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Time Decay for Some Degenerate Hyperbolic Systems with Strong Dissipations

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Abstract

Consider the initial-boundary value problem for the coupled degenerate strongly damped hyperbolic system of Kirchhoff type with a homogeneous Dirichlet boundary condition. We give the polynomially decay estimates of the solutions and their derivatives. Moreover, when either the wave coefficient or the initial data are appropriately small, we derive a lower decay rate for the solutions.

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1 Introduction

In this paper we consider the initial-boundary value problem for the coupled degenerate hyperbolic system with strong damping of Kirchhoff type :

$$\rho u_{tt} - \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)^{\gamma} \Delta u - \Delta u_t = 0 \qquad \text{in} \quad \Omega \times (0, \infty)\,, \qquad (1.1)$$

$$\rho v_{tt} - \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)^{\gamma} \Delta v - \Delta v_t = 0 \qquad \text{in} \quad \Omega \times (0, \infty)\,, \qquad (1.2)$$

with

$$u(x,0) = u_0(x)\,,\ u_t(x,0) = u_1(x)\,,\ v(x,0) = v_0(x)\,,\ v_t(x,0) = v_1(x)\,, \qquad (1.3)$$

$$u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty)\,, \qquad (1.4)$$

where $u = u(x,t)$ and $v = v(x,t)$ are unknown real value functions, Ω is an open boundary domain in N-dimensional Euclidean space $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta = \nabla\cdot\nabla = \sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplacian, $\|\cdot\| = \|\cdot\|_{L^2}$ is the usual norm of $L^2(\Omega)$, and ρ and γ are positive constants. The coupled hyperbolic system

(1.1)–(1.4) will be useful for the research of amplitude vibrations of two kinds of elastic stretched strings.

The coupled hyperbolic system (1.1)–(1.4) traces back to the single Kirchhoff type wave equation :

$$\rho u_{tt} - \left(\mu + \int_\Omega |\nabla u(x,t)|^2\,dx\right)^\gamma \Delta u - \delta\Delta u_t = 0 \qquad \text{in} \quad \Omega\times(0,\infty) \tag{1.5}$$

with $u(0) = u_0$, $u_t(0) = u_1$, and $u|_{\partial\Omega} = 0$, which is called a non-degenerate equation when $\mu > 0$ and a degenerate one when $\mu = 0$. When the dimension N is one, it is well-known that (1.5) describes small amplitude vibrations of an elastic stretched string. In the case of non-damping $\delta = 0$, (1.5) was introduced by Kirchhoff [5] (also see [3], [4]), and the problem of local-in-time solvability has been studied by several authors (see [1], [2] and the references cited there). In the case of damping $\delta > 0$, the problem of global-in-time solvability has been solved by several authors (see [6], [7], [11], [14] and the references cited there).

By the similar way as in [6] and [7], we see that the problem (1.1)–(1.4) admits a unique global solution $[u(t), v(t)]$ in the class $(C^0([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)))^2$ if the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H_0^1(\Omega))^2\times(L^2(\Omega))^2$. On the other hand, when $\gamma = 1$, in previous paper [16], we have derived the upper decay estimates of the solutions and their derivatives of the coupled hyperbolic system (1.1)–(1.4) (see [8], [18], [12], [13], [15] for single equations).

Our purpose in this paper is to derive the upper decay estimates of the solutions and their derivatives of the coupled hyperbolic system (1.1)–(1.2) when any $\gamma > 0$. Moreover, we will derive a lower decay estimate of the solutions when either the coefficient ρ or the initial data are appropriate small, and we will show a decay property for the $H^2(\Omega)$ norm of the solutions.

In order to derive the energy decay estimate and a lower decay estimate of the solutions, we will use the following energy and functional associated with (1.1)–(1.2) :

$$E(u,v,u_t,v_t) \equiv \rho\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{\gamma+1}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \tag{1.6}$$

and

$$H(u,v,u_t,v_t) \equiv \rho\frac{\|u_t\|^2 + \|v_t\|^2}{\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{1+[\gamma-1]^+}} + \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{1-[1-\gamma]^+} \tag{1.7}$$

where $[a]^+ = \max\{0, a\}$, and we often write $E(t) \equiv E(u(t), v(t), u_t(t), v_t(t))$ and $H(t) \equiv H(u(t), v(t), u_t(t), v_t(t))$ for simplicity. In particular, we will use the following notations related with the initial data $[u_0, v_0, u_1, v_1]$:

$$E(0) \equiv \rho\left(\|u_1\|^2 + \|v_1\|^2\right) + \frac{1}{\gamma+1}\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\gamma+1}$$

and

$$H(0) \equiv \rho \frac{\|u_1\|^2 + \|v_1\|^2}{(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{1+[\gamma-1]^+}} + \left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{1-[1-\gamma]^+} .$$

Moreover, we denote the Poincaré constant by $c_* = c_*(\Omega)$, that is,

$$c_* = \inf \left\{ \|\phi\|/\|\nabla\phi\| \;\middle|\; \phi \in H_0^1(\Omega) \,,\ \phi \neq 0 \right\} .$$

Our main results are as follows.

Theorem 1.1 *Let the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$. Then the solutions $u(t)$ and $v(t)$ of (1.1)–(1.4) satisfy*

$$\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leq C(1+t)^{-\frac{1}{\gamma}} \,, \tag{1.8}$$

$$\|u_t(t)\|^2 + \|v_t(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \tag{1.9}$$

for $t \geq 0$ with some positive constant C.

Moreover, if the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H^2(\Omega) \cap H_0^1(\Omega))^4$, then

$$\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \,, \tag{1.10}$$

$$\|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 \leq C(1+t)^{-4-\frac{1}{\gamma}} \tag{1.11}$$

for $t \geq 0$ with some positive constant C.

Theorem 1.2 *Let the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$. Suppose that $u_0 \neq 0$ (or $v_0 \neq 0$) and*

$$((\gamma+2)c_*)^2 \rho H(0)^\gamma < 1 \qquad \text{if} \quad \gamma \geq 1 \,;$$

$$((2\gamma+1)c_*)^2 \rho (H(0) + E(0)^{\frac{1}{\gamma+1}} B(0)) < 1 \qquad \text{if} \quad 0 < \gamma < 1$$

with $B(0) \equiv (2^2(2^2 c_*^2 \rho + 1)(1-\gamma)(E(0)^{\frac{\gamma}{2(\gamma+1)}} + 1))^2$. Then the solutions $u(t)$ and $v(t)$ of (1.1)–(1.4) satisfy

$$C^{-1}(1+t)^{-\frac{1}{\gamma}} \leq \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \tag{1.12}$$

for $t \geq 0$ with some positive constant C.

Moreover, if the initial data $[u_0, v_0, u_1, v_1]$ belong $(H^2(\Omega) \cap H_0^1(\Omega))^4$, then

$$\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 \leq C(1+t)^{-\varepsilon} \tag{1.13}$$

$$\|u_t(t)\|_{H^2}^2 + \|v_t(t)\|_{H^2}^2 \leq C(1+t)^{-2-\varepsilon} \tag{1.14}$$

for $t \geq 0$ with some number $0 < \varepsilon \leq 1/\gamma$ and some positive constant C.

Theorems 1.1 and 1.2 follow from Theorems 3.4 and 4.4 in the subsequent sections.

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in the Hilbert space $L^2(\Omega)$ or sometimes duality between the space X and its dual X'. We put $[a]^+ = \max\{0,a\}$ where $1/[a]^+ = \infty$ if $[a]^+ = 0$. Positive constants will be denoted by C and will change from line to line.

2 Energy Decay

First we introduce the following functions associated with the coupled system (1.1)–(1.2) which we will use through this paper :

$$\begin{aligned}
&K(t) \equiv \|u(t)\|^2 + \|v(t)\|^2 , && L(t) \equiv \|u_t(t)\|^2 + \|v_t(t)\|^2 ,\\
&M(t) \equiv \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 , && X(t) \equiv \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 ,\\
&Y(t) \equiv \|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2 , && Z(t) \equiv \|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 ,\\
&\Phi(t) \equiv \|\nabla u_{tt}(t)\|^2 + \|\nabla v_{tt}(t)\|^2 , && \Psi(t) \equiv \|\Delta u_t(t)\|^2 + \|\Delta v_t(t)\|^2 ,
\end{aligned}$$

and then, we see from (1.6) and (1.7) that

$$E(t) = \rho L(t) + \frac{1}{\gamma+1} M(t)^{\gamma+1} \quad \text{and} \quad H(t) = \rho \frac{L(t)}{M(t)^{1+[\gamma-1]^+}} + M(t)^{1-[1-\gamma]^+} . \tag{2.1}$$

The energy $E(t)$ of (1.1)–(1.2) has the following decay rate.

Proposition 2.1 *Suppose that the initial data* $[u_0, v_0, u_1, v_1]$ *belong to* $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$. *The the energy* $E(t)$ *satisfies*

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \left(2(4c_*^2+1)^2(1+\gamma^{-1})(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)^2 \right)^{-1} [t-1]^+ \right)^{-\frac{\gamma+1}{\gamma}} \tag{2.2}$$

that is,

$$M(t) \le C(1+t)^{-\frac{1}{\gamma}} \quad \textit{and} \quad L(t) \le C(1+t)^{-1-\frac{1}{\gamma}} \quad \textit{for} \quad t \ge 0 . \tag{2.3}$$

Proof. Multiplying (1.1) and (1.2) by $2u_t$ and $2v_t$, respectively, and integrating them over Ω, we have

$$\begin{aligned}
&\rho \frac{d}{dt}\|u_t\|^2 + M(t)^\gamma \frac{d}{dt}\|\nabla u\|^2 + 2\|\nabla u_t\|^2 = 0 ,\\
&\rho \frac{d}{dt}\|v_t\|^2 + M(t)^\gamma \frac{d}{dt}\|\nabla v\|^2 + 2\|\nabla v_t\|^2 = 0 .
\end{aligned}$$

Adding these two equations, we obtain

$$\rho\frac{d}{dt}L(t)+M(t)^{\gamma}\frac{d}{dt}M(t)+2Y(t)=0 \quad \text{or} \quad \frac{d}{dt}E(t)+2Y(t)=0\,, \tag{2.4}$$

and integrating (2.4) over $[0,t]$, we have

$$E(t)+2\int_0^t Y(s)\,ds=E(0)\,. \tag{2.5}$$

For any $t>0$, integrating (2.4) over $[t,t+1]$, we have

$$2\int_t^{t+1} Y(s)\,ds=E(t)-E(t+1) \qquad \left(\equiv 2D(t)^2\right)\,. \tag{2.6}$$

Then, there exist $t_1\in[t,t+1/4]$ and $t_2\in[t+3/4,t+1]$ such that

$$Y(t_j)\le 4\int_t^{t+1} Y(s)\,ds=4D(t)^2 \qquad \text{for} \quad j=1,2\,. \tag{2.7}$$

On the other hand, integrating (1.1) and (1.2) by u and v, respectively, and integrating them over Ω, we have

$$\rho\frac{d}{dt}(u,u_t)-\rho\|u_t\|^2+M(t)^{\gamma}\|\nabla u\|^2+(\nabla u,\nabla u_t)=0\,,$$
$$\rho\frac{d}{dt}(v,v_t)-\rho\|v_t\|^2+M(t)^{\gamma}\|\nabla v\|^2+(\nabla v,\nabla v_t)=0\,.$$

Adding these two equations, we obtain

$$M(t)^{\gamma+1}=\rho L(t)-\frac{\rho}{2}\frac{d}{dt}K'(t)-\frac{1}{2}M'(t)\,. \tag{2.8}$$

Integrating (2.8) over $[t_1,t_2]$, we have from (2.6) and (2.7) that

$$\begin{aligned}\int_{t_1}^{t_2} M(s)^{\gamma+1}\,ds &\le \rho\int_{t_1}^{t_2} L(s)\,ds+\frac{\rho}{2}\sum_{j=1}^{2}|K'(t_j)|+\frac{1}{2}\int_{t_1}^{t_2}|M'(s)|\,ds\\ &\le c_*^2\rho\int_t^{t+1}Y(s)\,ds+c_*^2\rho\sum_{j=1}^{2}M(t_j)^{\frac12}Y(t_j)^{\frac12}+\int_t^{t+1}M(s)^{\frac12}Y(s)^{\frac12}\,ds \qquad (2.9)\\ &\le c_*^2\rho D(t)^2+(4c_*^2\rho+1)D(t)\sup_{t\le s\le t+1}M(s)^{\frac12}\,, \qquad (2.10)\end{aligned}$$

where we used the facts that $L(t)\le c_*^2Y(t)$, $|M'(t)|\le 2M(t)^{\frac12}Y(t)^{\frac12}$, and

$$|K'(t)|\le 2K(t)^{\frac12}L(t)^{\frac12}\le 2c_*^2M(t)^{\frac12}Y(t)^{\frac12}\,. \tag{2.11}$$

Moreover, since $M(t) \le ((\gamma+1)E(t))^{\frac{1}{\gamma+1}}$ and $E(t)$ is a non-increasing function, it follows that

$$\int_{t_1}^{t_2} M(s)^{\gamma+1}\,ds \le c_*^2\rho D(t)^2 + (4c_*^2\rho+1)D(t)((\gamma+1)E(t))^{\frac{1}{2(\gamma+1)}}\,,$$

and from (2.1) and (2.6) that

$$\begin{aligned}\int_{t_1}^{t_2} E(s)\,ds &= \rho\int_{t_1}^{t_2} L(s)\,ds + \frac{1}{\gamma+1}\int_{t_1}^{t_2} M(s)^{\gamma+1}\,ds \\ &\le c_*^2\rho\int_{t}^{t+1} Y(s)\,ds + \frac{1}{\gamma+1}\int_{t_1}^{t_2} M(s)^{\gamma+1}\,ds \\ &\le 2c_*^2\rho D(t)^2 + (4c_*^2\rho+1)D(t)E(t)^{\frac{1}{2(\gamma+1)}}\,. \end{aligned} \tag{2.12}$$

Integrating (2.4) over $[t, t_2]$, we have from (2.6) and (2.12) that

$$\begin{aligned} E(t) &= E(t_2) + 2\int_t^{t_2} Y(s)\,ds \\ &\le 2\int_{t_1}^{t_2} E(s)\,ds + 2\int_t^{t+1} Y(s)\,ds \\ &\le 2(2c_*^2\rho+1)D(t)^2 + 2(4c_*^2\rho+1)D(t)E(t)^{\frac{1}{2(\gamma+1)}}\,. \end{aligned}$$

Since $D(t) \le E(0)^{\frac{\gamma}{2(\gamma+1)}}E(t)^{\frac{1}{2(\gamma+1)}}$ by (2.5) and (2.6), we observe from the Young inequality that

$$\begin{aligned} E(t) &\le 2(4c_*^2\rho+1)(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)D(t)E(t)^{\frac{1}{2(\gamma+1)}} \\ &\le \frac{2\gamma+1}{2(\gamma+1)}\Big(2(4c_*^2\rho+1)(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)D(t)\Big)^{\frac{2(\gamma+1)}{2\gamma+1}} + \frac{1}{2(\gamma+1)}E(t) \end{aligned}$$

and from (2.6) that

$$\begin{aligned} E(t)^{1+\frac{\gamma}{\gamma+1}} = E(t)^{\frac{2\gamma+1}{\gamma+1}} &\le \Big(2(4c_*^2\rho+1)(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)D(t)\Big)^2 \\ &\le 2(4c_*^2\rho+1)(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)^2(E(t)-E(t+1))\,. \end{aligned} \tag{2.13}$$

Therefore, applying Lemma 2.2 below together with (2.5) to (2.13), we obtain the desired estimate (2.2). □

In order to derive the decay estimate of the energy $E(t)$, we used the following inequality in the proof of Proposition 2.1 (see [8], [9], [10] for the proof).

Lemma 2.2 *Let $\phi(t)$ be a non-increasing and non-negative function on $[0,\infty)$ and satisfy*

$$\phi(t)^{1+\alpha} \le k(\phi(t)-\phi(t+1))$$

with certain constants $k \ge 0$ and $\alpha > 0$. Then the function $\phi(t)$ satisfies

$$\phi(t) \le \big(\phi(0)^{-\alpha} + \alpha k^{-1}[t-1]^+\big)^{-\frac{1}{\alpha}} \qquad \textit{for} \quad t \ge 0\,.$$

3 Decay for first order derivatives

In what follows, we suppose that the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$. First we will improve the decay rate of $L(t)$ given by (2.3).

Proposition 3.1 *The function $L(t)$ satisfies*

$$L(t) \le C(1+t)^{-2-\frac{1}{\gamma}} \qquad \textit{for} \quad t \ge 0\,. \tag{3.1}$$

Proof. From (2.4) it follows that

$$\rho \frac{d}{dt} L(t) + 2Y(t) = -M(t)^{\gamma} M'(t) \le 2M(t)^{\gamma} M(t)^{\frac{1}{2}} Y(t)^{\frac{1}{2}}$$

and from the Young inequality and (2.3) that

$$\rho \frac{d}{dt} L(t) + Y(t) \le CM(t)^{2\gamma+1} \le C(1+t)^{-2-\frac{1}{\gamma}}$$

and

$$\rho \frac{d}{dt} L(t) + c_*^{-2} L(t) \le C(1+t)^{-2-\frac{1}{\gamma}}$$

and hence, we obtain the desired estimate (3.1). □

Proposition 3.2 *Let $M(0) > 0$. Suppose that $M(t) > 0$ for $0 \le t < T$, and*

$$(\gamma c_*)^2 \rho H(0)^{\gamma} < 1 \qquad \textit{if} \quad \gamma \ge 1\,; \tag{3.2}$$

$$(2c_*)^2 \rho (H(0) + E(0)^{\frac{1}{\gamma+1}} B(0)) < 1 \qquad \textit{if} \quad 0 < \gamma < 1 \tag{3.3}$$

with $B(0) \equiv (2^2(2^2 c_*^2 \rho + 1)(1-\gamma)(E(0)^{\frac{\gamma}{2(\gamma+1)}} + 1))^2$. Then it holds that

$$H(t) \le \begin{cases} H(0) & \textit{if} \quad \gamma \ge 1\,; \\ H(0) + E(0)^{\frac{1}{\gamma+1}} B(0) & \textit{if} \quad 0 < \gamma < 1 \end{cases} \tag{3.4}$$

for $0 \le t < T$.

Proof. Multiplying (2.4) by $M(t)^{-\gamma-k}$ with $k \ge 0$, we have

$$\begin{aligned} &\frac{d}{dt}\left(\rho \frac{L(t)}{M(t)^{\gamma+k}} + \frac{1}{M(t)^{k-1}}\right) + 2\frac{Y(t)}{M(t)^{\gamma+k}} \\ &\quad = -(\gamma+k)\rho \frac{M'(t)}{M(t)^{\gamma+k+1}} L(t) - k\frac{M'(t)}{M(t)^{k}}\,. \end{aligned} \tag{3.5}$$

(1) When $\gamma \ge 1$, we observe

$$\rho\left(\frac{L(t)}{M(t)}\right)^{\frac{1}{2}} = \rho^{\frac{1}{2}}\left(\frac{L(t)}{M(t)^\gamma}M(t)^{\gamma-1}\right)^{\frac{1}{2}} \le (\rho H(t)^\gamma)^{\frac{1}{2}} \tag{3.6}$$

where $H(t)$ is given by (2.1), that is, $H(t) \equiv \rho L(t)/M(t)^\gamma + M(t)$. From (3.5) with $k = 0$, it follows that

$$\frac{d}{dt}H(t) + 2\frac{Y(t)}{M(t)^\gamma} = -\gamma\rho\frac{M'(t)}{M(t)^{\gamma+1}}L(t) \le 2\gamma c_*\rho\left(\frac{L(t)}{M(t)}\right)^{\frac{1}{2}}\frac{Y(t)}{M(t)^\gamma}$$

and from (3.6) that

$$\frac{d}{dt}H(t) + 2\left(1 - \gamma c_*(\rho H(t)^\gamma)^{\frac{1}{2}}\right)\frac{Y(t)}{M(t)^\gamma} \le 0 \tag{3.7}$$

for $t \le t < T$.

If $\gamma c_*(\rho H(0)^\gamma)^{\frac{1}{2}} < 1$ (equivalent to (3.2)), then there exists T_1 such that $0 < T_1 \le T$ and

$$\gamma c_*(\rho H(t))^{\frac{1}{2}} \le 1 \tag{3.8}$$

for $0 \le t \le T_1$, and we observe from (3.7) that

$$\frac{d}{dt}H(t) \le 0 \quad \text{and} \quad H(t) \le H(0) \tag{3.9}$$

for $0 \le t \le T_1$. Therefore, we see that (3.8) and (3.9) hold true for $0 \le t < T$, and hence we obtain (3.4) with $\gamma \ge 1$.

(2) When $0 < \gamma < 1$, we observe

$$\rho\left(\frac{L(t)}{M(t)}\right)^{\frac{1}{2}} \le (\rho H(t))^{\frac{1}{2}} \tag{3.10}$$

where $H(t)$ is given by (2.1), that is, $H(t) \equiv \rho L(t)/M(t) + M(t)^\gamma$.

Moreover, from the energy decay (2.2) it follows that

$$\begin{aligned}
&\int_0^t E(s)^{\frac{2\gamma}{\gamma+1}}\,ds \\
&\le \int_0^t \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma}{\gamma+1}\left(2(4c_*^2\rho+1)^2(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)^2\right)^{-1}[t-1]^+\right)^{-\frac{\gamma+1}{\gamma}} ds \\
&\le E(0) + 2(\gamma+1)(4c_*^2\rho+1)^2 E(0)^{\frac{1}{\gamma+1}}(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)^2 \\
&\le 2^3(4c_*^2\rho+1)^2 E(0)^{\frac{1}{\gamma+1}}(E(0)^{\frac{\gamma}{2(\gamma+1)}}+1)^2 \qquad (\equiv K(0)\,)\,.
\end{aligned} \tag{3.11}$$

From (3.5) with $k = 1-\gamma$, it follows that

$$\begin{aligned}\frac{d}{dt}H(t) + 2\frac{Y(t)}{M(t)} &= -\rho\frac{M'(t)}{M(t)^2}L(t) - (1-\gamma)\frac{M'(t)}{M(t)^{1-\gamma}}\\ &\le 2c_*\rho\left(\frac{L(t)}{M(t)}\right)^{\frac{1}{2}}\frac{Y(t)}{M(t)} + 2(1-\gamma)M(t)^{\gamma}\left(\frac{Y(t)}{M(t)}\right)^{\frac{1}{2}}\end{aligned}$$

and from the Young inequality and (3.10) that

$$\begin{aligned}&\frac{d}{dt}H(t) + \left(1 - 2c_*(\rho H(t))^{\frac{1}{2}}\right)\frac{Y(t)}{M(t)}\\ &\quad\le (1-\gamma)^2\left((1+\gamma)E(t)\right)^{\frac{2\gamma}{\gamma+1}} \le 2(1-\gamma)^2 E(t)^{\frac{2\gamma}{\gamma+1}} \qquad (3.12)\end{aligned}$$

for $0 \le t < T$.

If $2c_*\left(\rho(H(0) + 2(1-\gamma)^2K(0))\right)^{\frac{1}{2}} < 1$ (equivalent to (3.3)), then there exists T_2 such that $0 < T_2 \le T$ and

$$2c_*(\rho H(t))^{\frac{1}{2}} < 1 \qquad (3.13)$$

for $0 \le t < T_2$, we observe from (3.11) and (3.12) that

$$\frac{d}{dt}H(t) \le 2(1-\gamma)^2E(t)^{\frac{2\gamma}{\gamma+1}} \quad \text{and} \quad H(t) \le H(0) + 2(1-\gamma)^2K(0) \qquad (3.14)$$

for $0 \le t \le T_2$. Therefore we see that (3.13) and (3.14) hold true for $0 \le t < T$, and hence we obtain (3.4) with $0 < \gamma < 1$. □

Proposition 3.3 *In addition to the assumption of Proposition 3.2, suppose that*

$$\left((\gamma+2)c_*\right)^2\rho H(0)^{\gamma} < 1 \qquad \textit{if} \quad \gamma \ge 1; \qquad (3.15)$$

$$\left((2\gamma+1)c_*\right)^2\rho\left(H(0) + E(0)^{\frac{1}{\gamma+1}}B(0)\right) < 1 \qquad \textit{if} \quad 0 < \gamma < 1 \qquad (3.16)$$

with $B(0) \equiv (2^2(2^2c_*^2\rho + 1)(1-\gamma)(E(0)^{\frac{\gamma}{2(\gamma+1)}} + 1))^2$. *Then it holds that*

$$M(t) \ge C'(1+t)^{-\frac{1}{\gamma}} \qquad (3.17)$$

for $0 \le t < T$ *with a positive constant* $C' > 0$.

Proof. (1) When $\gamma \ge 1$, from (3.5) with $k = 2$ it follows that

$$\begin{aligned}\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{\gamma+2}} + \frac{1}{M(t)}\right) + 2\frac{Y(t)}{M(t)^{\gamma+2}} &= -(\gamma+2)\rho\frac{M'(t)}{M(t)^{\gamma+3}}L(t) - 2\frac{M'(t)}{M(t)^2}\\ &\le 2(\gamma+2)c_*\rho\left(\frac{L(t)}{M(t)}\right)^{\frac{1}{2}}\frac{Y(t)}{M(t)^{\gamma+2}} + 4\left(\frac{Y(t)}{M(t)^{\gamma+2}}M(t)^{\gamma-1}\right)^{\frac{1}{2}}\end{aligned}$$

and from (3.6) and (3.4) that

$$\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{\gamma+2}}+\frac{1}{M(t)}\right)+2\left(1-(\gamma+2)c_*(\rho H(0)^\gamma)^{\frac{1}{2}}\right)\frac{Y(t)}{M(t)^{\gamma+2}}$$
$$\le 4\left(\frac{Y(t)}{M(t)^{\gamma+2}}M(t)^{\gamma-1}\right)^{\frac{1}{2}}$$

for $0 \le t < T$.

If $(\gamma+2)c_*(\rho H(0)^\gamma)^{\frac{1}{2}} < 1$ (equivalent to (3.15)), then we observe from the Young inequality and (2.3) that

$$\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{\gamma+2}}+\frac{1}{M(t)}\right)\le CM(t)^{\gamma-1}\le C(1+t)^{-1+\frac{1}{\gamma}}$$

and

$$\rho\frac{L(t)}{M(t)^{\gamma+2}}+\frac{1}{M(t)}\le C(1+t)^{\frac{1}{\gamma}}$$

for $0 \le t < T$, which implies the desired estimate (3.17) with $\gamma \ge 1$.

(2) When $0 < \gamma < 1$, from (3.5) with $k = \gamma + 1$ it follows that

$$\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\right)+2\frac{Y(t)}{M(t)^{2\gamma+1}}$$
$$=-(2\gamma+1)\rho\frac{M'(t)}{M(t)^{2\gamma+2}}L(t)-(\gamma+1)\frac{M'(t)}{M(t)^{\gamma+1}}$$
$$\le 2(2\gamma+1)c_*\rho\left(\frac{Y(t)}{M(t)}\right)^{\frac{1}{2}}\frac{Y(t)}{M(t)^{2\gamma+1}}+2(\gamma+1)\left(\frac{Y(t)}{M(t)^{2\gamma+1}}\right)^{\frac{1}{2}}$$

and from (3.10) and (3.4) that

$$\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\right)$$
$$+2\left(1-(2\gamma+1)c_*\left(\rho(H(0)+E(0)^{\frac{1}{\gamma+1}}B(0))\right)^{\frac{1}{2}}\right)\frac{Y(t)}{M(t)^{2\gamma+1}}$$
$$\le 2(\gamma+1)\left(\frac{Y(t)}{M(t)^{2\gamma+1}}\right)^{\frac{1}{2}}$$

for $0 \le t < T$.

If $(2\gamma+1)c_*\left(\rho(H(0)+E(0)^{\frac{1}{\gamma+1}}B(0))\right)^{\frac{1}{2}} < 1$ (equivalent to (3.16)), then we observe from the Young inequality that

$$\frac{d}{dt}\left(\rho\frac{L(t)}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\right)\le C \quad\text{and}\quad \rho\frac{L(t)}{M(t)^{2\gamma+1}}+\frac{1}{M(t)^\gamma}\le C(1+t)$$

for $0 \le t < T$, which implies the desired estimate (3.17) with $0 < \gamma < 1$. □

Theorem 3.4 *Let the initial data* $[u_0, v_0, u_1, v_1]$ *belong to* $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$. *Then the solutions* $u(t)$ *and* $v(t)$ *of* (1.1)–(1.4) *satisfy*

$$M(t) \le C(1+t)^{-\frac{1}{\gamma}} \quad \text{and} \quad L(t) \le C(1+t)^{-2-\frac{1}{\gamma}} \qquad \text{for} \quad t \ge 0\,. \tag{3.18}$$

Moreover, suppose that $M(0) > 0$ *and*

$$((\gamma+2)c_*)^2 \rho H(0)^\gamma < 1 \qquad \text{if} \quad \gamma \ge 1\ ;$$
$$((2\gamma+1)c_*)^2 \rho \left(H(0) + E(0)^{\frac{1}{\gamma+1}} B(0)\right) < 1 \qquad \text{if} \quad 0 < \gamma < 1$$

with $B(0) \equiv (2^2(2^2 c_*^2 \rho + 1)(1-\gamma)(E(0)^{\frac{\gamma}{2(\gamma+1)}} + 1))^2$. *Then*

$$M(t) \ge C'(t+1)^{-\frac{1}{\gamma}} \qquad \text{for} \quad t \ge 0 \tag{3.19}$$

with a positive constant $C' > 0$.

Proof. (3.18) follows (2.3) and (3.1). Since $M(0) > 0$, putting

$$T \equiv \sup\left\{ t \in [0, \infty) \;\middle|\; M(s) > 0 \text{ for } 0 \le s < t \right\}\ ,$$

we see that $T > 0$ and $M(t) > 0$ for $0 \le t < T$. If $T < \infty$, then $M(T) = 0$. However, from the lower estimate (3.17) we observe that $\lim_{t\to T} M(t) \ge C'(1+T)^{-\frac{1}{\gamma}} > 0$, and hence we obtain that $T = \infty$ and $M(t) > 0$ for all $t \ge 0$. Therefore (3.19) follows (3.17). □

4 Decay for second order derivatives

In this section we will derive the decay rate of the functions $X(t)$, $Y(t)$, $Z(t)$, and $\Psi(t)$.

In what follows, we suppose that the initial data $[u_0, v_0, u_1, v_1]$ belong to $(H^2(\Omega) \cap H_0^1(\Omega))^4$.

Proposition 4.1 *The functions* $Z(t)$ *and* $Y(t)$ *satisfy*

$$Z(t) \le C \quad \text{and} \quad Y(t) \le C(1+t)^{-2} \qquad \text{for} \quad t \ge 0\,. \tag{4.1}$$

Proof. Multiplying (1.1) and (1.2) by $-2\Delta u$ and $-2\Delta v$, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\frac{d}{dt} Z(t) + 2M(t)^\gamma Z(t) = 2\rho \frac{d}{dt}\left((u_t, \Delta u) + (v_t, \Delta v)\right) + 2\rho Y(t) \tag{4.2}$$

and

$$\begin{aligned} &Z(t) + 2\int_0^t M(s)^\gamma Z(s)\, ds \\ &\quad \le Z(0) + 2\rho\left(L(t)^{\frac{1}{2}} Z(t)^{\frac{1}{2}} + L(0)^{\frac{1}{2}} Z(0)^{\frac{1}{2}}\right) + 2\rho \int_0^t Y(s)\, ds\,. \end{aligned}$$

Thus, from the Young inequality and (2.5) we obtain that $Z(t) \leq C(Z(0) + L(0) + E(0))$ for $t \geq 0$.

Multiplying (1.1) and (1.2) by $-2\Delta u_t$ and $-2\Delta v_t$, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\rho \frac{d}{dt} Y(t) + 2\Psi(t) = -M(t)^{\gamma} Z'(t) \leq 2M(t)^{\gamma} Z(t)^{\frac{1}{2}} \Psi(t)^{\frac{1}{2}} .$$

From the Young inequality it follows that

$$\rho \frac{d}{dt} Y(t) + \Psi(t) \leq M(t)^{2\gamma} Z(t)$$

and from (2.3) that

$$\rho \frac{d}{dt} Y(t) + c_*^{-2} Y(t) \leq M(t)^{2\gamma} Z(t) \leq C(1+t)^{-2} ,$$

and hence, we obtain that $Y(t) \leq C(1+t)^{-2}$ for $t \geq 0$. □

Proposition 4.2 *The functions $X(t)$, $Y(t)$, and $\Psi(t)$ satisfy*

$$X(t) \leq C(1+t)^{-4-\frac{1}{\gamma}} , \quad Y(t) \leq C(1+t)^{-2-\frac{1}{\gamma}} , \tag{4.3}$$

$$\Psi(t) \leq C(1+t)^{-2} \qquad \textit{for} \quad t \geq 0 . \tag{4.4}$$

Proof. Multiplying (1.1) and (1.2) differentiated once with respect to t by $2u_{tt}$ and $2v_{tt}$, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\begin{aligned} \rho \frac{d}{dt} X(t) + 2\Phi(t) &= -M(t)^{\gamma} Y'(t) - 2\gamma M(t)^{\gamma-1} M'(t) \left((\nabla u, \nabla u_{tt}) + (\nabla v, \nabla v_{tt}) \right) \\ &\leq (2+4\gamma) M(t)^{\gamma} Y(t)^{\frac{1}{2}} \Phi(t)^{\frac{1}{2}} . \end{aligned}$$

From the Young inequality it follows that

$$\rho \frac{d}{dt} X(t) + \Phi(t) \leq C M(t)^{2\gamma} Y(t)$$

and from (2.3) and (4.1) that

$$\rho \frac{d}{dt} X(t) + c_*^{-2} X(t) \leq C M(t)^{2\gamma} Y(t) \leq C(1+t)^{-\theta_1} \tag{4.5}$$

with $\theta_1 = 2 + 2 = 4$, and hence, we have

$$X(t) \leq C(1+t)^{-\theta_1} , \qquad \theta_1 = 4 . \tag{4.6}$$

From (2.4) it follows that

$$Y(t) = -\frac{1}{2}\rho L'(t) - \frac{1}{2}M(t)^\gamma M'(t) \le c_* \rho Y(t)^{\frac{1}{2}} X(t)^{\frac{1}{2}} + M(t)^\gamma M(t)^{\frac{1}{2}} Y(t)^{\frac{1}{2}} \,,$$

and from the Young inequality and (2.3) and (4.6) that

$$Y(t) \le CX(t) + CM(t)^{2\gamma+1} \le C(1+t)^{-\omega_1} \,, \qquad \omega_1 = \min\{\theta_1, 2+1/\gamma\} \,. \tag{4.7}$$

Applying (2.3) and (4.7) to (4.5), we obtain that

$$X(t) \le C(1+t)^{-\theta_2} \,, \qquad \theta_2 = 2 + \omega_1 \tag{4.8}$$

and from (4.7) that

$$Y(t) \le C(1+t)^{-\omega_2} \,, \qquad \omega_2 = \min\{\theta_2, 2+1/\gamma\} \,.$$

By induction, for $m = 2, 3, \cdots$, we observe

$$X(t) \le C(1+t)^{-\theta_m} \,, \qquad \theta_m = 2 + \omega_{m-1}$$

and

$$Y(t) \le C(1+t)^{-\omega_m} \,, \qquad \omega_m = \min\{\theta_m, 2+1/\gamma\} \,.$$

Therefore, we arrive at the desired estimate (4.3) for large m.

Moreover, from (1.1) and (1.2) together with (2.3), (4.1), and (4.3) we have

$$\Psi(t) \le 2\left(\rho^2 X(t) + M(t)^{2\gamma} Z(t)\right) \le C(1+t)^{-2} \tag{4.9}$$

for $t \ge 0$. □

Proposition 4.3 *Suppose that the assumptions of Theorem 3.4 are fulfilled. Then the functions $Z(t)$ and $\Psi(t)$ satisfy*

$$Z(t) \le C(1+t)^{-\varepsilon} \quad \textit{and} \quad \Psi(t) \le C(1+t)^{-2-\varepsilon} \qquad \textit{for} \quad t \ge 0 \tag{4.10}$$

with some $0 < \varepsilon \le 1/\gamma$.

Proof. From (4.2) it follows that

$$\frac{d}{dt}Z(t) + 2M(t)^\gamma Z(t) = 2\rho\left((u_{tt}, \Delta u) + (v_{tt}, \Delta v)\right) \le 2\rho X(t)^{\frac{1}{2}} Z(t)^{\frac{1}{2}}$$

and

$$\frac{d}{dt}Z(t) + M(t)^\gamma Z(t) \le \rho^2 \frac{X(t)}{M(t)^\gamma} \,.$$

Since $M(t)^\gamma \geq \varepsilon(1+t)^{-1}$ with some $0 < \varepsilon \leq 1/\gamma$ by (2.3) and (3.19), we observe from (4.3) that

$$\frac{d}{dt}Z(t) + \varepsilon(1+t)^{-1}Z(t) \leq C(1+t)^{-3-\frac{1}{\gamma}}$$

and

$$\frac{d}{dt}\left((1+t)^\varepsilon Z(t)\right) \leq C(1+t)^{-3-\frac{1}{\gamma}+\varepsilon}$$

and hence,

$$Z(t) \leq C(1+t)^{-\varepsilon} \qquad \text{for} \quad t \geq 0\,. \tag{4.11}$$

Moreover, using (4.9) together with (2.3), (4.3), (4.11), we obtain

$$\Psi(t) \leq 2\left(\rho^2 X(t) + M(t)^{2\gamma} Z(t)\right) \leq C(1+t)^{-2-\varepsilon}$$

for $t \geq 0$. □

Theorem 4.4 *Let the initial data* $[u_0, v_0, u_1, v_1]$ *belong* $(H^2(\Omega)\cap H_0^1(\Omega))^4$. *The problem* (1.1)–(1.4) *admits a unique global solution* $[u(t), v(t)]$ *in the class*

$$\left(C^0([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))\right)^2$$

and it holds that

$$X(t) \leq C(1+t)^{-4-\frac{1}{\gamma}} \quad \textit{and} \quad Y(t) \leq C(1+t)^{-2-\frac{1}{\gamma}} \qquad \textit{for} \quad t \geq 0\,. \tag{4.12}$$

Moreover, suppose that the initial data $[u_0, v_0, u_1, v_1]$ *satisfy* $M(0) > 0$ *and*

$$((\gamma+2)c_*)^2 \rho H(0)^\gamma < 1 \qquad \textit{if} \quad \gamma \geq 1\,;$$
$$((2\gamma+1)c_*)^2 \rho (H(0) + E(0)^{\frac{1}{\gamma+1}} B(0)) < 1 \qquad \textit{if} \quad 0 < \gamma < 1$$

with $B(0) \equiv (2^2(2^2 c_*^2 \rho + 1)(1-\gamma)(E(0)^{\frac{\gamma}{2(\gamma+1)}} + 1))^2$. *Then*

$$Z(t) \leq C(1+t)^{-\varepsilon} \quad \textit{and} \quad \Psi(t) \leq C(1+t)^{-2-\varepsilon} \qquad \textit{for} \quad t \geq 0 \tag{4.13}$$

with some $0 < \varepsilon \leq 1/\gamma$.

Proof. Applying the Banach contraction mapping theorem, we can get a local existence theorem (see [1], [2], [17] and the references cited there), that is, there exists a unique local solution $[u(t), v(t)]$ of (1.1)–(1.4) in the class

$$\left(C^0([0,T); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^2([0,T); L^2(\Omega))\right)^2$$

for some $T > 0$. Moreover, if $\|u(t)\|_{H^2} + \|v(t)\|_{H^2} + \|u_t(t)\|_{H^2} + \|v_t(t)\|_{H^2} < \infty$ for $t \geq 0$, then we can take $T = \infty$.

On the other hand, Proposition 4.1 and Proposition 4.2 give the a-priori estimate for the local solution $[u(t), v(t)]$ of (1.1)–(1.4), and hence, the problem (1.1)–(1.4) admits a unique global solution $[u(t), v(t)]$. Moreover, (4.12) and (4.13) follow from (4.3) and (4.10), respectively. □

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