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On Global Solvability for Some Degenerate Dissipative Nonlinear Kirchhoff Strings

By

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Abstract

Consider the initial boundary value problem for degenerate dissipative wave equations of Kirchhoff type with attractive force terms. We are interested in the case of $0 < \gamma < 1$ for the degeneracy of nonlinear term $\Phi(r) = r^\gamma$. We prove the global solvability problem, provided that the initial data belong to the potential well and satisfy a suitable smallness condition. Moreover, we derive optimal decay estimates of the solutions.

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1 Introduction

In this paper, we investigate on the global existence and decay estimates of solutions to the initial boundary value problem for the following degenerate dissipative wave equations of Kirchhoff type with the attractive force term :

$$\begin{cases} u_{tt} + u_t = \Phi\left(\displaystyle\int_0^\ell |u_x(x,t)|^2\,dx\right) u_{xx} + f(u) \quad \text{in } (0,\ell)\times(0,\infty)\,, \\ u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u(0,t) = u(\ell,t) = 0\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, $u_t = \partial_t u = \partial u/\partial t$, $u_x = \partial_x u = \partial u/\partial x$, $\ell > 0$, and

$$\Phi(r) = r^\gamma \quad \text{with} \quad \gamma > 0 \quad \text{and} \quad f(u) = |u|^p u \quad \text{with} \quad p > 0\,.$$

Equation (1.1) describes small amplitude vibrations of an elastic stretched string. Kirchhoff [9] first studied such integrate-differential equations without any dissipation (see [3], [5], [13]).

We define the energy $E(u,u_t)$ and the potential $J(u)$ associated with the degenerate equation (1.1) by

$$E(u,u_t) \equiv \|u_t\|^2 + J(u) \tag{1.2}$$

and

$$J(u) \equiv \frac{1}{\gamma+1}\|u_x\|^{2(\gamma+1)} - \frac{2}{p+2}\|u\|_{p+2}^{p+2}, \tag{1.3}$$

respectively. We introduce the potential well $\mathcal{W}$ by

$$\mathcal{W} \equiv \{u \in H_0^1 \mid J(u) < d,\ K(u) \geq 0\}, \tag{1.4}$$

where

$$K(u) \equiv \|u_x\|^{2(\gamma+1)} - \|u\|_{p+2}^{p+2} \tag{1.5}$$

and the potential well depth d is defined by

$$d = \inf\{J(u) \mid K(u) = 0,\ u \neq 0\} \tag{1.6}$$

(see [8], [12], [19], [22]). If $p > 2\gamma$, it is easy to see that

$$\begin{aligned} J(u) &= \frac{1}{\gamma+1}K(u) + \frac{p-2\gamma}{(\gamma+1)(p+2)}\|u\|_{p+2}^{p+2} \\ &= \frac{2}{p+2}K(u) + \frac{p-2\gamma}{(\gamma+1)(p+2)}\|u_x\|^{2(\gamma+1)}, \end{aligned}$$

and hence,

$$J(u) \geq \max\left\{\frac{1}{\gamma+1}K(u),\ \frac{p-2\gamma}{(\gamma+1)(p+2)}\|u_x\|^{2(\gamma+1)}\right\}. \tag{1.7}$$

Moreover, when $u \in \mathcal{W}$, we have

$$K(u) \geq \left(1 - \left(\frac{J(u)}{d}\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\right)\|u_x\|^{2(\gamma+1)}. \tag{1.8}$$

Indeed, taking $\lambda > 0$ such that $K(\lambda u) = 0$ for $u \neq 0$, that is,

$$K(\lambda u) = \lambda^{2(\gamma+1)}\|u_x\|^{2(\gamma+1)} - \lambda^{p+2}\|u\|_{p+2}^{p+2} = 0,$$

we have

$$\lambda^{p-2\gamma}\|u\|_{p+2}^{p+2} = \|u_x\|^{2(\gamma+1)} \quad \text{and} \quad \lambda = \left(\frac{\|u_x\|^{2(\gamma+1)}}{\|u\|_{p+2}^{p+2}}\right)^{\frac{1}{p-2\gamma}} \tag{1.9}$$

and $\lambda \geq 1$ by $u \in \mathcal{W}$. On the other hand, we have

$$d \leq J(\lambda u) = \frac{\lambda^{2(\gamma+1)}}{\gamma+1}\|u_x\|^{2(\gamma+1)} - \frac{2\lambda^{p+2}}{p+2}\|u\|_{p+2}^{p+2} \leq \lambda^{2(\gamma+1)}J(u). \tag{1.10}$$

Thus, we observe from (1.9) and (1.10) that

$$K(u) = \left(1 - \left(\frac{1}{\lambda}\right)^{p-2\gamma}\right)\|u_x\|^{2(\gamma+1)} \geq \left(1 - \left(\frac{J(u)}{d}\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\right)\|u_x\|^{2(\gamma+1)}.$$

When the initial data belong to usual Sobolev spaces, Arosio and Garavaldi [1] have carried out detailed analysis about the existence of local solutions for the Kirchhoff type equations (also see [2], [4], [15] and the references cited therein).

In order to prove the existence of global solutions, we need to derive suitable a-priori estimates including the uniformly estimates for the second order derivatives in addition to the usual energy estimate, which is the main difficulty of problems for Kirchhoff type equations.

In the case of non-degenerate type $\Phi(r) \geq C_0 > 0$ (e.g. $\Phi(r) = 1 + r^\gamma$), Hosoya and Yamada [7] have proved the exponential decay estimates and the global existence of solutions under small data conditions (see also [16]).

In the case of degenerate type $\Phi(r) \geq 0$ (e.g. $\Phi(r) = r^\gamma$), the situations are more delicate and difficult. Fortunately, applying the general theory on the energy decay of hyperbolic equations in [11], we see that the energy decays at a certain algebraic rate. In particular, when $f(u) \equiv 0$, we have derived the detailed estimates of the solutions in previous paper [18] (also see [6], [14] and the references cited therein).

When $\Phi(r) = r^\gamma \in C^1([0,\infty))$ (i.e. $\gamma \geq 1$), under the conditions that $p > 2\gamma$, $u_0 \in \mathcal{W}$, $u_0 \neq 0$, and the initial data are small, we have proved the global existence of solutions for (1.1), and we have derived some upper decay estimates of the solutions in [16] (also see [17] for decay properties, and [15] for $f(u) = -|u|^p u$). In order to get the a-priori estimate in $H^2 \times H^1$, we used the function $H(t) \equiv \|u_{xt}(t)\|^2/\|u_x(t)\|^{2\gamma} + \|u_{xx}(t)\|^2$ when $\gamma \geq 1$.

However, in the case of $0 < \gamma < 1$, the method in [16] can not be applied directly to the problem (1.1). Since $\Phi(r)$ is not C^1 at the origin, this situation is more delicate and difficult. To prove the existence of global solutions of (1.1) for $\gamma > 0$, we need to modify the function $H(t)$ including the $H^2 \times H^1$ norm of $[u(t), u_t(t)]$. The main difficulty is generated by the degeneracy of $\Phi(r) \equiv r^\gamma$ with $0 < \gamma < 1$. A key point of the analysis is to show that the non-local term $\Phi(\|u_x(t)\|^2) > 0$ for each time t and the decay rate of the H^2 norm of the solution is $-1/\gamma$ which is optimal (see (1.11)).

In what follows, we denote $E(t) \equiv E(u(t), u_t(t))$, $J(t) \equiv J(u(t))$, $K(t) \equiv K(u(t))$ for simplicity of the notations. Moreover, we denote the Sobolev–Poincaré constant by c_*, that is, $\|v\|_p \leq c_*\|v_x\|$ for $1 \leq p \leq \infty$.

Our purpose in this paper is to the existence of global solutions of (1.1) in the case of $\gamma > 0$ (in particular $0 < \gamma < 1$) and to derive the detailed decay estimates of the solutions.

Our main result is as follows.

Theorem 1.1 *Let the initial data* $[u_0, u_1]$ *belong to* $H^2 \cap \mathcal{W} \times H_0^1$ *and* $u_0 \neq 0$. *Suppose that* $p > 2\gamma$. *There exists* ε_0 $(0 < \varepsilon_0 < d)$ *such that if* $E(0) \equiv E(u_0, u_1) \leq \varepsilon$ *for* $\varepsilon \leq \varepsilon_0$ *(see* (3.1) *and* (3.2)*), then the problem* (1.1) *admits a global solution* $u(t)$ *in the class* $C^0([0,\infty); H^2 \cap \mathcal{W}) \cap C^1([0,\infty); H_0^1) \cap C^2([0,\infty); L^2)$ *and the solution* $u(t)$ *satisfies*

$$C'(1+t)^{-\frac{1}{\gamma}} \leq \|\partial_x^k u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \quad \text{for} \quad k = 0, 1, 2, \tag{1.11}$$

$$\|\partial_x^j \partial_t u(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \quad \text{for} \quad j = 0, 1, \tag{1.12}$$

$$\|\partial_t^2 u(t)\|^2 \leq C(1+t)^{-3-\frac{1}{\gamma}} \quad \text{for} \quad t \geq 0, \tag{1.13}$$

where C *and* C' *are some positive constants depending on the initial data* $[u_0, u_1]$.

Theorem 1.1 follows from Theorems 3.1–4.4 in the continuing sections, and Theorem 1.1 can be applied to Equation (1.1) with the nonlinear term $f(u) = \pm|u|^{p+1}$.

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2 = L^2(\Omega)$ with $\Omega = (0, \ell)$ or sometimes duality between the space X and its dual X'. The spaces $H^k = H^k(\Omega)$ and $L^q = L^q(\Omega)$ have the usual norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_q$ ($\|\cdot\| = \|\cdot\|_2$ for $q = 2$), respectively. We put $(a)^+ = \max\{0, a\}$ where $1/(a)^+ = \infty$ if $(a)^+ = 0$. Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

The proof of the following local existence theorem is standard and we omit it here (see [2], [15], [20], [21]).

Theorem 2.1 *Suppose that the initial data* $[u_0, u_1]$ *belong to* $H^2 \cap H_0^1 \times H_0^1$ *and* $u_0 \neq 0$. *Then, the problem* (1.1) *admits a local solution* $u(t)$ *in the class* $C^0([0,T); H^2 \cap H_0^1) \cap C^1([0,T); H_0^1) \cap C^2([0,T); L^2)$ *for some* $T > 0$. *Moreover, if* $\|u_x(t)\| > 0$ *and* $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} < \infty$ *for* $0 \leq t \leq T$, *we can take* $T = \infty$.

In what follows, we denote $M(t) \equiv \|u_x(t)\|^2$ for simplicity of the notation.

Proposition 2.2 *Let* $u(t)$ *be a solution of* (1.1). *Suppose that* $u_0 \in \mathcal{W}$ *and* $E(0) < d$ *and* $p > 2\gamma$. *Then, it holds that*

$$\kappa^{-1} M(t)^{\gamma+1} \leq E(t) < d \tag{2.1}$$

and

$$(\gamma+1)\delta J(t) \le \delta M(t)^{\gamma+1} \le K(t) \le (\gamma+1)J(t) \tag{2.2}$$

where $\kappa > 0$ *and* $0 < \delta < 1$ *are defined by*

$$\kappa = \frac{(\gamma+1)(p+2)}{p-2\gamma} \quad \text{and} \quad \delta = \left(1 - \left(\frac{E(0)}{d}\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\right). \tag{2.3}$$

Proof. Multiplying (1.1) by u_t and integrating it over $\Omega = (0, \ell)$, we have

$$\frac{d}{dt}E(t) + 2\|u_t(t)\|^2 = 0 \tag{2.4}$$

and

$$E(t) + 2\int_0^t \|u_t(s)\|^2\, ds = E(0). \tag{2.5}$$

From (1.2), (1.7), and (2.5), we observe that

$$\frac{p-2\gamma}{(\gamma+1)(p+2)} M(t)^{\gamma+1} \le J(t) \le E(t) \le E(0) < d \tag{2.6}$$

which implies (2.1). Thus, from (1.8) and (2.6) we observe that

$$K(t) \ge \left(1 - \left(\frac{J(t)}{d}\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\right) M(t)^{\gamma+1} \ge \delta M(t)^{\gamma+1}, \tag{2.7}$$

and hence, from (1.3), (1.7), and (2.7) we obtain the desired estimate (2.2). □

In what follows, let $u(t)$ be a solution and we assume that

$$E(0) \le \min\{1, d\}. \tag{2.8}$$

Proposition 2.3 *Under the assumption of Proposition 2.2, the energy* $E(t)$ *satisfies that*

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(t-1)^+\right)^{-\frac{\gamma+1}{\gamma}} \tag{2.9}$$

where $d_1 = (\gamma+1)\gamma^{-1}(2(2+\delta^{-1}) + 5\delta^{-1}\kappa^{\frac{1}{2(\gamma+1)}})^2$ *is a positive constant.*

Proof. Integrating (2.4) over $[t, t+1]$, we observe

$$2\int_t^{t+1} \|u_t(s)\|^2\, ds = E(t) - E(t+1) \quad (\equiv 2D(t)^2). \tag{2.10}$$

There exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_j)\|^2 \le 4D(t)^2 \quad \text{for} \quad j = 1, 2. \tag{2.11}$$

On the other hand, multiplying (1.1) by $u(t)$ and integrating it over $\Omega \times [t_1, t_2]$, we have from (2.10) and (2.11) that

$$\begin{aligned}
\int_{t_1}^{t_2} K(s)\,ds &= \int_{t_1}^{t_2} \left(\|u_t(s)\|^2 - \frac{d}{dt}(u_t(s), u(s)) - (u_t(s), u(s)) \right) ds \\
&\le \int_t^{t+1} \|u_t(s)\|^2\,ds + \sum_{j=1}^{2} \|u_t(t_j)\| \|u(t_j)\| + \int_t^{t+1} \|u_t(s)\| \|u(s)\|\,ds \\
&\le D(t)^2 + 5D(t) \sup_{t \le s \le t+1} \|u(s)\|
\end{aligned}$$

and from (2.2), (2.10), and (2.11) that

$$\begin{aligned}
\int_{t_1}^{t_2} E(s)\,ds &= \int_{t_1}^{t_2} \left(\|u_t(s)\|^2 + J(s) \right) ds \\
&\le \int_t^{t+1} \|u_t(s)\|^2\,ds + \delta^{-1} \int_{t_1}^{t_2} K(s)\,ds \\
&\le (1+\delta^{-1}) D(t)^2 + 5\delta^{-1} D(t) \sup_{t \le s \le t+1} \|u(s)\|
\end{aligned} \tag{2.12}$$

Moreover, integrating (2.4) over $[t, t_2]$ we have from (2.10) and (2.12) that

$$\begin{aligned}
E(t) &= E(t_2) + 2\int_t^{t_2} \|u_t(s)\|^2\,ds \\
&\le 2\int_{t_1}^{t_2} E(s)\,ds + 2\int_t^{t+1} \|u_t(s)\|^2\,ds \\
&\le 2(2+\delta^{-1}) D(t)^2 + 5\delta^{-1} D(t) \sup_{t \le s \le t+1} \|u(s)\|\,.
\end{aligned}$$

Since it follows from the Sobolev–Poincaré inequality and (2.1) and (2.4) that

$$\sup_{t \le s \le t+1} \|u(s)\| \le \sup_{t \le s \le t+1} c_* M(s)^{\frac{1}{2}} \le c_* (\kappa E(t))^{\frac{1}{2(\gamma+1)}}\,, \tag{2.13}$$

and from (2.8) and (2.10) that

$$D(t) \le E(t)^{\frac{1}{2}} \le E(0)^{\frac{\gamma}{2(\gamma+1)}} E(t)^{\frac{1}{2(\gamma+1)}} \le E(t)^{\frac{1}{2(\gamma+1)}}\,,$$

we have

$$E(t) \le (2(2+\delta^{-1}) + 5\delta^{-1}\kappa^{\frac{1}{2(\gamma+1)}}) D(t) E(t)^{\frac{1}{2(\gamma+1)}}\,,$$

and from (2.10) that

$$E(t)^{1+\frac{\gamma}{\gamma+1}} \le (2(2+\delta^{-1})+5\delta^{-1}\kappa^{\frac{1}{2(\gamma+1)}})^2(E(t)-E(t+1)). \qquad (2.14)$$

Thus, applying Lemma 2.4 to (2.14), we obtain the desired estimate (2.9). □

In order to derive the energy decay, we used the following inequality (see Nakao [11] and [12] for the proof).

Lemma 2.4 *Let $\phi(t)$ be a non-increasing non-negative function on $[0,\infty)$ and satisfy*

$$\phi(t)^{1+\alpha} \le k\,(\phi(t)-\phi(t+1))$$

with certain constants $k \ge 0$ and $\alpha > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \le \left(\phi(0)^{-\alpha}+\alpha k^{-1}(t-1)^+\right)^{-\frac{1}{\alpha}} \quad \text{for } t \ge 0.$$

Corollary 2.5 *If $q > \gamma$, then it holds that*

$$\int_0^t M(s)^q\,ds \le d_2 E(0)^{\frac{q-\gamma}{\gamma+1}} \qquad (2.15)$$

where $d_2 = \kappa^{\frac{q}{(\gamma+1)}}(1+\gamma(q-\gamma)^{-1}d_1)$ is a positive constant.

Proof. From (2.1) and (2.9) we observe that

$$\begin{aligned}\int_0^t M(s)^q\,ds &\le \left(\int_0^1+\int_1^t\right)(\kappa E(s))^{\frac{q}{\gamma+1}}\,ds\\ &\le \kappa^{\frac{q}{\gamma+1}}\left(E(0)^{\frac{q}{\gamma+1}}+\int_1^t\left(E(0)^{-\frac{\gamma}{\gamma+1}}+d_1^{-1}(s-1)\right)^{-\frac{q}{\gamma}}ds\right)\\ &\le \kappa^{\frac{q}{\gamma+1}}\left(E(0)^{\frac{q}{\gamma+1}}+\frac{\gamma}{q-\gamma}d_1E(0)^{\frac{q-\gamma}{\gamma+1}}\right)\end{aligned}$$

and we obtain (2.15). □

We introduce the function $\mu(t)$ by

$$\mu(t) \equiv \sup_{0\le s\le t}\frac{\|u_t(s)\|^2}{M(s)^{2\gamma+1}}. \qquad (2.16)$$

Proposition 2.6 *Suppose that*

$$p > 2\gamma \quad \text{and} \quad \frac{|M'(t)|}{M(t)} \le \frac{1}{\gamma+1} \qquad (2.17)$$

and the initial energy $E(0)$ *satisfies*

$$2^2 c_*^{p+2}(\kappa E(0))^{\frac{1}{\gamma+1}} < 1\,. \tag{2.18}$$

Then, it holds that

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le G(t) \le 2\left(G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}}\mu(t)^{\frac{1}{2}}\right)^2 \tag{2.19}$$

where $d_0 = 2(\gamma+1)(p+1)c_*^p d_2$ *is a positive constant, and* $G(t)$ *is defined by*

$$G(t) \equiv \frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t) + \frac{2}{M(t)^{\gamma+1}}(f(u(t)), u_{xx}(t)) \tag{2.20}$$

with

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(M(t)\|u_{xt}(t)\|^2 - \frac{1}{4}|M'(t)|^2\right)\,. \tag{2.21}$$

Proof. We observe from the definition of $Q(t)$ that

$$\frac{\|u_{xt}(t)\|^2}{M(t)^\gamma} \ge Q(t) \ge 0\,, \tag{2.22}$$

and from the Sobolev-Poincaré inequality and (2.6) that

$$\begin{aligned} \frac{2|(f(u(t)), u_{xx}(t))|}{M(t)^{\gamma+1}} &\le \frac{2c_*^{p+2}}{M(t)^{\gamma+1}}\|u_x(t)\|^p\|u_{xx}(t)\|^2 \\ &\le 2c_*^{p+2}M(t)^{\frac{1}{2}(p-2\gamma)}\frac{\|u_{xx}(t)\|^2}{M(t)} \\ &\le 2c_*^{p+2}(\kappa E(0))^{\frac{1}{\gamma+1}}\frac{\|u_{xx}(t)\|^2}{M(t)}\,. \end{aligned} \tag{2.23}$$

If $E(0)$ is small such that

$$2c_*^{p+2}(\kappa E(0))^{\frac{1}{\gamma+1}} < \frac{1}{2}\,, \tag{2.24}$$

we have

$$\frac{1}{2}\frac{\|u_{xx}(t)\|^2}{M(t)} \le G(t) \le 2\frac{\|u_{xx}(t)\|^2}{M(t)} + \frac{\|u_{xt}(t)\|^2}{M(t)^\gamma}\,. \tag{2.25}$$

Using Equation (1.1), we observe

$$\begin{aligned} \frac{d}{dt}\frac{\|u_{xx}(t)\|^2}{M(t)} &= \frac{1}{M(t)^{\gamma+2}}\left(2(M(t)^\gamma u_{xx}, u_{xxt})M(t) - (M(t)^\gamma u_{xx}, u_{xx})M'(t)\right) \\ &= \frac{-2}{M(t)^{\gamma+2}}\left(\|u_{xt}(t)\|^2 + (u_{xtt}, u_{xt}) + \frac{d}{dt}(f(u), u_{xx}) - ((f(u))_t, u_{xx})\right)M(t) \\ &\quad + \frac{1}{M(t)^{\gamma+2}}\left(\frac{1}{2}M'(t) - \|u_{xt}(t)\|^2 + \frac{1}{2}M''(t) - ((f(u))_x, u_x)\right)M'(t) \end{aligned}$$

and

$$\frac{d}{dt}\frac{(f(u),u_{xx})}{M(t)^{\gamma+1}} = \frac{1}{M(t)^{\gamma+1}}\frac{d}{dt}(f(u),u_{xx}) + (\gamma+1)\frac{M'(t)}{M(t)^{\gamma+2}}((f(u))_x,u_x).$$

Thus, we have

$$\frac{d}{dt}\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + \frac{2((f(u))_x,u_x)}{M(t)^{\gamma+1}}\right) = -2Q(t) - R(t) + S(t), \tag{2.26}$$

where $Q(t)$ is defined by (2.21) and

$$R(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(2(u_{xtt},u_{xt})M(t) + \left(\|u_{xt}(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)\right),$$

$$S(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left((2\gamma+1)((f(u))_x,u_x)M'(t) + 2((f(u))_t,u_{xx})M(t)\right).$$

On the other hand, we observe

$$\frac{d}{dt}Q(t) = -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) + R(t). \tag{2.27}$$

Summing up (2.26) and (2.27), we have

$$\frac{d}{dt}G(t) + 2\left(1 + \frac{\gamma+2}{2}\frac{M'(t)}{M(t)}\right)Q(t) = S(t), \tag{2.28}$$

where $G(t)$ is defined by (2.20).

Moreover, we observe from (2.17) that

$$1 + \frac{\gamma+2}{2}\frac{M'(t)}{M(t)} \geq 0$$

and from the Sobolev-Poincaré inequality that

$$\begin{aligned}
|S(t)| &\leq \frac{2(2\gamma+1)(p+1)}{M(t)^{\gamma+2}}\|u(t)\|_\infty^p\|u_x(t)\|^2\|u_t(t)\|\|u_{xx}(t)\| \\
&\quad + \frac{2(p+1)}{M(t)^{\gamma+1}}\|u(t)\|_\infty^p\|u_t(t)\|\|u_{xx}(t)\| \\
&\leq \frac{4(\gamma+1)(p+1)c_*^p}{M(t)^{\gamma+1}}\|u_t(t)\|\|u_{xx}(t)\|\|u_x(t)\|^p \\
&\leq 4(\gamma+1)(p+1)c_*^p\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}G(t)^{\frac{1}{2}}M(t)^{\frac{p}{2}}.
\end{aligned}$$

Thus, we have from (2.28) that

$$\frac{d}{dt}G(t) \leq 4(\gamma+1)(p+1)c_*^p\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}G(t)^{\frac{1}{2}}M(t)^{\frac{p}{2}}$$

or

$$\frac{d}{dt}G(t)^{\frac{1}{2}} \le 2(\gamma+1)(p+1)c_*^p \frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}} M(t)^{\frac{p}{2}}.$$

If $p > 2\gamma$, we observe from Corollary 2.5 that

$$\begin{aligned} G(t)^{\frac{1}{2}} &\le G(0)^{\frac{1}{2}} + 2(\gamma+1)(p+1)c_*^p \mu(t)^{\frac{1}{2}} \int_0^t M(s)^{\frac{p}{2}}\, ds \\ &\le G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}} \mu(t)^{\frac{1}{2}}, \quad d_0 = 2(\gamma+1)(p+1)c_*^p d_2, \qquad (2.29) \end{aligned}$$

and hence, from (2.25) and (2.29) we obtain the desired estimate (2.19). □

Proposition 2.7 *Under the assumption of Proposition 2.6, suppose that the initial energy $E(0)$ satisfies*

$$2^7(\gamma+1)^2 d_0^2 E(0)^{\frac{p-2\gamma}{\gamma+1}} < 1. \qquad (2.30)$$

Then, it holds that

$$\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} \le B(0) \qquad (2.31)$$

and

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le 2\left(G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}} B(0)^{\frac{1}{2}}\right)^2, \qquad (2.32)$$

where $B(0)$ is defined by

$$B(0) \equiv \max\left\{ \frac{\|u_1\|^2}{M(0)^{2\gamma+1}},\ 2^7(\gamma+1)^2 G(0) \right\}. \qquad (2.33)$$

Proof. Multiplying (1.1) by $2u_t$ and $M(t)^{-\gamma-1}$ and integrating it over Ω, we have from the Sobolev-Poincaré inequality and (2.6) that

$$\begin{aligned} &\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} + 2\left(1 + \frac{2\gamma+1}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} \\ &= -\frac{M'(t)}{M(t)^{\gamma+1}} + \frac{2}{M(t)^{2\gamma+1}}(f(u), u_t) \\ &\le \frac{2}{M(t)^{\gamma+1}}\|u_t(t)\|\|u_{xx}(t)\| + \frac{2c_*^{p+2}}{M(t)^{\gamma+1}}\|u_x(t)\|^p\|u_{xx}(t)\|\|u_t(t)\| \\ &\le 2\left(1 + c_*^{p+2}M(t)^{\frac{1}{2}(p-2\gamma)}\right)\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}} \\ &\le 2\left(1 + c_*^{p+2}(\kappa E(0))^{\frac{p-2\gamma}{2(\gamma+1)}}\right)\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}} \\ &\le 2^2\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}}, \end{aligned}$$

where we used (2.18) at the last inequality. Since it follows from (2.17) that

$$1 + \frac{2\gamma+1}{2}\frac{M'(t)}{M(t)} \geq \frac{1}{2(\gamma+1)},$$

we observe from the Young inequality and (2.19) that

$$\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{2(\gamma+1)}\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} \leq 2^3(\gamma+1)\frac{\|u_{xx}(t)\|^2}{M(t)}$$
$$\leq 2^5(\gamma+1)\left(G(0) + d_0^2 E(0)^{\frac{p-2\gamma}{\gamma+1}}\mu(t)\right).$$

Thus, by the standard calculation for ODE, we obtain

$$\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} \leq \max\left\{ \frac{\|u_1\|^2}{M(0)^{2\gamma+1}},\, 2^6(\gamma+1)^2\left(G(0) + d_0^2 E(0)^{\frac{p-2\gamma}{\gamma+1}}\mu(t)\right) \right\}.$$

If $E(0)$ is small such that

$$2^6(\gamma+1)^2 d_0^2 E(0)^{\frac{p-2\gamma}{\gamma+1}} < \frac{1}{2},$$

we have that

$$\mu(t) \leq \max\left\{ \frac{\|u_1\|^2}{M(0)^{2\gamma+1}},\, 2^7(\gamma+1)^2 G(0) \right\} \tag{2.34}$$

which gives the desired estimate (2.31).

Moreover, from (2.19) and (2.34) we obtain

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \leq 2\left(G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}} B(0)^{\frac{1}{2}}\right)^2$$

which implies (2.32) □

Proposition 2.8 *Under the assumption of Proposition 2.7, the function $M(t)$ satisfies*

$$M(t) \equiv \|u_x(t)\|^2 \geq C'(1+t)^{-\frac{1}{\gamma}} \qquad \textit{for} \quad t \geq 0 \tag{2.35}$$

with some positive constant C'.

Proof. Multiplying (1.1) by $2u_t$ and $M(t)^{-2\gamma-1}$, and integrating it over Ω, we have

$$\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^{\gamma}}\right)+2\left(1+\frac{2\gamma+1}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}}$$
$$=-(\gamma+1)\frac{M'(t)}{M(t)^{\gamma+1}}+\frac{2}{M(t)^{2\gamma+1}}(f(u),u_t)$$
$$\le 2(\gamma+1)\frac{\|u_t(t)\|}{M(t)^{\frac{2\gamma+1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}}+2c_*^{p+1}\frac{\|u_t(t)\|}{M(t)^{\frac{2\gamma+1}{2}}}M(t)^{\frac{p-2\gamma}{2}}$$
$$\le C\frac{\|u_t(t)\|}{M(t)^{\frac{2\gamma+1}{2}}},$$

where we used the facts that $\|u_{xx}(t)\|^2/M(t)\le C$ and $M(t)\le C$ at the last inequality. Since it follows from (2.17) that

$$1+\frac{2\gamma+1}{2}\frac{M'(t)}{M(t)}\ge\frac{1}{\gamma+1}>0,$$

we observe from the Young inequality that

$$\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^{\gamma}}\right)\le C \qquad \text{or} \qquad \frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^{\gamma}}\le C(1+t)$$

which gives the desired estimate (2.35). □

3 Global Solutions

Theorem 3.1 *Let the initial data* $[u_0,u_1]$ *belong to* $H^2\cap\mathcal{W}\times H_0^1$ *and* $M(0)>0$ *and* $E(0)<d$. *Suppose that* $p>2\gamma$ *and the initial data* $[u_0,u_1]$ *satisfy*

$$\max\left\{2^2c_*^{p+2}(\kappa E(0))^{\frac{1}{\gamma+1}}\,,\,2^7(\gamma+1)^2d_0^2E(0)^{\frac{p-2\gamma}{\gamma+1}}\right\}<1 \tag{3.1}$$

and

$$2(\gamma+1)^{\frac{2(\gamma+1)}{(\gamma+1)}}\left(G(0)^{\frac{1}{2}}+d_0E(0)^{\frac{p-2\gamma}{2(\gamma+1)}}B(0)^{\frac{1}{2}}\right)B(0)^{\frac{1}{2}}E(0)^{\frac{\gamma}{\gamma+1}}<1. \tag{3.2}$$

where d_0 *is a positive constant given by* (2.19), *and* $G(0)$ *and* $B(0)$ *are defined by* (2.20) *and* (2.33), *respectively.*

Then, the problem (1.1) *admits a global solution* $u(t)$ *in the class* $C^0([0,\infty);$

$H^2 \cap \mathcal{W}) \cap C^1([0,\infty); H_0^1) \cap C^2([0,\infty); L^2)$ and the solution $u(t)$ satisfies

$$\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1}, \tag{3.3}$$

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le C, \quad \frac{\|u_t(t)\|^2}{M(t)^{2\gamma+1}} \le C, \tag{3.4}$$

$$C'(1+t)^{-\frac{1}{\gamma}} \le \|u_x(t)\|^2 \le C(1+t)^{-\frac{1}{\gamma}}, \tag{3.5}$$

$$C'(1+t)^{-\frac{1}{\gamma}} \le \|u_{xx}(t)\|^2 \le C(1+t)^{-\frac{1}{\gamma}}, \tag{3.6}$$

$$\|u_t(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}} \qquad \text{for} \quad t \ge 0, \tag{3.7}$$

where C and C' are some positive constants.

Proof. Let $u(t)$ be a solution on $[0,T]$. Since $M(0) > 0$, putting

$$T_1 \equiv \{\, t \in [0,\infty) \mid M(s) > 0 \text{ for } 0 \le s < t \,\},$$

we have that $T_1 > 0$. If $T_1 < T$, then

$$M(t) > 0 \quad \text{for} \quad 0 \le t < T_1, \quad M(T_1) = 0. \tag{3.8}$$

We observe

$$\begin{aligned} \frac{|M'(t)|}{M(t)} &\le 2\frac{\|u_t(t)\|\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}} = 2\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}} M(t)^{\gamma} \\ &\le 2\frac{\|u_t(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}}((\gamma+1)E(0))^{\frac{\gamma}{\gamma+1}}. \end{aligned} \tag{3.9}$$

Since it follows from (2.20), (2.33), and (3.2) that

$$\frac{|M'(0)|}{M(0)} \le 2B(0)^{\frac{1}{2}}\left(G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}} B(0)^{\frac{1}{2}}\right)((\gamma+1)E(0))^{\frac{\gamma}{\gamma+1}} < \frac{1}{\gamma+1},$$

putting

$$T_2 \equiv \sup\left\{ t \in [0,\infty) \mid \frac{|M'(s)|}{M(s)} < \frac{1}{\gamma+1} \text{ for } 0 \le s < t \right\},$$

we see that $T_1 > 0$. If $T_2 < T_1$, then we have that

$$\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \text{for} \quad 0 \le t < T_2, \quad \frac{|M'(T_2)|}{M(T_2)} = \frac{1}{\gamma+1}. \tag{3.10}$$

On the other hand, we observe from (3.2), (3.9), and Proposition 2.7 that

$$\begin{aligned} \frac{|M'(t)|}{M(t)} &\le 2B(0)^{\frac{1}{2}}\left(G(0)^{\frac{1}{2}} + d_0 E(0)^{\frac{p-2\gamma}{2(\gamma+1)}} B(0)^{\frac{1}{2}}\right)((\gamma+1)E(0))^{\frac{\gamma}{\gamma+1}} \\ &< \frac{1}{\gamma+1} \quad \text{for} \quad 0 \le t \le T_2, \end{aligned}$$

which is a contradiction to (3.10), and hence, we have that $T_2 \geq T_1$. Then, we observe from Proposition 2.8 that

$$M(t) \geq C'(1+t)^{-\frac{1}{\gamma}} > 0 \quad \text{for} \quad 0 \leq t \leq T_1\,,$$

which is a contradiction to (3.8), and hence, we have that $T_1 \geq T$.

Multiplying (1.1) by $(-2u_{xxt})$ and $M(t)^{-\gamma}$ and integrating it over Ω, we have

$$\begin{aligned}
\frac{d}{dt}H(t) + 2\left(1+\frac{\gamma}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u_{xt}(t)\|^2}{M(t)^\gamma} &= -\frac{1}{M(t)^\gamma}((f(u))_x, u_{xt}) \\
&\leq \frac{2(p+1)}{M(t)^\gamma}\|u(t)\|_\infty^p\|u_x(t)\|\|u_{xt}(t)\| \\
&\leq 2(p+1)c_*^p M(t)^{\frac{1}{2}(p+1-\gamma)}\frac{\|u_{xt}(t)\|}{M(t)^{\frac{\gamma}{2}}}\,,
\end{aligned}$$

where $H(t)$ is defined by

$$H(t) = \frac{\|u_{xt}(t)\|^2}{M(t)^\gamma} + \|u_{xx}(t)\|^2\,.$$

Since it follows from (3.10) that

$$1+\frac{\gamma}{2}\frac{M'(t)}{M(t)} \geq \frac{\gamma+2}{2(\gamma+1)} \geq 0\,,$$

we observe from the Young inequality that

$$\frac{d}{dt}H(t) \leq CM(t)^{p+1-\gamma}$$

and from Corollary 2.5 that if $p+1 > 2\gamma$,

$$H(t) \leq H(0) + CE(0)^{\frac{p+1-2\gamma}{\gamma+1}}\,. \tag{3.11}$$

Thus, we obtain that $M(0) > 0$ and $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} \leq C$ for $0 \leq t \leq T$. Therefore, the local solution $u(t)$ of (1.1) in the sense of Proposition 2.2 can be continued globally in time. Then, the estimates (2.9), (2.31), (2.32), and (2.35) hold true for $t \geq 0$, and hence, (3.5) follows from (2.9) and (2.35), (3.6) follows from (2.32) and (2.35), (3.7) follows from (2.31) and (3.5). □

4 Decay Estimates

Proposition 4.1 *Under the assumption of Theorem 3.1, it holds that*

$$\frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} + \|u_{xt}(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}}\,. \tag{4.1}$$

Proof. Multiplying (1.1) differentiated with respect to t by $2u_{tt}$ and $M(t)^{-\gamma}$, and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}F(t) + 2\left(1 + \frac{\gamma}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} \qquad (4.2)\\
&= 2\gamma\frac{M'(t)}{M(t)}(u_{xx}, u_{tt}) + \frac{2}{M(t)^\gamma}((f(u))_t, u_{tt})\\
&\leq \frac{4\gamma}{M(t)}\|u_t(t)\|\|u_{xx}(t)\|^2\|u_{tt}(t)\| + \frac{2(p+1)c_*^p}{M(t)^\gamma}\|u_x(t)\|^p\|u_t(t)\|\|u_{tt}(t)\|\\
&\leq C\frac{\|u_{tt}(t)\|}{M(t)^{\frac{\gamma}{2}}}\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + M(t)^{\frac{1}{2}(p-2\gamma)}\right)M(t)^{\frac{\gamma}{2}}\|u_t(t)\|,
\end{aligned}$$

where $F(t)$ is defined by

$$F(t) \equiv \frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} + \|u_{xt}(t)\|^2 .$$

Since it follows from (3.3) that

$$1 + \frac{\gamma}{2}\frac{M'(t)}{M(t)} \geq \frac{\gamma+2}{2(\gamma+1)} > \frac{1}{2},$$

we observe from the Young inequality and (2.6) and (3.3) that

$$\begin{aligned}
\frac{d}{dt}F(t) + \frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} &\leq C\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + M(t)^{\frac{1}{2}(p-2\gamma)}\right)^2 M(t)^\gamma\|u_t(t)\|^2\\
&\leq Cf(t)^2, \qquad f(t)^2 \equiv M(t)^\gamma\|u_t(t)\|^2 . \qquad (4.3)
\end{aligned}$$

Integrating (4.3) over $[t, t+1]$, we have

$$\int_t^{t+1}\frac{\|u_{tt}(s)\|^2}{M(s)^\gamma}\,ds \leq F(t) - F(t+1) + C\sup_{t\leq s\leq t+1} f(s)^2 \qquad (\equiv D(t)^2)\,. \tag{4.4}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\frac{\|u_{tt}(t_j)\|^2}{M(t_j)^\gamma} \leq 4D(t)^2 \qquad \text{for} \quad j = 1, 2 . \tag{4.5}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$F(t_*) \leq 2\int_{t_1}^{t_2} F(s)\,ds . \tag{4.6}$$

On the other hand, multiplying (1.1) differentiated with respect to t by u_t and $M(t)^{-\gamma}$, and integrating it over Ω, we have

$$\begin{aligned}
&\|u_{xt}(t)\|^2 + \frac{\gamma}{2}\frac{|M'(t)|}{M(t)} \\
&\quad = \frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} - \frac{d}{dt}\frac{(u_t, u_{tt})}{M(t)^\gamma} - \left(1 + \gamma\frac{M'(t)}{M(t)}\right)\frac{(u_t, u_{tt})}{M(t)^\gamma} + \frac{((f(u))_t, u_t)}{M(t)^\gamma},
\end{aligned}$$

and integrating the resulting equation over $[t_1, t_2]$, we obtain from (3.3), (3.7), (4.4), and (4.5) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \|u_{xt}(s)\|^2\, ds \\
&\le \int_t^{t+1} \frac{\|u_{tt}(s)\|^2}{M(s)^\gamma}\, ds + \sum_{j=1}^{2} \frac{\|u_t(t_j)\|}{M(t_j)^{\frac{\gamma}{2}}}\frac{\|u_{tt}(t_j)\|}{M(t_j)^{\frac{\gamma}{2}}} + C\int_t^{t+1} \frac{\|u_t(s)\|}{M(s)^{\frac{\gamma}{2}}}\frac{\|u_{tt}(s)\|}{M(s)^{\frac{\gamma}{2}}}\, ds \\
&\quad + C\int_t^{t+1} M(s)^{\frac{1}{2}(p-2\gamma)}\|u_t(s)\|^2\, ds \\
&\le CD(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) + C\sup_{t\le s\le t+1} h(s)^2
\end{aligned}$$

with

$$g(t)^2 \equiv \frac{\|u_t(t)\|^2}{M(t)^\gamma} \quad \text{and} \quad h(t)^2 \equiv M(t)^{\frac{1}{2}(p-2\gamma)}\|u_t(t)\|^2,$$

and

$$\begin{aligned}
&\int_{t_1}^{t_2} F(s)\, ds = \int_{t_1}^{t_2}\left(\frac{\|u_{tt}(s)\|^2}{M(s)^\gamma} + \|u_{xt}(s)\|^2\right) ds \\
&\quad \le CD(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) + C\sup_{t\le s\le t+1} f(s)^2 + C\sup_{t\le s\le t+1} h(s)^2. \qquad (4.7)
\end{aligned}$$

Moreover, for $\tau \in [t, t+1]$, integrating (4.2) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (4.6) that

$$\begin{aligned}
F(\tau) &= F(t_*) + \int_\tau^{t_*}\left(\left(2 + \gamma\frac{M'(s)}{M(s)}\right)\frac{\|u_{tt}(s)\|^2}{M(s)^\gamma} - 2\gamma\frac{M'(s)}{M(s)}(u_{xx}, u_{tt})\right. \\
&\qquad \left. - \frac{2}{M(s)^\gamma}((f(u))_t, u_{tt})\right) ds \\
&\le 2\int_{t_1}^{t_2} F(s)\, ds + C\int_t^{t+1}\frac{\|u_{tt}(s)\|^2}{M(s)^\gamma}\, ds \\
&\qquad + C\int_t^{t+1}\frac{\|u_{tt}(s)\|}{M(s)^{\frac{\gamma}{2}}}\left(\frac{\|u_{xx}(s)\|^2}{M(s)} + M(s)^{\frac{1}{2}(p-2\gamma)}\right) M(s)^{\frac{\gamma}{2}}\|u_t(s)\|\, ds
\end{aligned}$$

and from (3.4), (3.5), (3.7), (4.4), and (4.7) that

$$\sup_{t\le s\le t+1} F(s) \le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) + C \sup_{t\le s\le t+1} f(s)^2 + C \sup_{t\le s\le t+1} h(s)^2 .$$

Moreover, we observe from (4.4) that

$$\begin{aligned}\sup_{t\le s\le t+1} F(s)^2 &\le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 + C \sup_{t\le s\le t+1} f(s)^4 + C \sup_{t\le s\le t+1} h(s)^4 \\ &\le C\left(F(t) + \sup_{t\le s\le t+1} g(s)^2\right)\left(F(t) - F(t+1) + C \sup_{t\le s\le t+1} f(s)^2\right) \\ &\quad + C \sup_{t\le s\le t+1} f(s)^4 + C \sup_{t\le s\le t+1} h(s)^4\end{aligned}$$

and from the Young inequality that

$$\begin{aligned}\sup_{t\le s\le t+1} F(s)^2 \le C&\left(F(t) + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1)) \\ &+ C\left(\sup_{t\le s\le t+1} g(s)^2 + \sup_{t\le s\le t+1} f(s)^2\right) \sup_{t\le s\le t+1} f(s)^2 + C \sup_{t\le s\le t+1} h(s)^4 .\end{aligned}$$

On the other hand, since it follows from (3.5) and (3.7) that

$$\begin{aligned} f(t)^2 &\equiv M(t)^\gamma \|u_t(t)\|^2 \le C(1+t)^{-3-\frac{1}{\gamma}} , \\ g(t)^2 &\equiv \frac{\|u_t(t)\|^2}{M(t)^\gamma} \le C(1+t)^{-1-\frac{1}{\gamma}} , \\ h(t)^2 &\equiv M(t)^{\frac{1}{2}(p-2\gamma)} \|u_t(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}} ,\end{aligned}$$

we have

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + (1+t)^{-1-\frac{1}{\gamma}}\right)(F(t) - F(t+1)) + C(1+t)^{-4-\frac{2}{\gamma}} . \tag{4.8}$$

Thus, applying Lemma 4.2 below to (4.8), we obtain the desired estimate (4.1). □

In order to derive the decay estimate of the function $G(t)$, we used the following inequality (see [10], [11], [18] for the proof).

Lemma 4.2 *Let $\phi(t)$ be a non-negative function on $[0,\infty)$ and satisfy*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\alpha} \le \left(k_0\phi(t)^\alpha + k_1(1+t)^{-\beta}\right)(\phi(t) - \phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants k_0, k_1, $k_2 \ge 0$, $\alpha > 0$, $\beta > 0$, *and* $\gamma > 0$. *Then, the function* $\phi(t)$ *satisfies*

$$\phi(t) \le C_0(1+t)^{-\theta}, \qquad \theta = \min\left\{ \frac{1+\beta}{\alpha}, \frac{\gamma}{1+\alpha} \right\}$$

for $t \ge 0$ *with some constant* C_0 *depending on* $\phi(0)$.

Proposition 4.3 *Under the assumption of Theorem 3.1, it holds that*

$$\|u(t)\|^2 \ge C'(1+t)^{-\frac{1}{\gamma}} \tag{4.9}$$

with some positive constant C'.

Proof. From Equation (1.1), we observe

$$\begin{aligned} \frac{d}{dt}\frac{M(t)}{\|u(t)\|^2} &= \frac{-2}{\|u(t)\|^2}(u_{xx} + \frac{M(t)}{\|u(t)\|^2}u, u_t) \\ &= \frac{-2M(t)^\gamma}{\|u(t)\|^2}(u_{xx} + \frac{M(t)}{\|u(t)\|^2}u, u_{xx}) + \frac{2}{\|u(t)\|^2}(u_{xx} + \frac{M(t)}{\|u(t)\|^2}u, u_{tt} - f(u)) \\ &= \frac{-2M(t)^\gamma}{\|u(t)\|^2}\|u_{xx} + \frac{M(t)}{\|u(t)\|^2}u\|^2 + \frac{2}{\|u(t)\|^2}(u_{xx} + \frac{M(t)}{\|u(t)\|^2}u, u_{tt} - f(u)). \end{aligned}$$

Moreover, the Young inequality yields

$$\begin{aligned} \frac{d}{dt}\frac{M(t)}{\|u(t)\|^2} &\le C\frac{\|u_{tt} - f(u)\|^2}{\|u(t)\|^2 M(t)^\gamma} \\ &\le C\left(\frac{1}{M(t)}\frac{\|u_{tt}(t)\|^2}{M(t)^\gamma} + M(t)^{2\gamma}M(t)^{p-2\gamma}\right)\frac{M(t)}{\|u(t)\|^2} \\ &\le (1+t)^{-2}\frac{M(t)}{\|u(t)\|^2} \end{aligned}$$

where we used (2.35), (3.5), and (4.1) at the last inequality. Thus, we obtain

$$\frac{M(t)}{\|u(t)\|^2} \le C \quad \text{and} \quad \|u(t)\|^2 \ge C^{-1}M(t)$$

which gives the desired estimate (4.9). □

Summing up Propositions 4.1 and 4.3, we conclude the following theorem.

Theorem 4.4 *Under the assumption of Theorem 3.1, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\begin{aligned} &\|u_{xt}(t)\| \le C(1+t)^{-2-\frac{1}{\gamma}}, \quad \|u_{tt}(t)\|^2 \le C(1+t)^{-3-\frac{1}{\gamma}}, \\ &\|u(t)\|^2 \ge C'(1+t)^{-\frac{1}{\gamma}} \qquad \text{for} \quad t \ge 0, \end{aligned}$$

where C *and* C' *are certain positive constants.*

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