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Additive Reverses of Schwarz and Grüss Type Trace Inequalities for Operators in Hilbert Spaces

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Abstract

Some reverse of Schwarz trace inequality for operators in Hilbert spaces are provided. Applications in connection to Grüss inequality are also given.

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Introduction

Let $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ be two positive *n-tuples* with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{0.1}$$

for each $i \in \{1, \ldots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [50]:

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

b) *Shisha-Mond's inequality* [54]:

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \le \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

c) *Ozeki's inequality* [47]:

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \frac{n^2}{3} \left(M_1 M_2 - m_1 m_2 \right)^2 .$$

d) *Diaz-Metcalf's inequality* [17]:

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^{n} a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^{n} a_k b_k .$$

If $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

e) *Cassels' inequality* [57]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \dots, n\}, \tag{0.2}$$

then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2 \right) \left(\sum_{k=1}^{n} w_k b_k^2 \right)}{\left(\sum_{k=1}^{n} w_k a_k b_k \right)^2} \leq \frac{(M+m)^2}{4mM}.$$

f) *Greub-Reinboldt's inequality* [37]. We have

$$\left(\sum_{k=1}^{n} w_k a_k^2 \right) \left(\sum_{k=1}^{n} w_k b_k^2 \right) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{k=1}^{n} w_k a_k b_k \right)^2 ,$$

provided $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition (0.1).

g) *Generalized Diaz-Metcalf's inequality* [17], see also [45, p. 123]. If u, $v \in [0,1]$ and $v \leq u$, $u+v = 1$ and (0.2) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vMm \sum_{k=1}^{n} w_k a_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k a_k b_k .$$

h) *Klamkin-McLenaghan's inequality* [39]. If $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$ satisfy (0.2), then

$$\left(\sum_{i=1}^{n} w_i a_i^2 \right) \left(\sum_{i=1}^{n} w_i b_i^2 \right) - \left(\sum_{i=1}^{n} w_i a_i b_i \right)^2 \leq \left(M^{\frac{1}{2}} - m^{\frac{1}{2}} \right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2 . \tag{0.3}$$

For other recent results providing discrete reverse inequalities, see the monograph online [19].

The following reverse of Schwarz's inequality in inner product spaces holds [20].

Theorem 1 (Dragomir, 2003, [20]). Let $A,\ a \in \mathbb{C}$ and $x,\ y \in H$, a complex inner product space with the inner product $\langle \cdot,\cdot \rangle$. If

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{0.4}$$

or equivalently,

$$\left\| x - \frac{a+A}{2}\cdot y \right\| \leq \frac{1}{2}\left|A-a\right| \left\|y\right\|, \tag{0.5}$$

holds, then we have the inequality

$$0 \leq \left\|x\right\|^2 \left\|y\right\|^2 - \left|\langle x,y\rangle\right|^2 \leq \frac{1}{4}\left|A-a\right|^2 \left\|y\right\|^4. \tag{0.6}$$

The constant $\frac{1}{4}$ is sharp in (0.6).

In 1935, G. Grüss [38] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{0.7}$$

where $f,\ g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma \tag{0.8}$$

for each $x \in [a,b]$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [22], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

Theorem 2 (Dragomir, 1999, [22]). Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R},\mathbb{C}$) and $e \in H$, $\|e\|=1$. If $\varphi,\ \gamma,\ \Phi,\ \Gamma$ are real or complex numbers and $x,\ y$ are vectors in H such that the conditions

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{0.9}$$

hold, then we have the inequality

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right| \leq \frac{1}{4}\left|\Phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{0.10}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [25] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [11]-[13], [21]-[28], [34], [48], [61] and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

1 Some Facts on Trace of Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.11}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.12}$$

showing that the definition (1.11) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.13}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.12) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 3. We have

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Hilbert space with inner product

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.14}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.15}$$

for any $A \in \mathcal{B}_2(H)$ and

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.16}$$

for any $A \in \mathcal{B}_2(H)$ and $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.17}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 4. If $A \in \mathcal{B}(H)$, then the following are equivalent:

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;

(ii) A (or $|A|$) is the product of two elements of $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 5. With the above notations:

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{1.18}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1=\sup\{\langle A,B\rangle_2 \mid B\in\mathcal{B}_2(H),\ \|B\|\le 1\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ is a Banach space.
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H)\cong K(H)^* \text{ and } \mathcal{B}_1(H)^*\cong\mathcal{B}(H),$$

where $K(H)^*$ is the dual space of $K(H)$ and $\mathcal{B}_1(H)^*$ is the dual space of $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.19}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.19) converges absolutely and it is independent from the choice of basis.
The following result collects some properties of the trace:

Theorem 6. We have
(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.20}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then $AT,\ TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.21}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\|=1$;
(iv) If $A,\ B\in\mathcal{B}_2(H)$ then $AB,\ BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A,B\rangle_2=\operatorname{tr}(B^*A)=\operatorname{tr}(AB^*) \text{ and } \|A\|_2^2=\operatorname{tr}(A^*A)=\operatorname{tr}\left(|A|^2\right)$$

for any $A,\ B\in\mathcal{B}_2(H)$.
The following Hölder's type inequality has been obtained by Ruskai in [51]

$$|\operatorname{tr}(AB)|\le\operatorname{tr}(|AB|)\le\left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{1.22}$$

where $\alpha \in (0,1)$ and $A,\ B \in \mathcal{B}(H)$ with $|A|^{1/\alpha},\ |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{1.23}$$

with $A,\ B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [55].

For some classical trace inequalities see [14], [16], [46] and [60], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [35], [40], [41], [43], [52] and [56].

We denote by

$$\mathcal{B}_1^+(H) := \{P :\ P \in \mathcal{B}_1(H),\ P \text{ and is selfadjoint and } P \ge 0\}.$$

We obtained recently the following result [33]:

Theorem 7. For any $A,\ C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot 1_H\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}, \end{aligned} \tag{1.24}$$

where $\|\cdot\|$ is the operator norm.

We also have [33]:

Corollary 8. Let $\alpha,\ \beta \in \mathbb{C}$ and $A \in B(H)$ such that

$$\left\|A - \frac{\alpha+\beta}{2}\cdot 1_H\right\| \le \frac{1}{2}|\beta-\alpha|.$$

For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}. \end{aligned} \tag{1.25}$$

In particular, if $C \in \mathcal{B}(H)$ is such that

$$\left\| C - \frac{\alpha+\beta}{2} \cdot 1_H \right\| \le \frac{1}{2} |\beta - \alpha| ,$$

then

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \\ &\le \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ &\le \frac{1}{2}|\beta-\alpha| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \le \frac{1}{4}|\beta-\alpha|^2 . \end{aligned} \tag{1.26}$$

Also

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \right| \\ &\le \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ &\le \frac{1}{2}|\beta-\alpha| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \le \frac{1}{4}|\beta-\alpha|^2 . \end{aligned} \tag{1.27}$$

For other related results see [33].

2 Additive Reverses of Schwarz Trace Inequality

In order to simplify writing, we use the following notation

$$\mathcal{B}_+(H) := \{P \in \mathcal{B}(H),\ P \text{ is selfadjoint and } P \ge 0\} .$$

The following result holds:

Theorem 9. Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$.

(i) We have

$$
\begin{aligned}
0 &\le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) - |\operatorname{tr}(PB^*A)|^2 \qquad (2.28)\\
&= \operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B) - \overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right]\\
&- \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right)\\
&\le \frac{1}{4}|\Gamma - \gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2\\
&- \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right).
\end{aligned}
$$

(ii) If

$$
\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right) \ge 0 \qquad (2.29)
$$

or, equivalently

$$
\operatorname{tr}\left(P\left|A - \frac{\gamma + \Gamma}{2}B\right|^2\right) \le \frac{1}{4}|\Gamma - \gamma|^2\operatorname{tr}\left(P|B|^2\right), \qquad (2.30)
$$

then

$$
\begin{aligned}
0 &\le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) - |\operatorname{tr}(PB^*A)|^2 \qquad (2.31)\\
&\le \operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B) - \overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right]\\
&\le \frac{1}{4}|\Gamma - \gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) - |\operatorname{tr}(PB^*A)|^2 \qquad (2.32)\\
&\le \frac{1}{4}|\Gamma - \gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2\\
&- \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right)\\
&\le \frac{1}{4}|\Gamma - \gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2.
\end{aligned}
$$

Proof. Observe that, by the trace properties, we have

$$\begin{aligned} I_1 &:= \operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B) - \overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right] \quad (2.33)\\ &= \operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\overline{\operatorname{tr}(PB^*A)} - \overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right]\\ &= \operatorname{Re}\left[\Gamma \operatorname{tr}\left(P|B|^2\right)\overline{\operatorname{tr}(PB^*A)} + \overline{\gamma}\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|B|^2\right)\right.\\ &\left. - |\operatorname{tr}(PB^*A)|^2 - \Gamma\overline{\gamma}\left[\operatorname{tr}\left(P|B|^2\right)\right]^2\right.\\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)} + \overline{\gamma}\operatorname{tr}(PB^*A)\right]\\ &- |\operatorname{tr}(PB^*A)|^2 - \left[\operatorname{tr}\left(P|B|^2\right)\right]^2\operatorname{Re}(\Gamma\overline{\gamma}) \end{aligned}$$

and

$$\begin{aligned} I_2 &:= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right)\\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\operatorname{tr}\left(\Gamma PA^*B + \overline{\gamma}PB^*A - \overline{\gamma}\Gamma PB^*B - PA^*A\right)\right]\\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\Gamma\operatorname{tr}(PA^*B) + \overline{\gamma}\operatorname{tr}(PB^*A)\right.\\ &\left. - \overline{\gamma}\Gamma\operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P|A|^2\right)\right]\\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)} + \overline{\gamma}\operatorname{tr}(PB^*A)\right]\\ &- \left[\operatorname{tr}\left(P|B|^2\right)\right]^2\operatorname{Re}(\overline{\gamma}\Gamma) - \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right), \end{aligned}$$

for P a selfadjoint operator with $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and γ, $\Gamma \in \mathbb{C}$.

Then we have

$$I_1 - I_2 = \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - |\operatorname{tr}(PB^*A)|^2,$$

which proves the equality in (2.28).

Utilising the elementary inequality for complex numbers

$$\operatorname{Re}(u\overline{v}) \leq \frac{1}{4}|u+v|^2, \; u, v \in \mathbb{C},$$

we have

$$\begin{aligned} &\operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B) - \overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right] \quad (2.34)\\ &= \operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right)\left(\overline{\operatorname{tr}(PB^*A) - \gamma\operatorname{tr}\left(P|B|^2\right)}\right)\right]\\ &\leq \frac{1}{4}\left[\Gamma \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A) + \operatorname{tr}(PB^*A) - \gamma\operatorname{tr}\left(P|B|^2\right)\right]^2\\ &= \frac{1}{4}|\Gamma - \gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2, \end{aligned}$$

which proves the last inequality in (2.28).

We have the equalities

$$\begin{aligned}
&\frac{1}{4}\left|\Gamma-\gamma\right|^2 P\left|B\right|^2 - P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2 \qquad (2.35)\\
&=P\left[\frac{1}{4}\left|\Gamma-\gamma\right|^2\left|B\right|^2-\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right]\\
&=P\left[\frac{1}{4}\left|\Gamma-\gamma\right|^2\left|B\right|^2-\left(A-\frac{\gamma+\Gamma}{2}B\right)^*\left(A-\frac{\gamma+\Gamma}{2}B\right)\right]\\
&=P\left[\frac{1}{4}\left|\Gamma-\gamma\right|^2\left|B\right|^2\right.\\
&\left.-\left|A\right|^2+\overline{\frac{\gamma+\Gamma}{2}}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\left|\frac{\gamma+\Gamma}{2}\right|^2\left|B\right|^2\right]
\end{aligned}$$

$$\begin{aligned}
&=P\left[-\left|A\right|^2+\overline{\frac{\gamma+\Gamma}{2}}B^*A+\frac{\gamma+\Gamma}{2}A^*B\right.\\
&\left.+\left(\frac{1}{4}\left|\Gamma-\gamma\right|^2-\left|\frac{\gamma+\Gamma}{2}\right|^2\right)\left|B\right|^2\right]\\
&=P\left[-\left|A\right|^2+\overline{\frac{\gamma+\Gamma}{2}}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\operatorname{Re}\left(\Gamma\overline{\gamma}\right)\left|B\right|^2\right]
\end{aligned}$$

for any bounded operators A, B, P and the complex numbers γ, $\Gamma \in \mathbb{C}$.

Let P be a selfadjoint operator with $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and γ, $\Gamma \in \mathbb{C}$. Taking the trace in (2.35) we get

$$\begin{aligned}
&\frac{1}{4}\left|\Gamma-\gamma\right|^2\operatorname{tr}\left(P\left|B\right|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right) \qquad (2.36)\\
&=-\operatorname{tr}\left(P\left|A\right|^2\right)-\operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right)\\
&+\overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}\left(PB^*A\right)+\frac{\gamma+\Gamma}{2}\operatorname{tr}\left(PA^*B\right)\\
&=-\operatorname{tr}\left(P\left|A\right|^2\right)-\operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right)\\
&+\overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}\left(PB^*A\right)+\frac{\gamma+\Gamma}{2}\overline{\operatorname{tr}\left(PB^*A\right)}
\end{aligned}$$

$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A) + \overline{\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + 2\operatorname{Re}\left[\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\overline{\overline{\Gamma}\operatorname{tr}(PB^*A)}}\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right].$$

Utilising the equality for I_2 above, we conclude that (2.29) holds if and only if (2.30) holds, and the inequalities (2.31) and (2.32) thus follow from (2.28).

The case $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ goes likewise and the details are omitted. □

For two given operators T, $U \in B(H)$ and two given scalars α, $\beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [31] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}\operatorname{Re}\langle \mathcal{C}_{\alpha,\beta}(T,U)x, x\rangle &= \operatorname{Re}\langle \mathcal{C}_{\beta,\alpha}(T,U)x, x\rangle \\ &= \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\ &= \frac{1}{4}|\beta-\alpha|^2\left\langle |U|^2 x, x\right\rangle - \left\langle \left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 x, x\right\rangle\end{aligned} \tag{2.37}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 10. For α, $\beta \in \mathbb{C}$ and T, $U \in B(H)$ the following statements are equivalent:

(i) The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;

(ii) We have the norm inequality

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \le \frac{1}{2}\left|\beta-\alpha\right|\left\|Ux\right\| \tag{2.38}$$

for any $x \in H$;

(iii) We have the following inequality in the operator order

$$\left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 \le \frac{1}{4}\left|\beta-\alpha\right|^2\left|U\right|^2.$$

As a consequence of the above lemma we can state:

Corollary 11. Let α, $\beta \in \mathbb{C}$ and T, $U \in B(H)$. If $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, then

$$\left\|T - \frac{\alpha+\beta}{2}\cdot U\right\| \le \frac{1}{2}\left|\beta-\alpha\right|\left\|U\right\|. \tag{2.39}$$

Remark 1. In order to give examples of linear operators T, $U \in B(H)$ and numbers α, $\beta \in \mathbb{C}$ such that the transform $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w $(w \neq 0)$ with the property that $\|Sx - zVx\| \le |w|\,\|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T and U satisfy (2.38), i.e., $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive.

Corollary 12. Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$. If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then we have the inequalities (2.31) and (2.32).

The case of selfadjoint operators is as follows.

Corollary 13. Let P, A, B be selfadjoint operators with either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $M > m$. If $(A - mB)(MB - A) \ge 0$, then

$$\begin{aligned} 0 &\le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \\ &\le \left[\left(M\operatorname{tr}\left(PB^2\right) - \operatorname{tr}(PBA)\right)\left(\operatorname{tr}(PAB) - m\operatorname{tr}\left(PB^2\right)\right)\right] \\ &\le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} 0 &\le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \\ &\le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 - \operatorname{tr}\left(PB^2\right)\operatorname{tr}\left[P(A-mB)(MB-A)\right] \\ &\le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2. \end{aligned} \tag{2.41}$$

We also have the following result:

Theorem 14. Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$.

(i) We have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - |\operatorname{tr}(PB^*A)|^2 \qquad (2.42) \\ &= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) \\ &- \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda - \operatorname{tr}(PB^*A)\right|^2. \end{aligned}$$

(ii) If there is $r > 0$ such that

$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) \leq r^2\left[\operatorname{tr}\left(P|B|^2\right)\right],$$

then we have the reverse of Schwarz inequality

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - |\operatorname{tr}(PB^*A)|^2 \qquad (2.43) \\ &\leq r^2\left[\operatorname{tr}\left(P|B|^2\right)\right] - \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda - \operatorname{tr}(PB^*A)\right|^2 \\ &\leq r^2\left[\operatorname{tr}\left(P|B|^2\right)\right]. \end{aligned}$$

Proof. Using the properties of trace, we have for $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and

$\lambda \in \mathbb{C}$ that

$$\begin{aligned}
J_1 &:= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right)\\
&= \operatorname{tr}\left(P\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right)^*\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right)\right)\\
&= \operatorname{tr}\left(P\left[\operatorname{tr}\left(P|B|^2\right)|A|^2+|\lambda|^2|B|^2\right.\right.\\
&\left.\left.-\overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}B^*A-\lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A^*B\right]\right)\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right)\\
&-\overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}(PB^*A)-\lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}(PA^*B)\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right)\\
&-\overline{\lambda}\operatorname{tr}(PB^*A)\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}-\overline{\overline{\lambda}\operatorname{tr}(PB^*A)}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\\
&= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right)\\
&-2\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{Re}\left(\overline{\lambda}\operatorname{tr}(PB^*A)\right)
\end{aligned}$$

and

$$\begin{aligned}
J_2 \quad : &= \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right|^2\\
&= \left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right)\overline{\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right)}\\
&= \operatorname{tr}\left(P|B|^2\right)|\lambda|^2-2\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{Re}\left(\overline{\lambda}\operatorname{tr}(PB^*A)\right)+|\operatorname{tr}(PB^*A)|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&J_1-J_2\\
&= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right)-\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right|^2
\end{aligned}$$

and the equality (2.42) is proved.

The inequality (2.43) follows from (2.42).

The other case is similar. □

Corollary 15. Let, either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and δ, $\Delta \in \mathbb{C}$.

If
$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(C^* - \bar{\delta}D^*\right)(\Delta D - C)\right]\right) \geq 0 \tag{2.44}$$
or, equivalently
$$\operatorname{tr}\left(P\left|C - \frac{\delta+\Delta}{2}D\right|^2\right) \leq \frac{1}{4}|\Delta-\delta|^2 \operatorname{tr}\left(P|D|^2\right), \tag{2.45}$$
then
$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|C|^2\right)\operatorname{tr}\left(P|D|^2\right) - |\operatorname{tr}(PD^*C)|^2 \\ &\leq \frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|D|^2\right)\right]^2 - \left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|D|^2\right) - \operatorname{tr}(PD^*C)\right|^2 \\ &\leq \frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|D|^2\right)\right]^2. \end{aligned} \tag{2.46}$$

Proof. The equivalence of the inequalities (2.44) and (2.45) follows from Theorem 9 (ii).

If we write the inequality (2.45) for $C = A$ and $D = B$, we have
$$\operatorname{tr}\left(P\left|A - \frac{\delta+\Delta}{2}B\right|^2\right) \leq \frac{1}{4}|\Delta-\delta|^2 \operatorname{tr}\left(P|B|^2\right).$$

If we multiply this inequality by $\operatorname{tr}\left(P|B|^2\right) \geq 0$ we get
$$\begin{aligned} &\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} A - \frac{\delta+\Delta}{2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} B\right|^2\right) \\ &\leq \frac{1}{4}|\Delta-\delta|^2 \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|B|^2\right). \end{aligned} \tag{2.47}$$

Let
$$\lambda = \frac{\delta+\Delta}{2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} \text{ and } r = \frac{1}{2}|\Delta-\delta|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}.$$

Then by (2.47) we have
$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} A - \lambda B\right|^2\right) \leq r^2 \operatorname{tr}\left(P|B|^2\right),$$
and by (2.43) we get
$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - |\operatorname{tr}(PB^*A)|^2 \\ &\leq \frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2 - \left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right|^2 \\ &\leq \frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2, \end{aligned}$$
and the inequality (2.46) is proved. □

Corollary 16. Let, either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and δ, $\Delta \in \mathbb{C}$. If the transform $\mathcal{C}_{\delta,\Delta}(C, D)$ is accretive, then we have the inequalities (2.46).

The case of selfadjoint operators is as follows.

Corollary 17. Let P, C, D be selfadjoint operators with either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and n, $N \in \mathbb{R}$ with $N > n$. If $(C - nD)(ND - C) \geq 0$, then

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(PC^2\right)\operatorname{tr}\left(PD^2\right) - [\operatorname{tr}(PDC)]^2 \\ &\leq \frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2 - \left(\frac{n+N}{2}\operatorname{tr}\left(PD^2\right) - \operatorname{tr}(PDC)\right)^2 \\ &\leq \frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2. \end{aligned} \tag{2.48}$$

3 Trace Inequalities of Grüss Type

Let P be a selfadjoint operator with $P \geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle\cdot,\cdot\rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle\cdot,\cdot\rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle_{2,P}}$ for any A, $B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A|^2\right) = \operatorname{tr}(APA^*) = \operatorname{tr}\left(|A|^2 P\right)$$

and

$$\langle A, B\rangle_{2,P} = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

for any A, $B \in \mathcal{B}_2(H)$.

For a pair of complex numbers (α, β) and $P \in \mathcal{B}_+(H)$, in order to simplify the notations, we say that the pair of operators $(U, V) \in \mathcal{B}_2(H) \times \mathcal{B}_2(H)$ has the *trace* P-(α,β)-*property* if

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(U^* - \overline{\alpha}V^*\right)(\beta V - U)\right]\right) \geq 0$$

or, equivalently

$$\operatorname{tr}\left(P\left|U - \frac{\alpha+\beta}{2}V\right|^2\right) \leq \frac{1}{4}|\beta - \alpha|^2 \operatorname{tr}\left(P|V|^2\right).$$

The above definitions can be also considered in the case when $P \in \mathcal{B}_1^+(H)$ and A, $B \in \mathcal{B}(H)$.

Theorem 18. Let, either $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ and λ, Γ, δ, $\Delta \in \mathbb{C}$. If (A, C) has the trace P-(λ, Γ)-*property and* (B, C) has the trace P-(δ, Δ)-*property, then*

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \qquad (3.49)\\
&\le \operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right.\\
&- \left[\operatorname{Re}\left(\operatorname{tr}\left[P(A^*-\overline{\gamma}C^*)(\Gamma C - A)\right]\right)\right]^{1/2}\\
&\times \left.\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right]^{1/2}\right]\\
&\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2.
\end{aligned}$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and A, B, $C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A, B\rangle_{2,P}\right|^2 \le \langle A, A\rangle_{2,P}\langle B, B\rangle_{2,P}$$

for any A, $B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H)\times\mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned}
&\left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right|^2\\
&\le \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right]
\end{aligned}$$

for any A, $B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \qquad (3.50)\\
&\le \left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right]\\
&\times \left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*B)|^2\right],
\end{aligned}$$

where for the last term we used the equality $\left|\langle B, C\rangle_{2,P}\right|^2 = \left|\langle C, B\rangle_{2,P}\right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (2.32) we have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \\ &\leq \operatorname{tr}\left(P|C|^2\right) \\ &\times \left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C - A)\right]\right)\right] \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*B)|^2 \\ &\leq \operatorname{tr}\left(P|C|^2\right) \\ &\times \left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right]. \end{aligned} \tag{3.52}$$

If we multiply (3.51) with (3.52) and use (3.50), then we get

$$\begin{aligned} &\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\ &\leq \left[\operatorname{tr}\left(P|C|^2\right)\right]^2 \\ &\times \left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C - A)\right]\right)\right] \\ &\times \left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right]. \end{aligned} \tag{3.53}$$

Utilising the elementary inequality for positive numbers m, n, p, q

$$\left(m^2-n^2\right)\left(p^2-q^2\right) \leq (mp-nq)^2,$$

we can state that

$$\begin{aligned} &\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C - A)\right]\right)\right] \\ &\times \left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right] \\ &\leq \left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right. \\ &- \left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C - A)\right]\right)\right]^{1/2} \\ &\left.\times \left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right]^{1/2}\right)^2 \end{aligned} \tag{3.54}$$

with the term in the right hand side in the brackets being nonnegative.

Making use of (3.53) and (3.54) we then get

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \tag{3.55}$$
$$\leq\left[\operatorname{tr}\left(P|C|^2\right)\right]^2\left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right.$$
$$-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2}$$
$$\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right)^2.$$

Taking the square root in (3.55) we obtain the desired result (3.49). □

Corollary 19. Let, either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ and λ, Γ, δ, $\Delta\in\mathbb{C}$. If the transforms $\mathcal{C}_{\lambda,\Gamma}(A,C)$ and $\mathcal{C}_{\delta,\Delta}(B,C)$ are accretive, then the inequality (3.49) is valid.

We have:

Corollary 20. Let P, A, B, C be selfadjoint operators with either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ and m, M, n, $N\in\mathbb{R}$ with $M>m$ and $N>n$. If $(A-mC)(MC-A)\geq 0$ and $(B-nC)(NC-B)\geq 0$ then

$$\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \tag{3.56}$$
$$\leq\operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right.$$
$$-\left[\operatorname{Re}\left(\operatorname{tr}(A-mC)(MC-A)\right)\right]^{1/2}$$
$$\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P(B-nC)(NC-B)\right]\right)\right]^{1/2}\right]$$
$$\leq\frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2.$$

Finally, we have:

Theorem 21. With the assumptions of Theorem 18 we have

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \tag{3.57}$$
$$\leq\operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right.$$
$$-\left|\frac{\Gamma+\gamma}{2}\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\right|$$
$$\left.\times\left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*B)\right|\right]$$
$$\leq\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2.$$

If the transforms $\mathcal{C}_{\lambda,\Gamma}(A,C)$ and $\mathcal{C}_{\delta,\Delta}(B,C)$ are accretive, then the inequality (3.57) also holds.

The proof is similar to the one for Theorem 18 via the Corollary 15 and the details are omitted.

Corollary 22. With the assumptions of Corollary 20 we have

$$\begin{aligned}
&\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \\
&\leq \operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right. \\
&-\left|\frac{M+m}{2}\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\right| \\
&\times\left.\left|\frac{n+N}{2}\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCB)\right|\right] \\
&\leq \frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2 .
\end{aligned} \tag{3.58}$$

4 Some Examples in the Case of $P\in\mathcal{B}_1(H)$

Utilising the above results in the case when $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $B=1_H$ we can also state the following inequalities that complement the earlier results obtained in [33]:

Proposition 23. Let $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$.

(i) We have

$$\begin{aligned}
0&\leq\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \\
&=\operatorname{Re}\left[\left(\Gamma-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)}-\overline{\gamma}\right)\right] \\
&-\frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right) \\
&\leq\frac{1}{4}|\Gamma-\gamma|^2-\frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right).
\end{aligned} \tag{4.59}$$

(ii) If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right)\geq 0 \tag{4.60}$$

or, equivalently

$$\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\leq\frac{1}{4}|\Gamma-\gamma|^2, \tag{4.61}$$

and we say for simplicity that A has the trace P-(λ,Γ)-*property*, then

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(P\left|A\right|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \\ &\le \operatorname{Re}\left[\left(\Gamma - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)} - \overline{\gamma}\right)\right] \le \frac{1}{4}\left|\Gamma-\gamma\right|^2\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(P\left|A\right|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \\ &\le \frac{1}{4}\left|\Gamma-\gamma\right|^2 - \frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \le \frac{1}{4}\left|\Gamma-\gamma\right|^2.\end{aligned} \tag{4.63}$$

(iii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := (A^* - \overline{\gamma}1_H)(\Gamma 1_H - A)$ is accretive, then the inequalities (4.62) and (4.63) also hold.

Corollary 24. Let $P \in \mathcal{B}_1^+(H)$, A be a selfadjoint operator and m, $M \in \mathbb{R}$ with $M > m$.

(i) If $(A - m1_H)(M1_H - A) \ge 0$, then

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \\ &\le \left[\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)\right] \le \frac{1}{4}(M-m)^2\end{aligned} \tag{4.64}$$

and

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \\ &\le \frac{1}{4}(M-m)^2 - \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[P\left(A - mB\right)\left(MB - A\right)\right] \le \frac{1}{4}(M-m)^2.\end{aligned} \tag{4.65}$$

(ii) If $m1_H \le A \le M1_H$, then (4.64) and (4.65) also hold.

We have the following reverse of Schwarz inequality as well:

Proposition 25. Let $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$.

(i) If A has the trace P-(λ,Γ)-*property*, then

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(P\left|A\right|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \\ &\le \frac{1}{4}\left|\Gamma-\gamma\right|^2 - \left|\frac{\Gamma+\gamma}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}\left|\Gamma-\gamma\right|^2.\end{aligned} \tag{4.66}$$

(ii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := (A^* - \overline{\gamma}1_H)(\Gamma 1_H - A)$ is accretive, then the inequality (4.66) also holds.

Corollary 26. Let $P \in \mathcal{B}_1^+(H)$, A be a selfadjoint operator and $m, M \in \mathbb{R}$ with $M > m$.

(i) If $(A - m1_H)(M1_H - A) \geq 0$, then

$$0 \leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \leq \frac{1}{4}(M-m)^2 - \left|\frac{m+M}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}(M-m)^2. \tag{4.67}$$

(ii) If $m1_H \leq A \leq M1_H$, then (4.67) also holds.

Finally, we have the following Grüss type inequality as well:

Proposition 27. Let $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$.

(i) If A has the trace P-(λ, Γ)-*property and* B has the trace P-(δ, Δ)-*property, then*

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right| \\ &\leq \left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta| \right.\\ &- \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right)\right]^{1/2} \\ &\left.\times \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^* - \overline{\delta}1_H\right)\left(\Delta 1_H - B\right)\right]\right)\right]^{1/2}\right] \leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\end{aligned} \tag{4.68}$$

and

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right| \\ &\leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta| - \left|\frac{\Gamma+\gamma}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{\delta+\Delta}{2} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \\ &\leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|.\end{aligned} \tag{4.69}$$

(ii) If the transforms $\mathcal{C}_{\lambda,\Gamma}(A)$ and $\mathcal{C}_{\delta,\Delta}(B)$ are accretive then (4.68) and (4.69) also hold.

The case of selfadjoint operators is as follows:

Corollary 28. Let P, A, B be selfadjoint operators with $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and $m, M, n, N \in \mathbb{R}$ with $M > m$ and $N > n$.

(i) If $(A - m1_H)(M1_H - A) \geq 0$ and $(B - n1_H)(N1_H - B) \geq 0$ then

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \tag{4.70}$$
$$\leq \left[\frac{1}{4}(M-m)(N-n) - \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}(\operatorname{tr}(A - m1_H)(M1_H - A))\right]^{1/2} \times \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}(\operatorname{tr}[P(B - n1_H)(N1_H - B)])\right]^{1/2}\right]$$
$$\leq \frac{1}{4}(M-m)(N-n)$$

and

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \tag{4.71}$$
$$\leq \frac{1}{4}(M-m)(N-n) - \left|\frac{m+M}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{n+N}{2} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|$$
$$\leq \frac{1}{4}(M-m)(N-n).$$

(ii) If $m1_H \leq A \leq M1_H$ and $n1_H \leq B \leq N1_H$ then (4.70) and (4.71) also hold.

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