J. Math. Tokushima Univ.
Vol. 49 (2015), 39–58

Fourier Transformation of L^2_{loc}-functions

By

Yoshifumi Ito

Professor Emeritus, The University of Tokushima
Home Address : *209-15 Kamifukuman Hachiman-cho*
Tokushima 770-8073, Japan
e-mail address : *yoshifumi@md.pikara.ne.jp*

(Received September 30, 2015)

Abstract

In this paper, we study the Fourier transformation of L^2_{loc}-functions and L^2_c-functions in order to investigate the natural statistical phenomena by using the theory of natural statistical physics. Thereby we prove the structure theorems of the image spaces $\mathcal{F}L^2_{\mathrm{loc}}$ and $\mathcal{F}L^2_c$. We study the convolution $f * g$ of a L^2_c-function f and a L^2_{loc}-function g. Further, we characterize the local Sobolev spaces and the space of solutions of Schrödinger equations. Here assume $d \geq 1$. These results are the English version of Ito [17], chapter 5.

2000 Mathematics Subject Classification. Primary 42B10; Secondary 42A38, 42A85, 46E30, 46E35, 46F20.

Introduction

In this paper, we study the Fourier transformation of L^2_{loc}-functions and L^2_c-functions and some applications.

In section 1, we define the Fourier transformation and the inverse Fourier transformation of L^2_{loc}-functions. We show some examples of Fourier transformation of L^2_{loc}-functions. We prove the inversion formulas of the Fourier transformation and the inverse Fourier transformation of L^2_{loc}-functions.

In section 2, using Paley-Wiener theorem for L^2-functions, we prove the structure theorems of the function spaces L^2_{loc} and L^2_c and the structure theorems of the Fourier images $\mathcal{F}L^2_{\mathrm{loc}}$ and $\mathcal{F}L^2_c$.

In section 3, we study the convolution $f * g$ of a function f in $L^2_c = L^2_c(\boldsymbol{R}^d)$ and a function g in $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\boldsymbol{R}^d)$.

In section 4, we define the local Sobolev space $H^s_{\mathrm{loc}}(\boldsymbol{R}^d)$, $(-\infty < s < \infty)$, and study its fundamental properties.

In section 5, we determine the space of solutions of Schrödinger equations which describe the law of natural statistical phenomena in the space $\boldsymbol{R}^d$. This space is determined by virtue of the framework of my theory of natural statistical physics.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Fourier transformation of L^2_{loc}-functions

In this section, at first we define the Fourier transformation of L^2_{loc}-functions and its fundamental properties.

Let $\boldsymbol{R}^d$ be the d-dimensional Euclidean space. Here assume $d \geq 1$. Further we denote $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\boldsymbol{R}^d)$ as usual.

For the points in $\boldsymbol{R}^d$

$$x = {}^t(x_1,\ x_2,\ \cdots,\ x_d),\ p = {}^t(p_1,\ p_2,\ \cdots,\ p_d),$$

we define

$$\begin{aligned} px &= (p,\ x) = p_1x_1 + p_2x_2 + \cdots + p_dx_d, \\ |x| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}, \\ |p| &= \sqrt{p_1^2 + p_2^2 + \cdots + p_d^2}. \end{aligned}$$

Let $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ be the space of all C^∞-functions with compact support in $\boldsymbol{R}^d$.

Here we define the Fourier transformation $\mathcal{F}$ by the relation

$$(\mathcal{F}\varphi)(p) = \frac{1}{(\sqrt{2\pi})^d}\int \varphi(x)e^{-ipx}dx,\ (p \in \boldsymbol{R}^d)$$

for $\varphi \in \mathcal{D}$. $\mathcal{F}\mathcal{D}$ denotes the space of the Fourier image of $\mathcal{D}$ by the Fourier transformation $\mathcal{F}$.

Further, let $\mathcal{D}' = \mathcal{D}'(\boldsymbol{R}^d)$ be the space of Schwartz distributions on $\boldsymbol{R}^d$.

Here, for the dual pair $\mathcal{D}'$ and $\mathcal{D}$ of two TVS's, we denote the dual inner product of $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ as $< T,\ \varphi >$ and, for the dual pair $(\mathcal{F}\mathcal{D})'$ and $\mathcal{F}\mathcal{D}$, we denote its dual inner product of $S \in (\mathcal{F}\mathcal{D})'$ and $\varphi \in \mathcal{F}\mathcal{D}$ as $< S,\ \varphi >$.

Now assume $T \in \mathcal{D}'$. Then, since we have $\mathcal{F}^{-1}\varphi \in \mathcal{D}$ for $\varphi \in \mathcal{F}\mathcal{D}$, we can define a continuous linear functional

$$S:\ \varphi \to < T,\ \mathcal{F}^{-1}\varphi >,\ (\varphi \in \mathcal{F}\mathcal{D})$$

and we have $S \in (\mathcal{F}\mathcal{D})'$. Namely, we have the equality

$$< S,\ \varphi > = < T,\ \mathcal{F}^{-1}\varphi > .$$

Then we define that S is a Fourier transform of T and denote it as $S = \mathcal{F}T$.

This is the new definition of the Fourier transformation of $\mathcal{D}'$. Since a Schwartz distribution is a generalized concept of functions, we had better to define the Fourier transformation of Schwartz distributions as in the same direction as the Fourier transformation of classical functions. Thus we define the new type of Fourier transformation of Schwartz distributions.

Therefore, for the Fourier transform $\mathcal{F}T \in \mathcal{F}\mathcal{D}'$ of $T \in \mathcal{D}'$, we have the relation

$$< \mathcal{F}T,\ \mathcal{F}\varphi > = < T,\ \varphi >,\ (\varphi \in \mathcal{D}).$$

This is a generalization of Parseval's foumula for L^2-functions. Then the Fourier transformation $\mathcal{F}$ is a topological isomorphism from $\mathcal{D}'$ to $\mathcal{F}\mathcal{D}'$.

Thus we have the isomorphisms

$$\mathcal{D}' \cong \mathcal{F}\mathcal{D}' \cong (\mathcal{F}\mathcal{D})'.$$

Here we denote the dual mapping of the Fourier transformation $\mathcal{F}:\ \mathcal{D} \to \mathcal{F}\mathcal{D}$ as $\mathcal{F}^*:\ (\mathcal{F}\mathcal{D})' \to \mathcal{D}'$. Then we have the equality

$$\mathcal{F}^*\mathcal{F} = \text{the identity mapping of } \mathcal{D}'.$$

We define the Fourier transformation of $f \in L^2_{\mathrm{loc}}$ considering it as an element of $\mathcal{D}'$.

We say that the limit in the sense of the topologies of $\mathcal{D}'$ or $\mathcal{F}\mathcal{D}'$ is **the limit in the sense of generalized functions**.

Then we give the following definition.

Definition 1.1 We define the Fourier transform $(\mathcal{F}f)(p)$ of $f \in L^2_{\mathrm{loc}}$ by the relation

$$(\mathcal{F}f)(p) = \lim_{R\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int_{|x|\le R} f(x)e^{-ipx}dx$$

in the sense of generalized functions.

Then we denote $\mathcal{F}f(p)$ as

$$(\mathcal{F}f)(p) = \frac{1}{(\sqrt{2\pi})^d} \int f(x)e^{-ipx}dx.$$

Here, when the integration domain is equal to the entire space $\boldsymbol{R}^d$, we omit the symbol of the integration domain.

Let $\mathcal{C} = \mathcal{C}(\boldsymbol{R}^d)$ be the function space of all continuous functions on $\boldsymbol{R}^d$. Then we have the inclusion relation

$$\mathcal{C} \subset L^2_{\text{loc}}.$$

Therefore, we can define the Fourier transformation of continuous functions which are not necessarily L^2-functions considering that they are L^2_{loc}-functions.

Example 1.1 We have the following equality:

$$(\mathcal{F}(-ix)^\alpha)(p) = \frac{1}{(\sqrt{2\pi})^d} \int (-ix)^\alpha e^{-ipx} dx = (\sqrt{2\pi})^d \delta^{(\alpha)}(p).$$

Here $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ denotes a multi-index of natural numbers.

Especially, for $\alpha = 0 = (0,\ 0,\ \cdots, 0)$, we have the equality

$$(\mathcal{F}1)(p) = \frac{1}{(\sqrt{2\pi})^d} \int e^{-ipx} dx = (\sqrt{2\pi})^d \delta(p).$$

Therefore, the Fourier transform of the constant function $\frac{1}{(\sqrt{2\pi})^d}$ is equal to the Dirac measure δ. Thereby, in general, the Fourier transform $\mathcal{F}f$ of a L^2_{loc}-function f is not necessarily a L^2_{loc}-function. As for this fact, my classmate Dr Kôzô Yabuta gives me this advice.

Remark 1.1 The L^2_{loc}-function which determines the natural statistical distribution of a certain physical system must be a solution of a certain Schrödinger equation.

In general, there is no non-constant polynomial solution of a certain Schrödinger equation . Therefore, in order to determine a natural statistical distribution, we have not to consider the Fourier transformation of non-constant polynomial functions.

Now we give some examples of Fourier transforms of continuous functions.

Example 1.2 Assume $-\infty < p,\ q < \infty$. Then we have the following (1) and (2):

(1) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin qx e^{-ipx} dx = \sqrt{\frac{\pi}{2}} \frac{1}{i} (\delta(p-q) - \delta(p+q)).$$

(2) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos qx e^{-ipx} dx = \sqrt{\frac{\pi}{2}} (\delta(p-q) + \delta(p+q)).$$

In the following Example 1.3 $\sim$ Example 1.5, the convergence of series is considered to be the convergence in the sense of generalized functions.

Example 1.3 The Fourier transform $\hat{f}(p)$ of Riemann's function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}, \ (-\infty < x < \infty)$$

is equal to

$$\hat{f}(p) = \sqrt{\frac{\pi}{2}} \frac{1}{i} \sum_{n=1}^{\infty} \frac{1}{n^2} (\delta(p - n^2) - \delta(p + n^2)), \ (-\infty < p < \infty).$$

Example 1.4 We assume that two constants a, b satisfy the following conditions (i)$\sim$(iii):

(i) $0 < a < 1$. (ii) b is a odd number. (iii) We have $ab > 1 + \frac{3}{2}\pi$.

Then the Fourier transform $\hat{f}(p)$ of Weierstrass function

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x), \ (-\infty < x < \infty)$$

is equal to

$$\hat{f}(p) = \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} (\delta(p - b^n \pi) + \delta(p + b^n \pi)), \ (-\infty < p < \infty).$$

Example 1.5 Assume that a is an even number. Then the Fourier transform $\hat{f}(p)$ of Cellérier function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(a^n x)}{a^n}, \ (-\infty < x < \infty)$$

is equal to

$$\hat{f}(p) = \sqrt{\frac{\pi}{2}} \frac{1}{i} \sum_{n=1}^{\infty} \frac{1}{a^n} (\delta(p - a^n) - \delta(p + a^n)), \ (-\infty < p < \infty).$$

Example 1.6 Assume $d \geq 1$. The constant function 1 belongs to $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\boldsymbol{R}^d)$. For $R > 0$, we put $\chi_R(x) = \chi_{|x| \leq R}(x)$. Then we have $\chi_R \in L^2_{\mathrm{loc}}$ and we have

$$\chi_R \to 1, \ (R \to \infty)$$

in the topology of L^2_{loc}-convergence. Thus we have

$$\chi_R \to 1,\ (R \to \infty)$$

in the topology of $\mathcal{D}'$. Then we have, for $R \to \infty$,

$$\hat{\chi}_R(p) = \frac{1}{(\sqrt{2\pi})^d}\int \chi_R(x)e^{-ipx}dx \to \frac{1}{(\sqrt{2\pi})^d}\int e^{-ipx}dx = \hat{1}(p) = (\sqrt{2\pi})^d\delta(p)$$

in the topology of $\mathcal{FD}'$.

Example 1.7 For $n \geq 1$, we put

$$\chi_n(x) = \chi_{[-n,\, n]}(x),\ (x \in \boldsymbol{R}).$$

Then we have $\chi_n \in L^2_{\mathrm{loc}}$ and we have

$$\chi_n \to 1, (n \to \infty)$$

in the topology of L^2_{loc}-convergence.
Thus we have

$$\chi_n \to 1, (n \to \infty)$$

in the topology of $\mathcal{D}'$. Then we have, for $n \to \infty$,

$$\hat{\chi}_n((p) = \frac{1}{\sqrt{2\pi}}\int \chi_n(x)e^{-ipx}dx \to \frac{1}{\sqrt{2\pi}}\int e^{-ipx}dx = \hat{1}(p) = \sqrt{2\pi}\delta(p)$$

in the topology of $\mathcal{FD}'$.

Example 1.8 We have

$$\frac{1}{\pi}\frac{\sin pn}{p} \to \delta(p),\ (n \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof We have the equality

$$\frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{-ipx}dx = \frac{1}{ip\sqrt{2\pi}}(e^{ipn} - e^{-ipn}) = \sqrt{\frac{2}{\pi}}\frac{\sin pn}{p}.$$

Thus we have the conclusion by virtue of Example 1.7.//

Example 1.9 Assume $d \geq 1$. Let $n = (n_1,\ n_2,\ \cdots,\ n_d)$ be a multi-index of positive natural numbers. We denote $|n| = n_1 + n_2 + \cdots + n_d$. By using the notation of Example 1.7, we denote

$$\chi_n(x) = \chi_{n_1}(x_1)\chi_{n_2}(x_2)\cdots\chi_{n_d}(x_d),\ (x \in \boldsymbol{R}^d),$$

$$\hat{\chi}_n(p) = \hat{\chi}_{n_1}(p_1)\hat{\chi}_{n_2}(p_2)\cdots\hat{\chi}_{n_d}(p_d),\ (p \in \boldsymbol{R}^d).$$

Then we have

$$\hat{\chi}(p) \to (\sqrt{2\pi})^d\delta(p),\ (|n| \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof By virtue of Example 1,7, because we have

$$\hat{\chi}_{n_j}(p_j) \to \sqrt{2\pi}\delta(p_j)$$

for $1 \le j \le d$, we have the conclusion. //

Theorem 1.1 *We use the same notation as Example* 1.9. *Then, for*

$$\chi_n(x) = \chi_{n_1}(x_1)\chi_{n_2}(x_2)\cdots\chi_{n_d}(x_d),\ (x \in \boldsymbol{R}^d),$$

we denote

$$\hat{\chi}_n(p) = \hat{\chi}_{n_1}(p_1)\hat{\chi}_{n_2}(p_2)\cdots\hat{\chi}_{n_d}(p_d),\ (p \in \boldsymbol{R}^d).$$

For $f(x) \in L^2_{\rm loc}$, we put $f_n(x) = \chi_n(x)f(x)$. Then we have $f_n(x) \in L^2_{\rm loc}$. Now, when we consider that f_n and f are elements of $\mathcal{D}'$, we denote their Fourier transformations as $\mathcal{F}f_n = \hat{f}_n$ and $\mathcal{F}f = \hat{f}$. Then we have

$$\hat{f}_n \to \hat{f},\ (|n| \to \infty)$$

in the topology of $\mathcal{FD}'$.

Proof When $|n| \to \infty$, we have

$$f_n(x) \to f(x),\ (x \in \boldsymbol{R}^d)$$

in the topology of $L^2_{\rm loc}$. Therefore, when $|n| \to \infty$, we have

$$f_n \to f$$

in the topology of $\mathcal{D}'$.

Since we have $f_n = \chi_n f$, we have the equality

$$\hat{f}_n = (\chi_n f)^\wedge = \frac{1}{(\sqrt{2\pi})^d}\hat{\chi}_n * f$$

in $\mathcal{FD}'$. Here the symbol $*$ denotes the convolution. By virtue of Example 1.9, we have

$$\hat{\chi}_n \to (\sqrt{2\pi})^d\delta,\ (|n| \to \infty).$$

Thus, when $|n| \to \infty$, we have

$$\hat{f}_n = \frac{1}{(\sqrt{2\pi})^d}\hat{\chi_n} * \hat{f} \to \delta * \hat{f} = \hat{f}$$

in the topology of $\mathcal{FD}'$. //

When we use the notation in Theorem 1.1, we have $\hat{f}_n \in L^2$ and

$$\hat{f}_n(p) = \frac{1}{(\sqrt{2\pi})^d} \int f_n(x) e^{-ipx} dx.$$

Therefore we have the equality

$$\lim_{|n|\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int f_n(x) e^{-ipx} dx = \hat{f}(p)$$

in $\mathcal{FD}'$. In this sense, we use the notation

$$\hat{f}(p) = \frac{1}{(\sqrt{2\pi})^d} \int f(x) e^{-ipx} dx$$

for $\hat{f}(p) \in \mathcal{FD}'$. Here we consider this integral in the sense of convergence in the topology of $\mathcal{FD}'$.

In this case, we say that this integral converges in the sense of generalized functions.

Similarly, we define the Fourier inverse transformation as follows.

Definition 1.2(Fourier inverse transformation) We define the Fourier inverse transformation of $g(p) \in L^2_{\text{loc}}$ by the relation

$$(\mathcal{F}^{-1}g)(x) = \lim_{R\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int_{|p|\le R} g(p) e^{ipx} dp$$

in the sense of generalized functions.

We denote $(\mathcal{F}^{-1}g)(x)$ as

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(\sqrt{2\pi})^d} \int g(p) e^{ipx} dp.$$

Theorem 1.2 *Let $\alpha = (\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_d)$ be a multi-index of natural numbers. Assume that $f(x) \in L^2_{\text{loc}}$ and $D^\alpha f(x) \in L^2_{\text{loc}}$ hold. Then we have the following* (1) *and* (2):

(1) $\mathcal{F}((-ix)^\alpha f)(p) = D^\alpha(\mathcal{F}f)(p).$

(2) $\mathcal{F}(D^\alpha f)(p) = (ip)^\alpha(\mathcal{F}f)(p).$

In Theorem 1.2, the symbols x^α and D^α etc. are the same as usually used. Namely $D^\alpha f$ means a L^2_{loc}-derivatives, and $D^\alpha(\mathcal{F}f)$ means, in general, a partial derivative of $\mathcal{F}f$ in $\mathcal{FD}'$ and so on .

Next we prove the Fourier inversion formula.

Now we assume $f \in L^2_{\mathrm{loc}}$. Then, since we have

$$f_R(x) \in L^2, \ (0 < R < \infty), \ (\mathcal{F}f_R)(p) \in L^2, \ (0 < R < \infty),$$

we have

$$\|\mathcal{F}f_R\| = \|f_R\|, \ (0 < R < \infty), \ \mathcal{F}^{-1}\mathcal{F}f_R(x) = f_R(x), \ (0 < R < \infty).$$

Then, since we have

$$f_R(x) \to f(x), \ (R \to \infty)$$

is the sense of generalized functions, we have the equality

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Therefore we have the following inversion formula.

Theorem 1.3(Inversion formula) *For $f(x) \in L^2_{\mathrm{loc}}$, we have the following inversion formula*

$$f(x) = \lim_{R\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int (\mathcal{F}f_R)(p) e^{ipx} dp = \frac{1}{(\sqrt{2\pi})^d} \int e^{ipx} dp \int f(y) e^{-ipy} dy.$$

Here the integral converges in the sense of generalized functions. Namely we have

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Similarly, for $g(p) \in L^2_{\mathrm{loc}}$, we denote the restriction of g to the closed ball $|p| \leq T$ as g_T. Then we have

$$\|\mathcal{F}^{-1}g_T\| = \|g_T\|, \ (0 < T < \infty), \ \mathcal{F}\mathcal{F}^{-1}g_T(p) = g_T(p), \ (0 < T < \infty).$$

Then, in the sense of generalized functions, we have

$$g_T(p) \to g(p), \ (T \to \infty).$$

Thus we have the equality

$$\mathcal{F}\mathcal{F}^{-1}g(p) = g(p)$$

in the sense of generalized functions.

Therefore we have the following inversion formula.

Theorem 1.4 (Inversion formula) *For $g \in L^2_{\mathrm{loc}}$, we have the following inversion formula*

$$g(p) = \frac{1}{(\sqrt{2\pi})^d} \int (\mathcal{F}^{-1}g)(x) e^{-ipx} dx = \frac{1}{(2\pi)^d} \int e^{-ipx} dx \int g(q) e^{iqx} dq.$$

Here the integral converges in the sense of generalized functions. Namely we have the equality

$$\mathcal{F}\mathcal{F}^{-1}g = g.$$

Theorem 1.5 *For $f \in L^2_{\mathrm{loc}}$, we have the equalities*:

$$\mathcal{F}^2 f(x) = f(-x),\ \mathcal{F}^4 f(x) = f(x).$$

2 Structure theorems

In this section, using Paley-Wiener theorem for L^2-functions, we study the structure theorems of the function spaces L^2_{loc} and L^2_c and the structure theorems of the Fourier images $\mathcal{F}L^2_{\mathrm{loc}}$ and $\mathcal{F}L^2_c$.

Now we choose an exhausting sequence $\{K_j\}$ of compact sets in $\boldsymbol{R}^d$ which satisfies the following conditions (i) and (ii):

(i) $K_1 \subset K_2 \subset \cdots \subset \boldsymbol{R}^d,\ \boldsymbol{R}^d = \bigcup_{j=1}^{\infty} K_j.$

(ii) $K_j = \mathrm{cl}(\mathrm{int}(K_j)),\ K_j \subset \mathrm{int}(K_{j+1}),\ (j = 1,\ 2,\ 3,\ \cdots).$

Then we denote the projective limit of projective system $\{L^2(K_j)\}$ of Hilbert spaces as

$$\varprojlim L^2(K_j).$$

Then we have the isomorphism

$$L^2_{\mathrm{loc}} \cong \varprojlim L^2(K_j)$$

as TVS's. Here, since, for each j, the restriction mapping $L^2(K_{j+1}) \to L^2(K_j)$ is a weakly compact mapping, L^2_{loc} is a FS*-space.

Further, because the system $\{L^2(K_j)\}$ of Hilbert spaces can be considered as an inductive system, we denote the inductive limit as

$$\varinjlim L^2(K_j).$$

Then we have the isomorphism

$$L^2_c \cong \varinjlim L^2(K_j)$$

as TVS's. Here L^2_c denotes the TVS of all L^2-functions with compact support. Then, since, for each j, the inclusion mapping $L^2(K_j) \to L^2(K_{j+1})$ is a weakly compact mapping, L^2_c is a DFS*-space.

Since $L^2(K_j)$ is a self-dual space, we have the isomorphism

$$L^2_{\rm loc} \cong (L^2_c)'$$

as TVS's. Here$(L^2_c)'$ denotes the dual space of L^2_c and we define the dual inner product of $f \in L^2_{\rm loc}$ and $g \in L^2_c$ by the equality

$$< f,\ g >= \int f(x)g(x)dx.$$

Here the dual inner product is a bilinear functional which defines the duality relation of the pair of two TVS's $L^2_{\rm loc}$ and L^2_c.

Then, because we have the inclusion relation $L^2_c \subset L^2$, we define the Fourier transformation of a L^2_c-function $g(x)$ by using the Fourier transformation of L^2-functions

$$\mathcal{F}g(p) = \frac{1}{(\sqrt{2\pi})^d} \int g(x)e^{-ipx}dx.$$

Further we define the Fourier transformation of a $L^2_{\rm loc}$-function f by the relation

$$\mathcal{F}f(p) = \lim_{j\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int_{K_j} f(x)e^{-ipx}dx$$

in the sense of generalized functions in $\mathcal{D}'$ and $\mathcal{F}\mathcal{D}'$.

By virtue of the definition of the Fourier transformation of $f \in L^2_{\rm loc}$, we have the equality

$$< \mathcal{F}f,\ \mathcal{F}g >=< f,\ g >$$

for any $g \in \mathcal{D}$.

Since a L^2_c-function g has the compact support, there exists some K_j such that $\mathrm{supp}(g) \subset K_j$ holds by the definition of $\{K_j\}$. Therefore, for an arbitrary $k \geq j$, we have the equalities

$$< f_{K_k},\ g >= \int_{K_k} f_{K_k}(x)g(x)dx = \int_{K_j} f(x)g(x)dx =< f,\ g > .$$

Here $f_{K_k}(x)$ denotes the image of $f(x) \in L^2_{\rm loc}$ by the restriction mapping $L^2_{\rm loc} \to L^2(K_k)$.

Since we have the equality

$$\int \mathcal{F}f_{K_k}(p)\mathcal{F}g(-p)dp = \int f_{K_k}(x)g(x)dx$$

by virtue of Parseval's formula, we have the equality

$$\lim_{k\to\infty} \int \mathcal{F}f_{K_k}(p)\mathcal{F}g(-p)dp = \lim_{k\to\infty} \int f_{K_k}(x)g(x)dx$$

$$= \int f_{K_j}(x)g(x)dx = \int \mathcal{F}f_{K_j}(p)\mathcal{F}g(-p)dp.$$

Especially, supposing that we have

$$\mathcal{D}_{K_j} \subset L^2(K_j),\ g \in \mathcal{D}_{K_j},$$

we have the equality

$$\int \mathcal{F}f(p)\mathcal{F}g(-p)dp = \int f(x)g(x)dx.$$

We can choose a compact set K_j arbitrarily. Thus, if we consider that $g \in \mathcal{D}_{K_j}$ holds for an arbitrary $\mathcal{D}_{K_j}$, we have the equality in the above for an arbitrary $g \in \mathcal{D}$.

Then, because the dual inner product

$$< f,\ g >= \int f(x)g(x)dx$$

is defined for an arbitrary $f \in L^2_{\mathrm{loc}}$ and $g \in L^2_c$, we have the equality

$$< \mathcal{F}f,\ \mathcal{F}g >= \int \mathcal{F}f(p)\mathcal{F}g(-p)dp = \int f(x)g(x)dx =< f,\ g >$$

for an arbitrary $f \in L^2_{\mathrm{loc}}$ and an arbitrary $g \in L^2_c$.

Now we choose one exhausting sequence $\{K_j\}$ of compact sets in $\boldsymbol{R}^d$ as in the above.

Then, for the sequence

$$L^2(K_1) \subset L^2(K_2) \subset \cdots,$$

we have the isomorphisms

$$L^2_c \cong \varinjlim L^2(K_j),\ L^2_{\mathrm{loc}} \cong \varprojlim L^2(K_j).$$

Further we have the isomorphisms

$$L^2_c \cong \bigcup_{j=1}^{\infty} L^2(K_j),\ L^2_{\mathrm{loc}} \cong \bigcap_{j=1}^{\infty} L^2(K_j).$$

Then we have the isomorphisms

$$\mathcal{F}L^2(K_j) \cong L^2(K_j),\ (j = 1,\ 2,\ 3,\ \cdots).$$

Further, for the sequence

$$\mathcal{F}L^2(K_1) \subset \mathcal{F}L^2(K_2) \subset \cdots,$$

we have the isomorphisms

$$\mathcal{F}L^2_c \cong \varinjlim \mathcal{F}L^2(K_j) \cong \varinjlim L^2(K_j) \cong L^2_c,$$

$$\mathcal{F}L^2_{\mathrm{loc}} \cong \varprojlim \mathcal{F}L^2(K_j) \cong \varprojlim L^2(K_j) \cong L^2_{\mathrm{loc}}.$$

Then we have the relations

$$\mathcal{F}L^2_{\mathrm{loc}} \subset \mathcal{F}\mathcal{D}',\ \mathcal{F}L^2_{\mathrm{loc}} \neq L^2_{\mathrm{loc}}.$$

Therefore we have the following theorem.

Theorem 2.1 *We use the notation in the above. Then we have the following isomorphisms* (1) $\sim$ (4):

(1) $L^2_c \cong \varinjlim L^2(K_j) \cong \bigcup_{j=1}^{\infty} L^2(K_j).$

(2) $\mathcal{F}L^2_c \cong \varinjlim \mathcal{F}L^2(K_j).$

(3) $\mathcal{F}L^2(K_j) \cong L^2(K_j),\ (j = 1,\ 2,\ 3,\ \cdots).$

(4) $\mathcal{F}L^2_c \cong L^2_c,\ \mathcal{F}L^2_c \subset L^2,\ L^2_c \subset L^2.$

Further we have the following theorem.

Theorem 2.2 *We use the notation in the above. Then we have the following isomorphisms* (1) $\sim$ (3) *and the relation* (4):

(1) $L^2_{\mathrm{loc}} \cong \varprojlim L^2(K_j) \cong \bigcap_{j=1}^{\infty} L^2(K_j) \cong (L^2_c)'.$

(2) $\mathcal{F}L^2_{\mathrm{loc}} \cong \varprojlim \mathcal{F}L^2(K_j).$

(3) $\mathcal{F}L^2_{\mathrm{loc}} \cong L^2_{\mathrm{loc}}.$

(4) $\mathcal{F}L^2_{\mathrm{loc}} \subset \mathcal{F}\mathcal{D}',\ \mathcal{F}L^2_{\mathrm{loc}} \neq L^2_{\mathrm{loc}},\ L^2_{\mathrm{loc}} \subset \mathcal{D}'.$

3 Convolution

In this section, we study the convolution $f*g$ of a function f in $L^2_c = L^2_c(\boldsymbol{R}^d)$ and a function g in $L^2_{\rm loc} = L^2_{\rm loc}(\boldsymbol{R}^d)$. Here assume $d \geq 1$.

We define the convolution $f * g$ of $f \in L^2_c$ and $g \in L_{\rm loc}$ by the relation

$$(f * g)(x) = \int f(x-y)g(y)dy.$$

Then we have the equality

$$\int f(x-y)g(y)dy = \int g(x-y)f(y)dy.$$

Therefore we have the following theorem.

Theorem 3.1 *For $f \in L^2_c$ and $g \in L^2_{\rm loc}$, we have $f * g \in L^2_{\rm loc}$. Further we have the relation*

$$f * g = g * f.$$

Theorem 3.2 *Let $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ be a multi-index of natural numbers. Then, for $f \in L^2_c$ and $g \in L^2_{\rm loc}$, we have the equality*

$$D^\alpha (f * g) = (D^\alpha f) * g = f * (D^\alpha g).$$

Here the partial derivatives are considered in the sense of topologies of L^2_c and $L^2_{\rm loc}$.

Corollary 3.1 *Assume $f \in L^2_c$. Then the linear transformation of $L^2_{\rm loc}$ defined by the convolution*

$$T_f :\ g \to f * g,\ (g \in L^2_{\rm loc})$$

is continuous in $L^2_{\rm loc}$.

Now assume that $\{g_n\}$ is a sequence of $L^2_{\rm loc}$-functions and it converges to $g \in L^2_{\rm loc}$ in the topology of $L^2_{\rm loc}$. Namely, assume that $g_n \to g,\ (n \to \infty)$ in the topology of $L^2_{\rm loc}$. Then we have

$$T_f(g_n) \to T_f(g),\ (n \to \infty).$$

Corollary 3.2 *Assume $g \in L^2_{\rm loc}$. Then the linear mapping $T_g = f * g,\ (f \in L^2_c)$ defined by the convolution is a continuous linear mapping from L^2_c into $L^2_{\rm loc}$.*

Therefore, if a sequence $\{f_n\}$ of functions in L^2_c convergences to $f \in L^2_c$ in the topology of L^2_c, we have

$$T_g(f_n) \to T_g(f), \ (n \to \infty).$$

Here the convolution of a function f in L^2_c and a function g in $L^2_{\rm loc}$ is a separately continuous bilinear mapping $L^2_c \times L^2_{\rm loc} \to L^2_{\rm loc}$.

Theorem 3.3 *Assume $f \in L^2_c$ and $g \in L^2_{\rm loc}$. Then we have*

$$\mathcal{F}(f * g) = (\sqrt{2\pi})^d \mathcal{F}(f)\mathcal{F}(g).$$

4 Characterization of the local Sobolev spaces

In this section, we define the local Sobolev space $H^s_{\rm loc}(\boldsymbol{R}^d)$ and study its fundamental properties. As for the precise concerning these results, we refer to Ito [1], [15], [16], [17]. This problem is the characterization of the local Sobolev space by using the Fourier transformation.

For a real number s, we define $L^{2,\,s} = L^{2,\,s}(\boldsymbol{R}^d)$ to be the Hilbert space of all complex valued measurable functions f which satisfies the condition

$$\int (1 + |x|^2)^s |f(x)|^2 dx < \infty.$$

Assume that s is a real number and $\mathcal{F}$ is the Fourier transformation of $L^2 = L^2(\boldsymbol{R}^d)$. Then we define the **Solobev space** $H^s = H^s(\boldsymbol{R}^d)$ to be the Hilbert space

$$H^s(\boldsymbol{R}^d) = \{f \in L^2(\boldsymbol{R}^d);\ \mathcal{F}f \in L^{2,\,s}(\boldsymbol{R}^d)\}.$$

Especially when m is a natural number, the Solobev space $H^m = H^m(\boldsymbol{R}^d)$ is equal to the Sobolev space

$$W^{m,\,2}(\boldsymbol{R}^d) = \{f \in L^2(\boldsymbol{R}^d);\ D^\alpha f \in L^2,\ |\alpha| \le m\}.$$

Here, for a multi-index $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ of natural numbers, the simbol

$$D^\alpha f = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_d}\Big)^{\alpha_d} f$$

denotes the L^2-derivative. Further we put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Then, for a real number s, the **local Sobolev space** $H^s_{\rm loc} = H^s_{\rm loc}(\boldsymbol{R}^d)$ is assumed to be the TVS of all complex valued measurable functions $f(x)$ on

$\boldsymbol{R}^d$ such that, for an arbitrary compact set K in $\boldsymbol{R}^d$, the function $f_K(x) = f(x)\chi_K(x)$ belongs to H^s. Here $\chi_K(x)$ denotes the characteristic function of the set K.

Now, for a real number s, we define the vector space $L^{2,\,s}_{\mathrm{loc}}$ by the condition

$$L^{2,\,s}_{\mathrm{loc}} = L^{2,\,s}_{\mathrm{loc}}(\boldsymbol{R}^d) = \{\ f \in L^2_{\mathrm{loc}};\ \sqrt{(1+|x|^2)^s} f(x) \in L^2_{\mathrm{loc}}\ \}\ .$$

$L^{2,\,s}_{\mathrm{loc}}$ is equal to the vector space of all complex valued measurable functions f on $\boldsymbol{R}^d$ which satisfy the condition

$$\int_K (1+|x|^2)^s |f(x)|^2 dx < \infty$$

for an arbitrary compact set K in $\boldsymbol{R}^d$.

We define the seminorm $\|f\|_{2,\,s,\,K}$ by the relation

$$\|f\|_{2,\,s,\,K} = \{\ \int_K (1+|x|^2)^s |f(x)|^2 dx\ \}^{1/2}\ .$$

Here K denotes a compact set K in $\boldsymbol{R}^d$.

Then the topology of $L^{2,\,s}_{\mathrm{loc}}$ is defined by the system of seminorms

$$\{\ \|\cdot\|_{2,\,s,\,K};\ K \text{ is a compact set in } \boldsymbol{R}^d\ \}\ .$$

Thereby $L^{2,\,s}_{\mathrm{loc}}$ is a Fréchet space.

For $f \in L^{2,\,s}_{\mathrm{loc}}$, we have the inequalities

$$B_K \int_K |f(x)|^2 dx \le \int_K (1+|x^2|)^s |f(x)|^2 dx \le C_K \int_K |f(x)|^2 dx.$$

Here B_K and C_K are two positive constants depending on K.

Therefore, for an arbitrary real number s, we have the equality $L^{2,\,s}_{\mathrm{loc}} = L^2_{\mathrm{loc}}$ as the sets of functions. Then $L^{2,\,s}_{\mathrm{loc}}$ is equal to the LCV L^2_{loc} endowed with the topology defined by the system of seminorms $\{\|\cdot\|_{2,\,s,\,K};\ K$ is a compact set in $\boldsymbol{R}^d\}$.

Now, for $f \in L^2_{\mathrm{loc}}$ and a compact set K in R^d, we define the seminorm $\|f\|_K$ of L^2_{loc} by the relation

$$\|f\|_K = (\ \int_K |f(x)|^2 dx\ )^{1/2}\ .$$

Thereby L^2_{loc} is a Fréchet space. Here, because the topologies of $L^{2,\,s}_{\mathrm{loc}}$ and L^2_{loc} are equivalent, $L^{2,\,s}_{\mathrm{loc}}$ and L^2_{loc} are topologically isomorphic.

Then we have the following theorem.

Theorem 4.1 *For a real number s, we have the equality*

$$H^s_{\mathrm{loc}} = H^s_{\mathrm{loc}}(\boldsymbol{R}^d) = \{\ f \in L^2_{\mathrm{loc}};\ \mathcal{F}f \in L^{2,\,s}_{\mathrm{loc}}\ \}\ .$$

Here $\mathcal{F}f \in L^{2,\,s}_{\rm loc}$ *is the Fourier transform of* $f \in L^2_{\rm loc}$.

Since, in general, we happen to have $\mathcal{F}f \notin L^2_{\rm loc}$ for $f \in L^2_{\rm loc}, \mathcal{F}f \in L^{2,\,s}_{\rm loc}$ is one restriction condition. In fact, though we have $1 \in L^2_{\rm loc}$, we have $1 \notin H^s_{\rm loc}$.

Especially, for a natural number m, we have the equalities

$$H^m_{\rm loc} = H^m_{\rm loc}(\boldsymbol{R}^d) = W^{m,\,2}_{\rm loc}(\boldsymbol{R}^d) = \{\ f \in L^2_{\rm loc};\ D^\alpha f \in L^2_{\rm loc},\ |\alpha| \le m\ \}\ .$$

Here , for a multi-index $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d)$ of natural numbers, $D^\alpha f$ means the $L^2_{\rm loc}$-derivatives as same as in the case of L^2. Further we put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Then, for $f \in H^m_{\rm loc}$ and an arbitrary compact set K in $\boldsymbol{R}^d$, we define the seminorm $\|f\|_{m,\,K}$ of $H^m_{\rm loc}$ by the relation

$$\|f\|_{m,\,K} = \Big(\sum_{|\alpha|\le m} \|D^\alpha f\|^2_K\ \Big)^{1/2}\ .$$

Thereby, the topology of $H^m_{\rm loc}$ is defined by the system of seminorms

$$\{\ \|\cdot\|_{m,\,K};\ K \text{ is a compact set in } \boldsymbol{R}^d\ \}\ .$$

Therefore $H^m_{\rm loc}$ is a Fréchet space.

Especially we remark that $H^0_{\rm loc} \subsetneq W^{0,\,2}_{\rm loc} = L^2_{\rm loc}$ holds.

In the sequel, we denote $H^0_{\rm loc}$ as $H_{\rm loc}$. Then $H_{\rm loc}$ is the closed subspace of $L^2_{\rm loc}$.

Since, for all real number s, we have

$$L^{2,\,s}_{\rm loc} = L^2_{\rm loc}$$

as sets of functions, we have, for all real number s

$$H^s_{\rm loc} = H_{\rm loc}$$

similarly.

Therefore, for the Fourier transform $\mathcal{F}f(p) \in L^2_{\rm loc}$ of $f(x) \in H_{\rm loc}$, we have the equality

$$\mathcal{F}f(p) = \lim_{R\to\infty} \frac{1}{(\sqrt{2\pi})^d} \int_{|x|\le R} f(x)e^{-ipx}dx$$

in the topology of $L^2_{\rm loc}$. Further, since we have $\mathcal{F}f(p) \in L^2_{\rm loc}$ for $f(x) \in H^s_{\rm loc}$, we have the Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^d} \int e^{ipx}dp \int f(y)e^{-ipy}dy$$

in the topology of $L^2_{\rm loc}$.

Remark 4.1 Since the conditions of definitions H^s and H^s_{loc} are given by the integral estimates of the classical functions, we remark that H^s and H^s_{loc} are some classes of classical functions and are characterized without using the theory of distributions.

Further, the fact that $L^{2,\,s}_{\mathrm{loc}}$ and H^s_{loc} are different TVS's for some different real number s means that the definitions of the topologies of those TVS's are different

5 Characterization of solutions of Schrödinger equations

In this section, we determine the space of solutions of Schrödinger equations which describe the law of natural statistical phenomena in the space $\boldsymbol{R}^d$. Here assume $d \geq 1$.

Now assume that, for $p \in \boldsymbol{R}^d$, $\psi_p(x) \in L^2_{\mathrm{loc}}$ satisfies the condition $\hat{\psi}_p(q) = \delta_p(q)$. Then we have $\hat{\delta}_p(x) = \psi_{-p}(x)$.

When we denote the subspace of $\mathcal{D}'$ spanned by $\{\psi_p, \delta_p;\ p \in \boldsymbol{R}^d\}$ as $V\{\psi_p,\ \delta_p;\ p \in \boldsymbol{R}^d\}$, we define the subspace $\mathcal{N}$ of $\mathcal{D}'$ by the relation

$$\mathcal{N} = H_{\mathrm{loc}} \oplus V\{\psi_p,\ \delta_p;\ p \in \boldsymbol{R}^d\}.$$

Then we have the inclusion relation

$$L^2 \subset H^2_{\mathrm{loc}},\ L^2 \subset \mathcal{N}.$$

The function space $\mathcal{C}_0 = \mathcal{C}_0(\boldsymbol{R}^d)$ is the TVS of all continuous functions with compact support in $\boldsymbol{R}^d$.

We say that a continuous linear functional μ on $\mathcal{C}_0$ as a **Radon measure** on $\boldsymbol{R}^d$.

The TVS of all Radon measures on $\boldsymbol{R}^d$ is equal to the dual space $(\mathcal{C}_0)'$.

Then we have the inclusion relation

$$\mathcal{N} \subset (\mathcal{C}_0)' \subset \mathcal{D}'.$$

When $f \in \mathcal{N}$ and $f \neq \delta_p$, $(p \in \boldsymbol{R}^d)$, we define $\mu \in (\mathcal{C}_0)'$ by the relation

$$\mu(\varphi) = \int f(x)\varphi(x)dx,\ (\varphi \in \mathcal{C}_0).$$

Then, if we denote $\mu = \mu_f$, the correspondence $f \to \mu_f$ is one to one correspondence.

Thereby, when $f \in \mathcal{N}$ and $f \neq \delta_p$, $(p \in \boldsymbol{R}^d)$, we can identify f and $\mu = \mu_f \in (\mathcal{C}_0)'$.

When $f \in \mathcal{N}$ and $f \in H_{\rm loc}$, we have $f \in L^2$ or $f \in L^2_{\rm loc}$. Therefore, we have $\mathcal{F}f \in L^2$ or $\mathcal{F}f \in L^2_{\rm loc}$ respectively.

Further, because we have

$$\mathcal{F}\psi_p = \delta_p,\ \mathcal{F}\delta_p = \psi_{-p},\ (p \in \boldsymbol{R}^d),$$

the Fourier transformation $\mathcal{F}$ is the isomorphism

$$\mathcal{F}:\ \mathcal{N} \to \mathcal{F}\mathcal{N}.$$

Hence we have

$$\mathcal{F}\mathcal{N} \cong \mathcal{N},\ \mathcal{F}\mathcal{N} \subset L^2_{\rm loc} + V\{\psi_p,\ \delta_p;\ p \in \boldsymbol{R}^d\}.$$

Thus we have the following theorem.

Theorem 5.1 *We use the notations in the above. We define the subspace $\mathcal{N}$ of $\mathcal{D}'$ by the relation*

$$\mathcal{N} = H_{\rm loc} \oplus V\{\psi_p,\ \delta_p;\ p \in \boldsymbol{R}^d\}.$$

Denoting the Fourier transformation of $\mathcal{D}'$ as $\mathcal{F}$, we have the following (1) *and* (2):

(1) *We have the isomorphism*

$$\mathcal{F}\mathcal{N} \cong \mathcal{N}.$$

(2) *We have the inclusion relation*

$$\mathcal{F}\mathcal{N} \subset L^2_{\rm loc} + V\{\psi_p,\ \delta_p;\ p \in \boldsymbol{R}^d\}.$$

Then, we consider that, in the space $\boldsymbol{R}^d$, $(d \geq 1)$, the function in $\mathcal{N}$ which is a solution of a Schrödinger equation determines the natural statistical distribution state for the natural statistical phenomenon of some physical system. This fact is a restriction condition for a solution of a Schrödinger equation in $\boldsymbol{R}^d$. This is a restriction condition in order that a solution of a Schrödinger equation satisfies the condition postulated for the law of natural statistical physics. As for the laws of natural statistical physics, we refer to Ito [18].

References

[1] Y.Ito, *Linear Algebra*, Kyôritu, 1987, (in Japanese).

[2] ———, *Analysis*, Vol.**1**, Science House, 1991, (in Japanese).

[3] ———, *Axioms of Arithmetic*, Science House, 1999, (in Japanese).

[4] ———, *Foundation of Analysis*, Science House, 2002, (in Japanese).

[5] ———, *Theory of Measure and Integration*, Science House, 2002, (in Japanese).

[6] ———, *Analysis*, Vol.**2**, Revised Ed. Science House, 2002, (in Japanese).

[7] ———, *Vector Analysis*, Science House, 2008, (in Japanese).

[8] ———, *Differential Calculus of L^p-functions and L^p_{loc}- functions*, Real Analysis Symposium 2009, Sakato, pp.97-102, (in Japanese).

[9] ———, *Exercises of Vector Analysis*, Science House, 2010, (in Japanese).

[10] ———, *Differential Calculus of L^p-functions and L^p_{loc}-functions. Revisited*, J. Math. Univ. Tokushima, **45**(2011), 49-66.

[11] ———, *Fourier transformaition of L^2_{loc}-functions and its applications,* Real Analysis Symposium 2013, Okayama, pp.5-8, (in Japanese).

[12] ———, *Theory of Lebesgue Integral*, preprint, 2010.11.3, (in Japanese).

[13] ———, *Vector Analysis* (Rev. Ed.), preprint, 2011.3.6, (in Japanese).

[14] ———, RS-*integral and* LS-*integral*, preprint, 2011.6.27, (in Japanese).

[15] ———, *Theory of Function Spaces and Theory of Hyperfunctions*, preprint, 2011.9.30, (in Japanese).

[16] ———, *Mathematical Foundations of Natural Statistical Physics*, preprint, 2013.3.31, (in Japanese).

[17] ———, *Fourier Analysis*, preprint, 2014.8.25, (in Japanese).

[18] ———, *Laws of Natural Statistical Physics*, J.Math. Univ. Tokushima, **48**(2014), 41-70.