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On the Fibres of An Almost Paracontact Metric Submersion

By

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Abstract

We discuss geometric properties of Riemannian submersions whose total space is an almost paracontact metric manifold. The study focuses on the geometry of the fibres. After determining the structure of the fibres, their implications on the total and the base space of fibration are studied.

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1 Introduction

Almost paracontact metric submersions are Riemannian submersions whose total space is endowed with almost paracontact metric structure. They have been introduced by Gündüzalp and Sahin [5] who considered the case of semi-Riemannian manifolds. Their study focused on the transfer of the structure from the total to the base space, the later being also a paracontact metric manifold, extending the study of Watson [13] who refereed to O'Neill [7].

Regarding the similarity between contact and paracontact structure, as indicated by Sato [8, 9], it seems interesting to examine the same similarities via submersions. Our paper focuses on the geometry of the fibres where, after determining their structures, their implications on the total and the base space are studied.

This paper is organized as follows

Section §2 is devoted to the preliminaries on manifolds where we consider almost para-Hermitian structures. Following Gray and Hervella [4], we have adaptated the defining relations of almost para-Hermitian structures. Almost paracontact structures are reviewed.

In Section §3, we treate the case of almost paracontact metric submersions where, after recalling fundamental properties, we have examined the structure of the fibres and their superminimality.

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2 Preliminaries on manifolds

2.1 Almost para-Hermitian manifolds

Let M^{2m} be a smooth manifold of even dimension $2m$. Consider an almost para complex structure J such that $J^2 = \mathbb{I}$, where $\mathbb{I}$ is the identity transformation. If there exists on M a metric tensor g such that $g(JD, JE) = -g(D, E)$, then the couple (g, J) is called an almost para complex metric structure (or an almost para-Hermitian metric). So, (M^{2m}, g, J) is an almost para-Hermitian manifold.

As in the case of almost Hermitian manifolds, the fundamental 2−form Ω, of the structure (g, J) is given by $\Omega(D, E) = g(D, JE)$. If further, J is parallel along the Levi-Civita connection ∇, (meaning that $\nabla J = 0$), then the manifold is said to be para-Kählerian.

Let us note some remarkable classes of almost para-Hermitian structures susceptible to be used in this study.

Following Gray and Hervella [4], an almost para-Hermitian manifold is called:
(1) para-Kählerian if $\nabla J = 0$;
(2) almost para- Kählerian if $d\Omega(D, E, G) = 0$;
(3) quasi para- Kählerian if $(\nabla_D\Omega)(E, G) + (\nabla_{JD}\Omega)(JE, G) = 0$;
(4) nearly para- Kählerian if $(\nabla_D\Omega)(D, E) = 0$.

2.2 Almost paracontact metric manifolds

Let M be a differentiable manifold of dimension $2m+1$. An almost paracontact structure on M is a triple (φ, ξ, η), where:

(1) ξ is a characteristic vector field,

(2) η is a 1−form such that $\eta(\xi) = 1$, and

(3) φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 = \mathbb{I} - \eta \otimes \xi, \tag{2.1}$$

where $\mathbb{I}$ is the identity transformation. If M is equipped with a Riemannian metric g such that

$$g(\varphi D, \varphi E) = -g(D, E) + \eta(D)\eta(E), \tag{2.2}$$

then (g, φ, ξ, η) is called an almost paracontact metric structure. So, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost paracontact metric manifold. As in the case of almost contact metric manifolds, any almost paracontact metric manifold admits a fundamental 2−form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E).$$

For some remarkable classes, we have the following defining relations.

An almost paracontact manifold is said to be:

(1) *normal* if $N_\varphi - 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhuis tensor of φ.

(2) *para-contact* if $\phi = d\eta$,

(3) *para-K-contact* if it is para-contact and ξ is Killing,

(4) *para-cosymplectic* if $\nabla\eta = 0$ and $\nabla\phi = 0$,

(5) *almost para-cosymplectic* if $d\phi = 0$ and $d\eta = 0$,

(6) *para-Sasakian* if $\phi = d\eta$ and M is normal,

(7) *quasi para-Sasakian* if $d\phi = 0$ and M is normal,

(8) *quasi para-K-cosymplectic* if $(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)\varphi E - \eta(E)(\nabla_{\varphi D}\xi) = 0$;

(9) *almost para-Kenmotsu* if $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E, G)\}$, where $\mathcal{G}$ denotes the cyclic sum over D, E, G;

(10) *para-Kenmotsu* if $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E, G)\}$, $d\eta = 0$ and is normal;

(11) *quasi para-Kenmotsu* if $(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G, D) + 2\eta(G)\phi(D, E)$ and $d\eta = 0$;

(12) *nearly para-Kenmotsu* if $(\nabla_D\varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$;

(13) *nearly para-cosymplectic* if $(\nabla_D\varphi)D = 0$;

(14) *closely para-cosymplectic* if $(\nabla_D\varphi)D = 0$ and $d\eta = 0$;

Following [5, 10], it is known that

$N^{(1)}(D, E) = N_\varphi(D, E) - 2d\eta(D, E)\xi$,

$N^{(2)}(D, E) = (\mathcal{L}_{\varphi D}\eta)E - (\mathcal{L}_{\varphi E}\eta)D$, where $\mathcal{L}$ denotes the Lie derivative.

Moreover, if $N^{(1)} = 0$ then $N^{(2)} = 0$. The vanishing of the tensor $N^{(1)}$ means that the manifold is normal.

Proposition 2.1. *Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost paracontact metric manifold. Then, we have,*

$$\begin{aligned}2g((\nabla_D\varphi)E, G) &= -d\phi(D, \varphi E, \varphi G) - d\phi(D, E, G) - g(N^{(1)}(E, G), \varphi D)\\ &+N^{(2)}(E, G)\eta(D) + 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E).\end{aligned}$$

Proof. See Zamkovoy [15]. □

The above proposition leads to express the defining relations of some structures in the function of the covariant or the exterior derivative of the tensors. For instance

Proposition 2.2. *Let* $(M^{2m+1}, g, \varphi, \xi, \eta)$ *be an almost paracontact metric manifold. If it is*

(1) quasi para-Sasakian, then
$g((\nabla_D\varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$;

(2) para-Sasakian, then $(\nabla_D\varphi)E = g(D, E)\xi - \eta(E)D$;

(3) almost para-cosymplectic, then $2g((\nabla_D\varphi)E, G) = g(N_\varphi(E, G), \varphi D)$;

(4) para-cosymplectic, then $\nabla_D\varphi = 0$;

(5) para-Kenmotsu, then $(\nabla_D\varphi)E = g(\varphi D, E)\xi - \eta(E)\varphi D$.

Proof. (1) Recall that a quasi para-Sasakian manifold is defined by $d\phi = 0$ and $N^{(1)} = 0$. Using Proposition 2.1, we have,

$$2g((\nabla_D\varphi)E, G) = N^{(2)}(E, G)\eta(D) + 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi E, D)\eta(E).$$

On the other hand, it is known that $N^{(1)} = 0$ implies that $N^{(2)} = 0$, as established in [10], from which, the preceding relation reduces to

$$g((\nabla_D\varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$$

which is the proof of (1).

Concerning the statement (2), we claim that
$2g((\nabla_D\varphi)E, G) = d\phi(D, \varphi E, \varphi G) - d\phi(D, E, G)$
$+ 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E)$,
because a para-Sasakian manifold is normal. Since $\phi = d\eta$, we have $d\phi = 0$ so that the above relation reduces to

$$2g((\nabla_D\varphi)E, G) = 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E).$$

Thus, $g((\nabla_D\varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$, which becomes

$$g((\nabla_D\varphi)E, G) = \phi(\varphi E, D)\eta(G) - \phi(\varphi G, D)\eta(E).$$

On the other hand, $\phi(\varphi E, D) = g(\varphi E, \varphi D) = g(E, D) - \eta(E)\eta(D)$ and $\phi(\varphi G, D) = g(\varphi G, \varphi D) = g(G, D) - \eta(G)\eta(D)$. These lead to $g((\nabla_D\varphi)E, G) = g(D, E)\eta(G) - g(G, D)\eta(E)$, which is

$$g((\nabla_D\varphi)E, G) = g(D, E)g(G, \xi) - g(G, D)g(E, \xi),$$

from which $(\nabla_D\varphi)E = g(D, E)\xi - \eta(E)D$, follows. This is the defining relation currently used in the definition of a para-Sasakian structure.

Let us consider the case of statement (3) concerning the almost para-cosymplectic manifolds. From the relation $d\phi = 0$ and $d\eta = 0$, we get $2g((\nabla_D\varphi)E, G) = g(N^{(1)}(E, G), \varphi D)$ by Proposition 2.1. But

$$N^{(1)}(E, G) = N_\varphi(E, G) - 2d\eta(E, G)\xi.$$

Since $d\eta = 0$, the above relation becomes $N^{(1)}(E,G) = N_\varphi(E,G)$ and then

$$2g((\nabla_D\varphi)E, G) = g(N_\varphi(E,G), \varphi D),$$

as claimed in statement (3).

Considering (4), it is known that a para-cosymplectic manifold is normal and then $N^{(1)}(E,G) = 0$ which leads to $2g((\nabla_D\varphi)E,G) = 0$ from which $g((\nabla_D\varphi)E,G) = 0$. Using the non-degeneracy of g, the last relation implies that $(\nabla_D\varphi)E = 0$ which is the proof of (4). In the literature, this is the defining relation currently used to define a para-cosymplectic manifold.

Let us consider the case of para-Kenmotsu manifold. Since a para-Kenmotsu manifold is normal and $d\eta = 0$, we then have

$$2g((\nabla_D\varphi)E, G) = 3d\phi(D, \varphi E, \varphi G) - 3d\phi(D, E, G).$$

Considering $d\phi(D,E,G)$ in a para-Kenmotsu manifold we have

$$3d\phi(D,E,G) = 2\{\eta(D)g(E,\varphi G) + \eta(E)g(G,\varphi D) + \eta(G)g(D,\varphi E)\} \tag{2.3}$$

Similarly,

$$3d\phi(D,\varphi E,\varphi G) = 2\{\eta(D)g(E,\varphi G)\}, \tag{2.4}$$

because $\eta(\varphi E) = 0 = \eta(\varphi G)$. Making (2.4)- (2.3), leads to
$3d\phi(D,\varphi E,\varphi G) - 3d\phi(D,E,G) = -2\{\eta(E)g(G,\varphi D) + \eta(G)g(D,\varphi E)\}$
and with this, we get

$$2g((\nabla_D\varphi)E,G) = -2\{\eta(E)g(G,\varphi D) + \eta(G)g(D,\varphi E)\},$$

which is equivalent to

$$g((\nabla_D\varphi)E,G) = -g(D,\varphi E)g(G,\xi) - g(G,\varphi D)g(E,\xi)$$

from which we deduce $(\nabla_D\varphi)E = g(\varphi D, E)\xi - \eta(E)\varphi D$. This is the defining relation usually used for a para-Kenmotsu manifold see for instance Blaga [1]. □

Note that almost paracontact metric manifolds have been studied by Dacko [2], Dacko and Olszak [3], Kaneyuki and Williams [6], Sato [8, 9], Zamkovoy [15] among others.

Some Examples

Following A.M. Blaga [1], let $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$ and setting $\eta = -\frac{1}{z}dz$, $\xi = -z\frac{\partial}{\partial z}$;

Note by $\mathcal{M}_{2m+1}(\mathbb{R})$ the set of $(2m+1)$ real matrices.

Taking $\varphi \in \mathcal{M}_{2m+1}(\mathbb{R})$ such that

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easy to verify that (φ, ξ, η) is an almost paracontact structure.

Now, considering $\varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have that (φ, ξ, η) is an almost contact structure.

In [3], Dacko and Olszak have constructed an example of para-cosymplectic structure in the following way. Let (N, J, G) be a para-Kählerian manifold. Consider the structure (φ, ξ, η, g) defined on the product manifold $M = N \times \mathbb{R}$ by $\varphi = (J, 0)$, $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and $g = G \times dt^2$ where t is the Cartesian coordinate on $\mathbb{R}$. Then (φ, ξ, η, g) is para-cosymplectic.

Following the previous example, it can be constructed some other by taking (N, J, G) in the various classes of almost para-Hermitian manifolds such as: almost para-Kählerian, quasi para- Kählerian and so on. Thus, we obtain the defining relations of almost para-cosymplectic and quasi para-cosymplectic respectively.

3 Almost paracontact metric submersions

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ be two almost paracontact metric manifolds. By an almost paracontact metric submersion in the sense of [5], one understands a Riemannian submersion

$$\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$$

satisfying

(i) $\pi_* \varphi = \varphi' \pi_*$,

(ii) $\pi_* \xi = \xi'$.

Recall that the tangent bundle $T(M)$ of the total space M has an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ is the vertical distribution while $H(M)$ designates the horizontal one. In [7], O'Neill has defined two configuration tensors T and A, of the total space of a Riemannian submersion by setting

$$T_D E = \mathcal{H} \nabla_{\mathcal{V} D} \mathcal{V} E + \mathcal{V} \nabla_{\mathcal{V} D} \mathcal{H} E;$$

$$A_D E = \mathcal{V} \nabla_{\mathcal{H} D} \mathcal{H} E + \mathcal{H} \nabla_{\mathcal{H} D} \mathcal{V} E.$$

Here, $\mathcal{H}$ and $\mathcal{V}$ designate the horizontal and vertical projections respectively.

On the base space, tensors and other objects will be denoted by a prime ' while those tangent to the fibres will be specified by a carret ˆ. For instance, $\hat{N}_J$ denotes the Nijenhuis tensor of J on the fibres. Herein, vector fields tangent to the fibres will be denoted by U, V and W.

Next, we overview some of the fundamental properties of this type of submersions, which also appear in [13].

3.1 Fundamental properties

Proposition 3.1. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost paracontact metric submersion. We have the following,*

(1) *If* $U \in V(M)$ *is vertical then* $\varphi U \in V(M)$ *is also vertical;*

(2) $\xi \in H(M)$;

(3) $\eta(U) = 0$ *for all vertical* $U \in V(M)$;

(4) $\hat{N}_J(U, V) = N^{(1)}(U, V)$;

Proof. Let us consider assertion (4). Consider two vertical vector fields U and V tangent to the fibres. Remember that in this case,

$$N^{(1)}(U, V) = N_\varphi(U, V) - 2d\eta(U, V)\xi.$$

What we need is to show that $d\eta(U, V) = 0$. In fact
$d\eta(U, V) = \frac{1}{2}(U\eta(V) - V\eta(U) - \eta([U, V])) = 0$ because U, V and [U,V] are vertical. From this we have $N^{(1)}(U, V) = N_\varphi(U, V)$. But on the fibres, if denoted by F^{2r}, one has $J = \hat{\varphi} = \varphi_|(F^{2r})$. Thus, $N_\varphi(U, V) = N_{\hat{\varphi}}(U, V) = \hat{N}_J(U, V)$.
Other statements are established as by Watson [13]. □

Statement (1) means that the vertical distribution is invariant by φ.

With $J = \hat{\varphi}$, it is clear that, on the fibres, the fundamental 2−form is $\hat{\phi}(U, V) = \hat{g}(U, \hat{\varphi}V) = \hat{g}(U, JV) = \hat{\Omega}(U, V)$.

3.2 Structure of the fibres

Proposition 3.2. *The fibres of an almost paracontact metric submersion are almost para-Hermitian manifolds.*

Proof. See [5, Prop 3.5]. □

Proposition 3.3. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost paracontact metric submersion. Then the total space cannot be para-contact, para-K-contact or para-Sasakian.*

Proof. If the total space is a para-contact manifold, it is defined by $\phi = d\eta$. Let U and V two vertical vector fields tangent to the fibres; we have
$\phi(U, V) = d\eta(U, V)$. But $d\eta(U, V) = \frac{1}{2}(U\eta(V) - V\eta(U) - \eta([U, V])) = 0$ because U, V and [U,V] are vertical. Therefore, $\phi(U, V) = 0$ which means that on the fibres, the fundamental 2−form $\hat{\phi}(U, V) = \hat{\Omega}(U, V)$ is null and this is absurd.

The same procedure applies to para-K-contact and para-Sasakian cases which have also $\phi = d\eta$ in their defining relations. □

The same result is valid in the case of nearly para-Sasakian, nearly para-K-Sasakian and quasi para-K-Sasakian.

Note that a nearly para-Sasakian manifold is defined by

$$(\nabla_D\varphi)E + (\nabla_E\varphi)D = 2g(D,E)\xi - \eta(D)E - \eta(E)D;$$

A nearly para-K-Sasakian manifold is defined by

$$(\nabla_D\varphi)E + (\nabla_E\varphi)D = 2g(D,E)\xi - \eta(E)D - \eta(D)E,$$

and $\nabla_D\xi = -\varphi D$;

A quasi para-K-Sasakian manifold is defined by

$$(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)\varphi E = 2g(D,E)\xi + \eta(E)(\nabla_{\varphi D}\xi) - 2\eta(E)D.$$

In each of the under consideration manifolds, if we replace D by U and E by V the right hand side becomes $2g(U,V)\xi$; but this is absurd because ξ has not a counterpart in almost para-Hermitian geometry.

Proposition 3.4. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion. If the total space is para-cosymplectic, quasi para-Sasakian or para-Kenmotsu, then the fibres are para-Kählerian.*

Proof. In this proposition, the proof consist in showing that $d\hat{\Omega} = 0 = \hat{N}_J$. Let U, V and W be three vertical vector fields tangent to the fibres. For a para-cosymplectic manifold, we can refer to its defining relation $\nabla_U\varphi = 0$ which gives $(\hat{\nabla}_U)J = 0$ and this is the defining relation of a para-Kähler structure on the fibres.

Concerning the quasi para-Sasakian structure, the defining relation $d\phi = 0$ gives $d\hat{\phi} = 0$ so that $d\hat{\Omega} = 0$. Since $N^{(1)} = 0$ then $\hat{N}_J = 0$. We then reach the defining relation of a para-Kähler structure on the fibres.

Consider the case of a para-Kenmotsu manifold, which is defined by $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E,G)\}$, $d\eta = 0$ and $N^{(1)} = 0$,

These relations become $d\hat{\phi}(U,V,W) = \frac{2}{3}\mathcal{G}\left\{\eta(U)\hat{\phi}(V,W)\right\}$, and $\hat{N}_J = 0$ on the fibres. Since η vanishes on vertical vector fields, we have $d\hat{\phi}(U,V,W) = 0$, which gives $d\hat{\Omega} = 0$. On the other hand, $\hat{N}_J(U,V) = N^{(1)}(U,V) = 0$. Therefore, the fibres are defined by $d\hat{\Omega} = 0 = \hat{N}_J$, which are the defining relations of the para-Kähler structure. □

Proposition 3.5. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion. If the total space is almost para-cosymplectic or an almost para-Kenmotsu manifold, then the fibres are almost para-Kählerian.*

Proof. As in the preceding proposition, the problem is to show that $d\hat{\Omega} = 0$ which is the defining relation of an almost para-Kähler structure.

Let the total space M be endowed with an almost para-cosymplectic structure. As in the preceding proposition, on the fibres, its defining relation gives $d\hat{\Omega} = 0$ which defines an almost para-Kähler structure.

Concerning the case of almost para-Kenmotsu structure, we have also $d\hat{\Omega} = 0$ because $d\hat{\phi} = 0$ which gives $d\hat{\Omega} = 0$ as already established. □

Proposition 3.6. *Assume that $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ is an almost paracontact metric submersion. If the total space is nearly para-cosymplectic, nearly para-Kenmotsu or closely para-cosymplectic, then the fibres are nearly para-Kählerian.*

Proof. To establish that the fibres are nearly para-Kählerian, we have to show that $(\hat{\nabla}_U J)U = 0$.

Note that a nearly para-Kähler structure is defined by $(\nabla_D \Omega)(D, E) = 0$ which can be expressed as $g(E, (\nabla_D J)D) = 0$. With this, we see that $(\nabla_D J)D$ is orthogonal to E. But on the fibres, since V is vertical, $g(V, (\hat{\nabla}_U J)U) = 0$ implies that $(\hat{\nabla}_U J)U = 0$.

Let us consider the case of nearly para-cosymplectic, defined by $(\nabla_D \varphi)D = 0$. It is clear that on the fibres one has $(\hat{\nabla}_U J)U = 0$ defining a nearly para-Kähler structure.

In the same way, a closely para-cosymplectic structure is defined by $(\nabla_D \varphi)D = 0 = d\eta$ so that on the fibres we have $(\hat{\nabla}_U J)U = 0$.

Consider the case of nearly para-Kenmotsu structure which verifies $(\nabla_D \varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$. On the fibres, this condition becomes $(\hat{\nabla}_U J)U = 0$ because $\eta(U) = 0$, we then get the nearly para-Kähler structure. □

Proposition 3.7. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion. If the total space is quasi para-K-cosymplectic or a quasi para-Kenmotsu manifold, then the fibres are quasi para-Kählerian.*

Proof. A quasi para-Kähler structure means that $(\hat{\nabla}_U J)V + (\hat{\nabla}_{JU} J)JV = 0$.

If the total space is quasi para-K-cosymplectic, as in the preceding cases, the fibres verify $(\hat{\nabla}_U J)V + (\hat{\nabla}_{JU} J)JV = 0$ because of the vanishing of η on vertical vector fields. Thus one obtains the defining relation of a quasi para-Kähler structure.

Considering the case of a quasi para-Kenmotsu manifold, we have $(\hat{\nabla}_U \hat{\Omega})(V, W) + (\hat{\nabla}_{JU} \hat{\Omega})(JV, W) = 0$ which is the defining relation of a quasi para-Kähler structure.

□

3.3 Superminimality of the fibres

Now we want to examine the superminimality of the fibres. We would like to begin by investigating the classes of almost paracontact metric submersions whose fibres are, or are not, superminimal in a natural way.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost paracontact metric manifold and $\bar{M}$ a φ-invariant submanifold of M. If, $\nabla_V \varphi = 0$ for all V tangent to $\bar{M}$, then $\bar{M}$ is said to be superminimal.

In order to verify the superminimality of the fibres of an almost paracontact metric submersion, there are four components of $g((\nabla_V \varphi)D, E)$ to be considered on the total space M. From [11, 12] we recall that

SM-1) $g((\nabla_V\varphi)U,W)=g(\hat{\nabla}_V(\hat{J}U)-\hat{J}\hat{\nabla}_VU,W)$,

SM-2) $g((\nabla_V\varphi)U,X)=g(T_V(\varphi U)-\varphi(T_VU),X)$,

SM-3) $g((\nabla_V\varphi)X,U)=-g((\nabla_V\varphi)U,X)$,

SM-4) $g((\nabla_V\varphi)X,Y)=-g(A_{\varphi X}Y+A_X(\varphi Y),V)$.

Proposition 3.8. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion. If the total space is para-cosymplectic, then the fibres are superminimal.*

Proof. Obvious. □

Proposition 3.9. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion. If the total space is a para-Kenmotsu manifold, then the fibres cannot be superminimal.*

Proof. Suppose that the fibres are superminimal. This means that $\nabla_U\varphi=0$ for all vector fields U tangent to the fibres. But on para-Kenmotsu manifold we have $0=g((\nabla_U\varphi)\varphi U,\xi)=g(\varphi U,\varphi U)g(\xi,\xi)=\|U\|^2$. If $\|U\|^2=0$ then $U=0$ which is not true. Thus, the fibres cannot be superminimal. □

Now, let us consider the integrability of the horizontal distribution.

Recall that the horizontal distribution of a Riemannian submersion is said to be integrable if the O'Neill tensor A vanishes identically (i.e. A$\equiv 0$).

Proposition 3.10. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost paracontact metric submersion such that the total space is almost para-cosymplectic or quasi para-Sasakian. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. It is not difficult to show that $A_X\varphi Y=\varphi A_XY$ for the three mentioned almost paracontact metric submersions. If the fibres are superminimal, we have $g((\nabla_U\varphi)X,Y)=-g(A_{\varphi X}Y+A_X\varphi Y,U)$, which implies that A$\equiv 0$. □

As in [14], we are able to use the superminimality of the fibres to induce a specific almost paracontact metric structure onto the total space of an almost paracontact metric submersion, provided that certain necessary structures exist on the base space and the fibres.

We begin by proving a technical result.

Lemma 3.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m+1}$ be an almost paracontact metric submersion. Suppose that $d\eta'=0$ on the base space. If the fibres are superminimal, then $d\eta=0$ on the total space.*

Proof. In order to see that $d\eta = 0$, we begin by assuming that X and Y are basic vector fields on the total space. Then $d\eta(X,Y) = d\eta'(X_*, Y_*) = 0$. The vanishing of expression $SM-2)$ implies, along with $A_{\varphi X} = 0$ that $A \equiv 0$. Now

$$\begin{aligned} 2d\eta(X,U) &= (\nabla_X \eta)U - (\nabla_U \eta)X \\ &= g(X, \nabla_U \xi) - g(U, \nabla_X \xi) \\ &= g(X, \nabla_U \xi) - g(U, A_X \xi) \\ &= g(X, \nabla_U \xi). \end{aligned}$$

The superminimality of the fibres implies that

$$\begin{aligned} 0 &= g((\nabla_U \varphi)\xi, X) \\ &= g(\nabla_U \varphi\xi, X) - g(\varphi\nabla_U \xi, X) \\ &= g(\nabla_U \xi, \varphi X) \end{aligned}$$

.

Thus, $\nabla_U \xi$ is $g-$orthogonal to all vector fields except, perhaps, ξ. Recall that $\|\xi\|^2 = g(\xi,\xi)$ is constant 1, so that $g(\nabla_U \xi, \xi) = 0$. Hence $d\eta(X,U) = 0$ and $d\eta(U,X) = 0$. Recall, too, that the Lie bracket $[U,V]$ is vertical from the complete integrability of the vertical distribution. Then

$$d\eta(U,V) = \frac{1}{2}\{U\eta(V) - V\eta(U) - \eta([U,V])\} = 0,$$

because η vanishes on the vertical distribution.

□

Theorem 3.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m+1}$ be an almost paracontact metric submersion. Assume that the base space is nearly para-cosymplectic, nearly para-K-cosymplectic or nearly para-Kenmotsu. If the fibres are superminimal, then the total space is respectively nearly para-cosymplectic, nearly para-K-cosymplectic or nearly para-Kenmotsu.*

Proof. There are four expressions that must vanish in order to conclude that the total space is nearly para-cosymplectic:

$NPC-1)$ $g((\nabla_U \varphi)U, V)$;
$NPC-2)$ $g((\nabla_U \varphi)U, X)$;
$NPC-3)$ $g((\nabla_X \varphi)X, U)$;
$NPC-4)$ $g((\nabla_X \varphi)X, Y)$.

The superminimality of the fibres implies that the first two expressions are zero. We may assume that the horizontal vector fields X and Y are basic for expression $NPC-4)$, in which case that expression vanishes because the base space is nearly para-cosymplectic. Finally,

$$\begin{aligned} g((\nabla_X \varphi)X, U) &= g(\nabla_X \varphi X, U) - g(\varphi\nabla_X X, U) \\ &= g(\nabla_X \varphi X, U) \\ &= 0 \end{aligned}$$

yielding the vanishing of expression $NPC-3$). Concerning the case of nearly para-K-cosymplectic structure on the base space, we need only establish that $\nabla\eta = 0$ on the total space; that is, we must show that $\nabla_E\xi = 0$ for all vector fields, E, on M. But $\nabla_X\xi = 0$ by projection onto the base space. For $\nabla_U\xi$, we know that $0 = (\nabla_U\varphi)\xi$ by the superminimality of the fibres. Thus

$$\begin{aligned} 0 &= \nabla_U\varphi\xi - \varphi\nabla_U\xi \\ &= -\varphi\varphi\nabla_U\xi \\ &= \nabla_U\xi - \eta(\nabla_U\xi)\xi \end{aligned}$$

But, during the proof of Lemma 3.1, we established that

$$\eta(\nabla_U\xi) = g(\nabla_U\xi, \xi) = 0.$$

Therefore, $\nabla\eta = 0$ and M is nearly para-K-cosymplectic.

Now, let us consider the case of the nearly para-Kenmotsu structure. Lemma 3.1 implies that $d\eta = 0$ on the total space. Since η vanishes on the vertical distribution, we need only to show that $(\nabla_U\varphi)U = 0$ and that $0 = (\nabla_X\varphi)X + \eta(X)\varphi X$. Let X be basic, then

$$(\nabla_X\varphi)X + \eta(X)\varphi X = (\nabla'_{X_*}\varphi')X_* + \eta'(X_*)\varphi' X_* = 0.$$

Clearly, $(\nabla_U\varphi)U = 0$ because the fibres are superminimal. Therefore, the total space is nearly para-Kenmotsu. □

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