [4] and [5]. For instance, a ring R is called *regular* in the sense of von Neumann if, for any $r \in R$, there is $a \in R$ such that $r = rar$. Likewise, a ring R is called *strongly regular* if, for any $r \in R$, there is $x \in R$ with $r = r^2x$. It is well known that strongly regular rings are exactly the reduced regular rings which are a subdirect product of division rings. However, there exist even finite commutative rings which are *not* regular; e.g., this is the ring $\mathbb{Z}_4$. That is why, a substantial generalization of these two classes is needed as follows: A ring R is said to be π*-regular* if, for each $r \in R$, there are $n \in \mathbb{N}$ and $b \in R$ which both depend on r with $r^n = r^nbr^n$. Also, a ring R is said to be *strongly* π*-regular* if, for each $r \in R$, there are $n \in \mathbb{N}$ and $y \in R$ which both depend on r with $r^n = r^{n+1}y$. Now, all finite rings (and even much more, all artinian rings) are known to be (strongly) π-regular.

In [2] was studied *uniquely regular* rings as those rings R for which each elements $r \in R$ possesses a unique inner addition $a \in R$ such that $r = rar$. There was proved that these are precisely the division rings. Our aim here is to enlarge this affirmation to the classes of π-regular and strongly π-regular rings.

So, we come to our basic tools.

Definition 1. We shall say that a ring R is *uniquely π-regular* if, for every $r \in R$, there are $n \in \mathbb{N}$ and unique $b \in R$ both depending on r such that the equality $r^n = r^n b r^n$ is valid.

Definition 2. We shall say that a ring R is *uniquely strongly π-regular* if, for every $r \in R$, there are $n \in \mathbb{N}$ and unique $y \in R$ both depending on r such that the equality $r^n = r^{n+1} y$ is fulfilled.

The objective here is to find a necessary and sufficient condition when an arbitrary ring is uniquely π-regular as well as uniquely strongly π-regular. This will be successfully done in the next section.

2 Main Results

Here we proceed by proving the major assertion that motivated the writing of this article.

Theorem. *A ring is uniquely π-regular ring if, and only if, it is a division ring.*

Proof. "**Necessity**". Let R be a uniquely π-regular ring. We foremost claim that R is without non-trivial idempotents and nilpotents. In fact, if e is a non-zero idempotent, then for all $m \in \mathbb{N}$ one may write that

$$e = e^m = e^m \cdot 1 \cdot e^m = e^m \cdot e \cdot e^m$$

which enables us that $e = 1$, as required.

As for the freeness of nilpotent elements, take $z \in R$ for which $z^2 = 0$. Assume in a way of contradiction that there exists $n \in \mathbb{N}$ such that $z^n = z^n d z^n$ for some unique $d \in R$. If $n \geq 2$ it follows that $z^n = 0$ and hence $0 = 0.d.0 = 0.h.0$ for any $h \in R$ with $h \neq d$. But this is impossible. So, $n = 1$ and we write $z = zdz$. One sees that

$$z(d(1 - z(1 - zd)))z = zdz = z((1 - (1 - dz)z)d)z.$$

Consequently, using the uniqueness, one can deduce that

$$d - dz(1 - zd) = d = d - (1 - dz)zd.$$

These two relations allow us to conclude that $dz = zd$ whence $z = z^2 d = 0$, contrary to our assumption. This finally shows that R does not have non-zero nilpotents, as claimed.

Furthermore, given $0 \neq r \in R$, there are $n \in \mathbb{N}$ and unique $b \in R$ which both depend on r such that the equality $r^n = r^n b r^n$ holds. Writing

$$r^n(b(1-r^n(1-r^nb)))r^n = r^nbr^n = r^n((1-(1-br^n)r^n)b)r^n.$$

we extract with the aid of uniqueness of the inner element that

$$b - br^n(1-r^nb) = b = b - (1-br^n)r^nb.$$

which amounts to $br^n = r^nb$ yielding $r^n = r^{2n}b$. This, however, guarantees that R must be strongly π-regular, say $r^n = r^{n+1}y$ for $y = r^{n-1}b \in R$. According to [1] and [3], with no loss of generality we may assume that $ry = yr$. One next observes that $r^ny^n = (ry)^n$ is an idempotent. In fact, multiplying both sides of the equality $r^n = r^{n+1}y$ by r we obtain that $r^{n+1} = r^{n+2}y$ and so substituting it again in the initial equality, we infer that $r^n = r^{n+2}y^2$, etc., after a final number of steps, we get that $r^n = r^{2n}y^n$ (actually, in our situation, $b = y^n$). Now, by what we have detected so far, one checks that $r^ny^n.r^ny^n = (r^{2n}y^n)y^n = r^ny^n$, which substantiates our assertion. Next, utilizing the lack of non-trivial idempotents established above, it follows that either $r^ny^n = 0$ and hence $r^n = r^{2n}y^n = 0$, or $r^ny^n = y^nr^n = 1$. In the first case, the lack of non-trivial nilpotents assures that $r = 0$. The second case implies that r inverts in R, i.e., R is a division ring, as wanted.

"**Sufficiency**". Suppose now R is a division ring. It is self-evident that every non-zero element $r \in R$ can be uniquely written as $r^i = r^ir^{-i}r^i$ for all positive integers i, as required. Therefore, R is a uniquely π-regular ring, as stated. □

As an immediate consequence, we derive the following criterion.

Corollary. *The next four statements are equivalent for a ring R:*

(1) R is uniquely π-regular.

(2) R is uniquely strongly π-regular.

(3) R is uniquely regular.

(4) R is a division ring.

In closing, we give some additional comments: Recall that a ring R is π-*boolean* if, for any $r \in R$, there exists $k \in \mathbb{N}$ with $r^{2k} = r^k$. Apparently, boolean rings are themselves π-boolean choosing $k = 1$.

Let us now R be such a ring that for every its element r there exist $n \in \mathbb{N}$ and invertible $u \in R$ such that $r^n = r^nur^n$. Clearly, the well-known unit-regular rings are so by taking $n = 1$. What we can say about the structure of R if the existing invertible element u is unique for each that r? Is the ring R either a π-boolean ring or a division ring?

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