J. Math. Univ. Tokushima
Vol. 48 (2014), 41–70

Laws of Natural Statistical Physics

By

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(Received September 30, 2014)

Abstract

In this paper, we give the new formulations of the laws of natural statistical physics in Chapter 2.

These are the following three cases:

(1) The case where the Schrödinger operator has only the discrete spectrum.

(2) The case where the Schrödinger operator has only the continuous spectrum.

(3) The case where the physical system is composed of particles moving periodically.

In Chapter 1, we study the new formulations of the concepts of natural probability and natural random variable which are necessary for the study of Chapter 2.

2000 Mathematics Subject Classification. Primary 81P99.

Introduction

In Chapter 1, as the preparations for these new formulations, we study the concepts of natural probability and natural random variables.

For a given normalized L^2-function ψ on $\boldsymbol{R}^n$, $(n \geq 1)$, we define the orthogonal probability measure $\psi_A = \chi_A \psi$ on the probability space $(\boldsymbol{R}^n, \mathcal{M}_n, \mu)$

to be a natural probability measure. Here $\mathcal{M}_n$ is the σ-additive family of all Lebesgue measurable sets on $\boldsymbol{R}^n$ and μ is the σ-additive measure:

$$\mu(A) = \int_A |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}, \ (A \in \mathcal{M}_n).$$

We define the vector-valued natural random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$ on a certain probability space $\Omega(\mathcal{B}, P)$ whose probability distribution law is defined by the L^2-density ψ so that the fundamental relation

$$P\left(\{\rho \in \Omega; \ \boldsymbol{r}(\rho) \in A\} \right) = \mu(A)$$

holds. Further we study the fundamental properties of these concepts.

Further we study two other cases of these concepts.

In this paper, remaking the old formulations in Ito [41], we succeeded in obtaining the essential expressions for the laws of natural statistical physics. Thereby, in the theory of natural statistical physics, we can understand in the unified manner the natural probability distributions of position variables and momentum variables of the physical system by using the concepts of the orthogonal probability measure or the local orthogonal probability measure.

Then these laws of natural probability distributions of position variables and momentum variables characterize the natural statistical phenomena of the physical system. Namely the main problem of the natural statistical physics is to study the statistical properties of the physical quantities of the physical system such as the expectation values of the energy, the momentum and the angular momentum and etc. of this physical system.

Using the results of this paper, especially, we can study the Dirac measure as the orthogonal probability measure in the case where the Fourier transform $\hat{\psi}$ of a L^2_{loc}-density ψ is the Dirac measure. As for the references, we refer to those in the last of this paper. Especially we refer to Ito [41].

Here we show my heartfelt gratitude to my wife Mutuko for her help of typesetting of the TEX-file of this manuscript

1 Natural probability

In this chapter, at first, we remember the notions of the measure-theoretical probability and the probability space.

1.1 Definition of probability and random variable

In this section, we remember the definitions of probability and random variable.

Definition 1.1.1 Assume that Ω is an arbitrary space and $\mathcal{B}$ is a σ-algebra of the family of subsets of Ω and P is a σ-additive measure.

Then we say that the composite concept $\Omega = \Omega(\mathcal{B},\ P) = (\Omega,\ \mathcal{B},\ P)$ is a **probability space** when it satisfies the following conditions (I)$\sim$(III):

(I) Ω is a non-empty set.

(II) $\mathcal{B}$ is a σ-algebra of the family of subsets of Ω. Namely, it satisfies the following conditions (i)$\sim$(iii):

(i) $\Omega \in \mathcal{B}$ holds.

(ii) If $A \in \mathcal{B}$ holds, we have $A^c \in \mathcal{B}$. Here A^c is the complementary event of A.

(iii) If we have $A_n \in \mathcal{B},\ (n = 1,\ 2,\ \cdots)$, we have $\bigcup_n A_n \in \mathcal{B}$.

(III) The real-valued set function $P(A)$ is a **σ- additive probability measure** on the measurable space $\Omega(\mathcal{B})$. Namely, it satisfies the following conditions (i)$\sim$(iii):

(i) If $A \in \mathcal{B}$ holds, we have $0 \leq P(A) \leq 1$.

(ii) If every pair of sets $A_n \in \mathcal{B},\ (n = 1,\ 2,\ 3,\ \cdots)$ are mutually disjoint, we have

$$P(A) = \sum_n P(A_n)$$

for the event

$$A = \sum_n A_n \in \mathcal{B}.$$

(iii) We have $P(\Omega) = 1$.

In Definition 1.1.1, (III), the **measurable space** $\Omega(\mathcal{B})$ is the composite concept of Ω and σ-algebra $\mathcal{B}$ composed of subsets of Ω.

We say that an element ω of Ω is an elementary event and an element A of $\mathcal{B}$ is a probability event. A probability event is simply said to be an event.

Let $\Omega = \Omega(\mathcal{B},\ P)$ be a probability space. Then we say that a function $X = X(\omega)$ of ω is a **random variable** if, for an arbitrary real x, $\{\omega;\ X(\omega) < x\}$ always belongs to $\mathcal{B}$.

For a random variable $X = X(\omega)$, we say that the function

$$F(x) = P\ (\ \{\omega;\ X(\omega) < x\}\ ),\ (-\infty < x < \infty)$$

of a real number x is the **distribution function** of X.

We obtain all informations concerning the distribution state of the values of this random variable varying randomly by using this distribution function.

Let $\mathcal{M}$ be a σ-algebra of the family which includes the family of Borel sets in $\boldsymbol{R}$ and assume that, for $A \in \mathcal{M}$, we have $\{\omega;\ X(\omega) \in A\} \in \mathcal{B}$ and

$$\mu(A) = P\ (\ \{\omega;\ X(\omega) \in A\}\ )$$

is a σ-additive measure on $\mathcal{M}$. Then the measure space $(\boldsymbol{R},\ \mathcal{M},\ \mu)$ is a probability space. We call this the **probability distribution** of the random variable $X = X(\omega)$.

Now we assume that $\boldsymbol{R}^n$ is the n-dimensional Euclidean space. Here assume $n \geq 1$. Let $(\boldsymbol{R}^n,\ \mathcal{M}_n,\ \lambda)$ be the Lebesgue measure space on $\boldsymbol{R}^n$. Namely, $\mathcal{M}_n$ denotes the σ-algebra of the family of all Lebesgue measurable sets on $\boldsymbol{R}^n$ and λ denotes the Lebesgue measure on $\boldsymbol{R}^n$.

In the sequel, we say simply that the n-dimensional Euclidean space is the **n-dimensional space**. We denote the dual space of $\boldsymbol{R}^n$ as $\boldsymbol{R}_n$. Then $\boldsymbol{R}^n$ and $\boldsymbol{R}_n$ are isomorphic. Therefore, in the sequel, identifying $\boldsymbol{R}_n$ with $\boldsymbol{R}^n$, we denote the n-dimensional space and its dual space as the same symbol $\boldsymbol{R}^n$.

In the sequel, when we consider the n-dimensional space, we always consider that a certain orthogonal coordinate system is selected properly and fixed.

Then we define a $\boldsymbol{R}^n$-valued function $\boldsymbol{r} = \boldsymbol{r}(\omega)$ on Ω to be a **vector-valued random variable** if $\{\omega;\ \boldsymbol{r}(\omega) \in A\}$ always belongs to $\mathcal{B}$ for an arbitrary $A \in \mathcal{M}_n$. Then, if we put

$$\mu(A) = P\ (\ \{\omega;\ \boldsymbol{r}(\omega) \in A\}\ )$$

for $A \in \mathcal{M}_n$, $(\boldsymbol{R}^n,\ \mathcal{M}_n,\ \mu)$ is a probability space. We say that this probability space is the probability distribution of the vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$.

Theorem 1.1.1 *We assume that $\Omega = \Omega(\mathcal{B},\ P)$ is a probability space, $\boldsymbol{r} = \boldsymbol{r}(\omega)$ is a $\boldsymbol{R}^n$-valued random variable and $(\boldsymbol{R}^n,\ \mathcal{M}_n,\ \mu)$ is the probability distribution of this vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$. If μ is absolutely continuous with respect to the Lebesgue measure λ, there exists a certain non-negative Lebesgue integrable function $p(\boldsymbol{r})$ such that the equality*

$$\mu(A) = P\ (\ \{\omega;\ \boldsymbol{r}(\omega) \in A\}\ ) = \int_A p(\boldsymbol{r})d\boldsymbol{r}$$

holds for $A \in \mathcal{M}_n$.

Then we have

$$\int p(\boldsymbol{r})d\boldsymbol{r} = 1.$$

In Theorem 1.1.1, when the integration domain of the integral is not expressed explicitly, we mean that this is the integral on $\boldsymbol{R}^n$. In the sequel of this paper, we keep this rule.

In this case, we say that the real function $p(\boldsymbol{r})$ is a **probability density function** of the vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$. We temporally say the probability density function as the **probability density** shortly. This is the general property for the classical random variable.

In this sense, the probability density is a L^1-density in the classical meaning.

For a vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$, the probability distribution $(\boldsymbol{R}^n, \mathcal{M}_n, \mu)$ or the probability density $p(\boldsymbol{r})$ gives the information concerning the distribution state of the vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$.

Now we give the definition of the expectation value.

Definition 1.1.2 We assume that a function $\Phi(\boldsymbol{r})$ of a vector $\boldsymbol{r}$ is a Lebesgue measurable function on $\boldsymbol{R}^n$. Then we define the expectation value $E[\Phi(\boldsymbol{r}(\omega))]$ of the random variable $\Phi(\boldsymbol{r}(\omega))$ by the relation

$$E[\Phi(\boldsymbol{r}(\omega))] = \int_{\Omega} \Phi(\boldsymbol{r}(\omega)) dP(\omega).$$

This definition has the meaning when the integral on the right hand side converges absolutely.

Then we have the following theorem.

Theorem 1.1.2 *Assume that Ω and $\boldsymbol{r} = \boldsymbol{r}(\omega)$ satisfy the conditions in Theorem* 1.1.1. *Then, for the expectation value of the random variable $\Phi(\boldsymbol{r}(\omega))$, we have the relation*

$$E[\Phi(\boldsymbol{r}(\omega))] = \int \Phi(\boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r}.$$

Here, this relation has the meaning when the integral on the right hand side converges absolutely.

Here we show the outline of the proof of the relation in Theorem 1.1.2.

At first, we prove the relation in Theorem 1.1.2 when $\Phi(\boldsymbol{r})$ is a simple function. Next, we prove this relation when $\Phi(\boldsymbol{r})$ is a measurable function which is the limit of a certain sequence of simple functions in the sense of pointwise convergence.

Then we have the following theorem.

Theorem 1.1.3 *Let $\boldsymbol{r} = \boldsymbol{r}(\omega)$ be a $\boldsymbol{R}^n$-valued random variable on Ω. Let $\Phi(\boldsymbol{r})$ and $\Psi(\boldsymbol{r})$ be two Lebesgue measurable functions of $\boldsymbol{r}$. Then we have the following relations* (1) *and* (2):

(1) $E[\Phi(\boldsymbol{r}(\omega)) + \Psi(\boldsymbol{r}(\omega))] = E[\Phi(\boldsymbol{r}(\omega))] + E[\Psi(\boldsymbol{r}(\omega))]$.

(2) $E[\alpha\Phi(\boldsymbol{r}(\omega)) + \beta] = \alpha E[\Phi(\boldsymbol{r}(\omega))] + \beta$.

Here α and β are two real constants.

Corollary 1.1.1 *Let $\boldsymbol{r} = \boldsymbol{r}(\omega)$ be a $\boldsymbol{R}^n$-valued random variable on Ω. Let $\Phi_1(\boldsymbol{r})$, $\Phi_2(\boldsymbol{r})$, $\cdots$, $\Phi_m(\boldsymbol{r})$ be the Lebesgue measurable functions of the vector $\boldsymbol{r}$. Further, let α_0, α_1, $\cdots$, α_m be certain real constants. Then we have the equality:*

$$\begin{aligned} &E[\alpha_0 + \alpha_1\Phi_1(\boldsymbol{r}(\omega)) + \alpha_2\Phi_2(\boldsymbol{r}(\omega)) + \cdots + \alpha_m\Phi_m(\boldsymbol{r}(\omega))] \\ &= \alpha_0 + \alpha_1 E_1[\Phi_1(\boldsymbol{r}(\omega))] + \alpha_2 E_2[\Phi_2(\boldsymbol{r}(\omega))] + \cdots + \alpha_m E_m[\Phi_m(\boldsymbol{r}(\omega))]. \end{aligned}$$

1.2 Concept of natural probability and its fundamental properties

In this section, we study the concept of natural probability and its fundamental properties. As for their details, we refer to Ito [1], [9], [10].

Let $\boldsymbol{R}^n$ be the n-dimensional space. Here assume that $n \geq 1$ holds. Let $\boldsymbol{C}$ be the field of complex numbers. Let $L^2 = L^2(\boldsymbol{R}^n)$ be the Hilbert space composed of all complex-valued square integrable functions on $\boldsymbol{R}^n$.

For φ, $\psi \in L^2$, we define the inner product (φ, ψ) by the relation

$$(\varphi, \psi) = \int \overline{\varphi(\boldsymbol{r})}\psi(\boldsymbol{r})d\boldsymbol{r}.$$

Here $\overline{\varphi(\boldsymbol{r})}$ denotes the complex conjugate of $\varphi(\boldsymbol{r})$.

We define the norm $\|\psi\|$ of $\psi \in L^2$ by the relation

$$\|\psi\| = \sqrt{(\psi, \psi)} = \left\{ \int |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} \right\}^{1/2}.$$

We say that a L^2-function ψ is normalized when $\|\psi\| = 1$ holds.

Now we define the characteristic function $\chi_A(\boldsymbol{r})$ of a subset A of $\boldsymbol{R}^n$ by the relation

$$\chi_A(\boldsymbol{r}) = \begin{cases} 1, & (\boldsymbol{r} \in A), \\ 0, & (\boldsymbol{r} \notin A). \end{cases}$$

Here we remember the concept of natural probability.

Theorem 1.2.1 *Assume that the measure space $(\boldsymbol{R}^n, \mathcal{M}_n, \lambda)$ is the Lebesgue measure space on $\boldsymbol{R}^n$. Assume that a function ψ is a normalized L^2-function on $\boldsymbol{R}^n$. Then, if we put*

$$\mu(A) = \int_A |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}$$

for $A \in \mathcal{M}_n$, μ is a probability measure on $\mathcal{M}_n$. Then the measure space $(\boldsymbol{R}^n, \mathcal{M}, \mu)$ is a probability space.

We say that a normalized L^2-function ψ such as in Theorem 1.2.1 is a L^2-**density**. The probability measure μ is absolutely continuous with respect to the Lebesgue measure.

Theorem 1.2.2 *Assume that a function ψ is a L^2-density on $\boldsymbol{R}^n$ and the probability space $(\boldsymbol{R}^n, \mathcal{M}_n, \mu)$ satisfies the condition in Theorem* 1.2.1. *Now we put*

$$\psi_A(\boldsymbol{r}) = \psi(A;\ \boldsymbol{r}) = \chi_A(\boldsymbol{r})\psi(\boldsymbol{r})$$

for $A \in \mathcal{M}_n$. Then, for the L^2-valued set function $\psi : A \to \psi_A$ defined on $\mathcal{M}_n$, we have the following (1) *and* (2):

(1) *If every pair of sets $A_1, A_2, \cdots$ of $\mathcal{M}_n$ are mutually disjoint, we have*

$$\psi_A = \sum_{m=1}^{\infty} \psi_{A_m}$$

in the sense of L^2-convergence for

$$A = \sum_{m=1}^{\infty} A_m.$$

(2) *For arbitrary $A,\ B \in \mathcal{M}_n$, we have*

$$(\psi_A,\ \psi_B) = \mu(A \cap B).$$

Namely, we have

$$\int \overline{\psi_A(\boldsymbol{r})}\psi_B(\boldsymbol{r})d\boldsymbol{r} = \int_{A\cap B} |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} = \mu(A \cap B).$$

Especially, if the condition $A \cap B = \phi$ holds for $A,\ B \in \mathcal{M}_n$, we have $\psi_A \perp \psi_B$.

Definition 1.2.1 If the function ψ and the probability space $(\boldsymbol{R}^n, \mathcal{M}, \mu)$ satisfy the conditions of Theorem 1.2.2, we say that the L^2-valued set function $\psi_A = \chi_A\psi$ on $\mathcal{M}_n$ is an **orthogonal probability measure** on the probability space $(\boldsymbol{R}^n, \mathcal{M}, \mu)$. Then we say that the orthogonal probability measure ψ_A is a **natural probability measure** defined by the L^2-density ψ. We also say that this natural probability measure ψ_A is a natural probability defined by the L^2-density. Further, we also say that the L^2-density ψ is a **natural probability density**.

Definition 1.2.2 We use the notation in Theorem 1.2.1 and Theorem 1.2.2. Assume that $\Omega = \Omega(\mathcal{B},\ P)$ is a probability space. We say that a $\boldsymbol{R}^n$-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$ on Ω is a **vector-valued natural random variable**, if there exists a L^2-density ψ on $\boldsymbol{R}^n$ which determines the probability distribution law of $\boldsymbol{r}$ such that the following conditions (1) and (2) hold:

(1) The L^2-valued set function $\psi_A = \chi_A\psi,\ (A \in \mathcal{M}_n)$ is an orthogonal probability measure on the probability space $(\boldsymbol{R}^n,\ \mathcal{M},\ \mu)$.

(2) For $A \in \mathcal{M}_n$, we have the relation

$$P\left(\ \{\rho \in \Omega;\ \boldsymbol{r}(\rho) \in A\}\ \right) = \mu(A).$$

Then we say that the vector-valued random variable $\boldsymbol{r}$ is ruled by the law of **natural probability distribution** which is determined by the L^2-density ψ.

We say that the probability space $(\boldsymbol{R}^n,\ \mathcal{M},\ \mu)$ is the **probability distribution** of a vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$.

In Definition 1.2.2, the condition (2) means that the probability of the event "$\boldsymbol{r}(\omega)$ belongs to A" is equal to $\mu(A)$.

Theorem 1.2.3 *Assume that $\boldsymbol{P}_n = \{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ \cdots\}$ is a sequence of vectors in $\boldsymbol{R}^n$ and the function $\hat{\psi}$ on $\boldsymbol{P}_n$ satisfies the condition*

$$\sum_{\boldsymbol{p}\in\boldsymbol{P}_n} |\hat{\psi}(\boldsymbol{p})|^2 = 1.$$

Assume that $\mathcal{F}_n$ is a σ-algebra of the family of all subsets of $\boldsymbol{P}_n$ and the set function ν on $\mathcal{F}_n$ is defined by the condition

$$\nu(B) = \sum_{\boldsymbol{p}\in B} |\hat{\psi}(\boldsymbol{p})|^2$$

for $B \in \mathcal{F}_n$. Further assume that the condition

$$\nu(\boldsymbol{P}_n) = \sum_{\boldsymbol{p}\in\boldsymbol{P}_n} |\hat{\psi}(\boldsymbol{p})|^2 = 1$$

holds. Thereby the measure space $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$ is a probability space.

Theorem 1.2.4 *Assume that the set $\boldsymbol{P}_n$, the function $\hat{\psi}$ and the probability space $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$ are the same as in Theorem 1.2.3. If we put*

$$\hat{\psi}_A(\boldsymbol{p}) = \chi_A(\boldsymbol{p})\hat{\psi}(\boldsymbol{p})$$

for $A \in \mathcal{F}_n$, the l^2-valued set function $\hat{\psi}:\ A \to \hat{\psi}_A$ defined on $\mathcal{F}_n$ satisfies the following conditions (1) and (2):

(1) *If every pair of sets* $A_1,\ A_2,\ \cdots$ *in* $\mathcal{F}_n$ *are mutually disjoint, we have*

$$\hat{\psi}_A = \sum_{m=1}^{\infty} \hat{\psi}_{A_m}$$

in the sense of l^2*-convergence for*

$$A = \sum_{m=1}^{\infty} A_m.$$

(2) *For arbitrary* $A,\ B \in \mathcal{F}_n$, *we have*

$$(\hat{\psi}_A,\ \hat{\psi}_B) = \nu(A \cap B).$$

Namely, we have

$$\sum_{\boldsymbol{p} \in \boldsymbol{P}} \overline{\hat{\psi}_A(\boldsymbol{p})} \hat{\psi}_B(\boldsymbol{p}) = \sum_{\boldsymbol{p} \in A \cap B} |\hat{\psi}(\boldsymbol{p})|^2 = \nu(A \cap B).$$

Especially, if we have $A \cap B = \phi$ *for* $A,\ B \in \mathcal{F}_n$, $\hat{\psi}_A \perp \hat{\psi}_B$ *holds in* l^2.

Definition 1.2.3 Assume that the set $\boldsymbol{P}_n$, the function $\hat{\psi}$ and the probability space $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$ are the same as in Theorem 1.2.4. Then we say that a l^2-valued set function $\hat{\psi}_A = \chi_A \hat{\psi}$ on $\mathcal{F}_n$ is a **discrete orthogonal probability measure** on the **discrete probability space** $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$. We say that the discrete orthogonal probability measure $\hat{\psi}$ is the discrete natural probability measure determined by the l^2-density $\hat{\psi}$.

Definition 1.2.4 Assume that the set $\boldsymbol{P}_n$, the function $\hat{\psi}$ and the probability space $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$ are the same as Definition 1.2.3. We say that a vector-valued discrete random variable $\boldsymbol{p} = \{\boldsymbol{p}_n = \boldsymbol{p}_n(\omega);\ n = 1,\ 2,\ 3,\ \cdots\}$ on the probability space $\Omega = \Omega(\mathcal{B},\ P)$ is a **discrete natural random variable** if there exists a l^2-density $\hat{\psi}$ on $\boldsymbol{P}_n$ such that it determines the law of discrete probability distribution so that we have the following conditions (1) and (2):

(1) If $\mathcal{F}_n$ is a σ-algebra of the family of all subsets of $\boldsymbol{P}_n$, the l^2-valued set function $\hat{\psi}_A = \chi_A \hat{\psi},\ (A \in \mathcal{F}_n)$ is a discrete orthogonal probability measure on the discrete probability space $(\boldsymbol{P}_n,\ \mathcal{F}_n,\ \nu)$.

(2) For $A \in \mathcal{F}_n$, we have the condition

$$P\left(\ \{\omega \in \Omega;\ \boldsymbol{p}(\omega) \in A\}\ \right) = \nu(A).$$

Then, we say that this vector-valued discrete random variable $\boldsymbol{p} = \{\boldsymbol{p}_n(\omega)\}$ is ruled by the law of natural probability distribution determined by the l^2-density $\hat{\psi}$.

1.3 Natural random variables and their expectation values

In this section, we define the expectation value of a natural random variable.

Now we assume that $\Omega = \Omega(\mathcal{B}, P)$ is a probability space. Assume that $(\boldsymbol{R}^n, \mathcal{M}_n, \lambda)$ is the Lebesgue measure space on $\boldsymbol{R}^n$. Then, we assume that a $\boldsymbol{R}^n$-valued function $\boldsymbol{r} = \boldsymbol{r}(\omega)$ on Ω is a vector-valued natural random variable.

Then, if $\Phi(\boldsymbol{r})$ is a Lebesgue measurable function of $\boldsymbol{r}$, we can define the expectation value $E[\Phi(\boldsymbol{r}(\omega))]$ of the natural random variable $\Phi(\boldsymbol{r}(\omega))$ as in Definition 1.1.2. For this expectation value, the relation in Theorem 1.1.2 holds by using $|\psi(\boldsymbol{r})|^2$ instead of $p(\boldsymbol{r})$.

Namely, we have the following theorem.

Theorem 1.3.1 *We assume that $\boldsymbol{r} = \boldsymbol{r}(\omega)$ is a $\boldsymbol{R}^n$-valued natural random variable on Ω. We assume that $\Phi(\boldsymbol{r})$ is a Lebesgue measurable function on $\boldsymbol{R}^n$. Then, the relation*

$$E[\Phi(\boldsymbol{r}(\omega))] = \int \Phi(\boldsymbol{r})|\psi(\boldsymbol{r})|^2 d\boldsymbol{r}$$

holds for the expectation value of the random variable $\Phi(\boldsymbol{r}(\omega))$. Here this relation has the meaning when the integral in the right hand side converges absolutely.

In this sense, we can calculate the considered expectation value $E[\Phi(\boldsymbol{r}(\omega))]$ by the similar way as the expectation value of a classical random variable except the difference of the form of the probability density. Similarly we have the analogs of Theorem 1.1.3 and its Corollary 1.1.1. Namely we have the following theorem.

Theorem 1.3.2 *Assume that $\boldsymbol{r} = \boldsymbol{r}(\omega)$ is a $\boldsymbol{R}^n$-valued natural random variable on Ω. Assume that $\Phi(\boldsymbol{r})$ and $\Psi(\boldsymbol{r})$ are two Lebesgue measurable functions on $\boldsymbol{R}^n$. Then we have the following relations* (1) *and* (2):

(1) $E[\Phi(\boldsymbol{r}(\omega)) + \Psi(\boldsymbol{r}(\omega))] = E[\Phi(\boldsymbol{r}(\omega))] + E[\Psi(\boldsymbol{r}(\omega))]$.

(2) $E[\alpha\Phi(\boldsymbol{r}(\omega)) + \beta] = \alpha E[\Phi(\boldsymbol{r}(\omega)) + \beta]$.

Here, α and β are some real constants.

Corollary 1.3.1 *Assume that $\boldsymbol{r} = \boldsymbol{r}(\omega)$ is the same as in Theorem* 1.3.2. *Assume that $\Phi_1(\boldsymbol{r})$, $\Phi_2(\boldsymbol{r})$, , $\Phi_m(\boldsymbol{r})$ are Lebesgue measurable functions on $\boldsymbol{R}^n$ and α_0, α_1, $\cdots$, α_m are some real constants. Then we have the following relation:*

$$E[\alpha_0 + \alpha_1\Phi_1(\boldsymbol{r}(\omega)) + \alpha_2\Phi_2(\boldsymbol{r}(\omega)) + \cdots + \alpha_m\Phi_m(\boldsymbol{r}(\omega))]$$

$$= \alpha_0 + \alpha_1 E[\Phi_1(\boldsymbol{r}(\omega))] + \alpha_2 E[\Phi_2(\boldsymbol{r}(\omega))] + \cdots + \alpha_m E[\Phi_m(\boldsymbol{r}(\omega))].$$

Now we assume that p is a positive natural number and L^p_μ is the space of all complex-valued p-th integrable functions on the probability space $(\boldsymbol{R}_n,\ \mathcal{M}_n,\ \mu)$. This space is equal to

$$L^p_\mu = \{f(\boldsymbol{r});\ \int |f(\boldsymbol{r})|^p d\mu(\boldsymbol{r}) = \int |f(\boldsymbol{r})|^p |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} < \infty\}.$$

Then we say that an element in L^p_μ is a L^p_μ-natural random variable.

In this paper, we have only to consider the special cases $p = 1,\ 2$. When a L^2-valued set function $\psi_A = \chi_A \psi$ is a natural probability on the probability space $(\boldsymbol{R}_n,\ \mathcal{M}_n,\ \mu)$, we define the natural expectation value of μ- measurable function f belonging to $L^p_\mu = L^p_\mu(\boldsymbol{R}_n,\ \mathcal{M}_n,\ \mu)$ with respect to ψ_A in the following.

Definition 1.3.1 Assume that $p \geq 1$. Then we define the natural expectation values by the following conditions (1) and (2):

(1) When a function $f(\boldsymbol{r})$ is a L^p_μ-simple function, namely, when we have

$$f(\boldsymbol{r}) = \sum_{m=1}^{\infty} a_m \chi_{A_m}(\boldsymbol{r}),\ (a_m \in \boldsymbol{C},\ m \geq 1),$$

$$\boldsymbol{R}^n = A_1 + A_2 + \cdots,\ \text{(direct sum)},$$

we define the natural expectation value of the function $f(\boldsymbol{r})$ by the relation

$$\int f(q)\psi(dq;\ \boldsymbol{r}) = \sum_{m=1}^{\infty} a_m \psi(A_m;\ \boldsymbol{r}).$$

Its norm is equal to

$$\| \int f(q) d\psi(dq; \boldsymbol{r}) \|^p = \sum_{m=1}^{\infty} |a_m|^p \int_{A_m} |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} = \int |f(\boldsymbol{r})|^p |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}.$$

(2) When a function f is a general L^p_μ-function, there exists a sequence of L^p_μ-simple functions $\{f_m\}$ such that we have $f_m \to f$ in L^p_μ. Then we define the **natural expectation value** of the function $f(\boldsymbol{r})$ by the relation

$$\int f(q)\psi(dq;\ \boldsymbol{r}) = \lim_{m \to \infty} \int f_m(q)\psi(d\boldsymbol{q};\ \boldsymbol{r}).$$

Its norm is equal to

$$\| \int f(q)\psi(dq;\ \boldsymbol{r}) \|^p = \int |f(\boldsymbol{r})|^p |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}.$$

Theorem 1.3.3 *Assume that* $\boldsymbol{P}_n = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots\}$ *is a sequence of vectors in* $\boldsymbol{R}^n$. *Assume that* $\Omega = \Omega(\mathcal{B}, P)$ *is a probability space and* $\boldsymbol{p} = \boldsymbol{p}(\omega) = \{\boldsymbol{p}_n = \boldsymbol{p}_n(\omega);\ n = 1, 2, 3, \cdots\}$ *is a* $\boldsymbol{P}_n$*-valued discrete natural random variable on* Ω. *Assume that there exists a* l^2*-density* $\hat{\psi}$ *on* $\boldsymbol{P}_n$. *Then we define a discrete orthogonal probability measure* $\hat{\psi}_A = \chi_A\hat{\psi}$, $(A \in \mathcal{F}_n)$ *on* $(\boldsymbol{P}_n, \mathcal{F}_n, \nu)$ *in the similar way as Definition* 1.2.4.

Now, we assume $\Phi(\boldsymbol{p})$ *is a function on* $\boldsymbol{P}_n$. *Then, for the expectation value of the random variable* $\Phi(\boldsymbol{p}(\omega))$, *we have the relation*

$$E[\Phi(\boldsymbol{P}(\omega))] = \sum_{n=1}^{\infty} \Phi(\boldsymbol{p}_n)|\hat{\psi}(\boldsymbol{p}_n)|^2.$$

Here this relation has the meaning when the series in the right hand side converges absolutely.

We remark that, for the expectation value considered in Theorem 1.3.3, we have the similar results as Theorem 1.3.2, Corollary 1.3.1 and Definition 1.3.2.

1.4 Concepts of local natural probability and definition of expectation value of local natural random variables

In this section, we study the local natural probability and the expectation value of local natural random variable. This is a very new result.

Theorem 1.4.1 *Assume that a function* ψ *is a* L^2_{loc}*-density on* $\boldsymbol{R}^n$. *Further, let* $\mathbb{S}$ *be a family of all compact subsets in* $\boldsymbol{R}^n$ *such that the condition*

$$\int_S |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} > 0$$

is satisfied. Now, for an arbitrary compact set S *in* $\mathbb{S}$, $\psi_S(\boldsymbol{r}) = \chi_S(\boldsymbol{r})\psi(\boldsymbol{r})$ *denotes the section of* ψ *over* S. *Then, if we define*

$$\mu_S(A) = \frac{\displaystyle\int_{A\cap S} |\psi_S(\boldsymbol{r})|^2 d\boldsymbol{r}}{\displaystyle\int_S |\psi_S(\boldsymbol{r})|^2 d\boldsymbol{r}}$$

for a Lebesgue measurable set A *in* $\boldsymbol{R}^n$, μ_S *is a probability measure on* $\boldsymbol{R}^n \cap S$. *Then, the measure space* $(\boldsymbol{R}^n \cap S, \mathcal{M}_n \cap S, \mu_S)$ *is a* **relative probability space**.

Theorem 1.4.2 *Assume the function* ψ *and the relative probability space* $(\boldsymbol{R}^n \cap S, \mathcal{M}_n \cap S, \mu_S)$ *are the same as in Theorem* 1.4.1.

Now, if we define

$$\psi_{S,\,A}(\boldsymbol{r}) = \chi_A(\boldsymbol{r})\psi_S(\boldsymbol{r})$$

for $A \in \mathcal{M}_n \cap S$, *the* $L^2(S)$*-valued set function* $\psi_S : A \to \psi_{S,\,A}$ *on* $\mathcal{M}_S \cap S$ *satisfies the following conditions* (1) *and* (2):

(1) *If every pair of sets* $A_1,\ A_2,\ \cdots$ *in* $\mathcal{M}_S \cap S$ *are mutually disjoint, we have the equality*

$$\psi_{S,\,A} = \sum_{m=1}^{\infty} \psi_{S,\,A_m}$$

in the sense of $L^2(S)$*-convergence for*

$$A = \sum_{m=1}^{\infty} A_m.$$

(2) *For arbitrary* $A,\ B \in \mathcal{M}_n \cap S$, *we have the relation*

$$(\psi_{S,\,A},\ \psi_{S,\,B}) = \mu_S(A \cap B).$$

Namely, we have the relation

$$\int \overline{\psi_{S,\,A}(\boldsymbol{r})}\psi_{S,\,B}(\boldsymbol{r})d\boldsymbol{r} = \int_{A\cap B} |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} = \mu_S(A \cap B).$$

Especially, if $A \cap B = \phi$ *holds for* $A,\ B \in \mathcal{M}_n \cap S$, *we have* $\psi_{S,\,A} \perp \psi_{S,\,B}$ *in* $L^2(S)$.

Theorem 1.4.3 *Assume that a function* ψ *is a* L^2_{loc}*-density on* $\boldsymbol{R}^n$, $\mathbb{S}$ *is the family of all compact sets* S *in* $\boldsymbol{R}^n$ *such that the condition*

$$\int_S |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} > 0$$

is satisfied. Assume that the relative probability space $(\boldsymbol{R} \cap S,\ \mathcal{M}_n \cap S,\ \mu_S)$ *is the same as in Theorem* 1.4.1. *Then we define* $\mu(A)$ *by the relation*

$$\mu(A) = \lim_S \mu_S(A)$$

for $A \in \mathcal{M}_n$ *as a Moore-Smith limit. Then we have*

$$\mu(\boldsymbol{R}^n) = 1.$$

Therefore, this measure space $(\boldsymbol{R}^n,\ \mathcal{M}_n, \mu)$ is a probability space.

Then we say that this probability space $(\boldsymbol{R}^n,\ \mathcal{M}_n,\ \mu)$ is the probability distribution of the vector-valued random variable $\boldsymbol{r} = \boldsymbol{r}(\omega)$.

Theorem 1.4.4 *Assume that $\Phi(\boldsymbol{r})$ is a Lebesgue measurable function of $\boldsymbol{r}$. Assume that the probability space $(\boldsymbol{R}^n, \mathcal{M}, \mu)$ is the same as in Theorem* 1.4.3. *Then we define the expectation value of $\Phi(\boldsymbol{r}(\omega))$ by the relation*

$$E[\Phi(\boldsymbol{r}(\omega))] = \lim_S E_S[\Phi(\boldsymbol{r}(\omega))] = \lim_S \frac{\int_S \Phi(\boldsymbol{r})|\psi(\boldsymbol{r})|^2 d\boldsymbol{r}}{\int_S |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}}$$

as a Moore-Smith limit. This has the meaning only when this limit exists. Here we define the local expectation value of $\Phi(\boldsymbol{r}(\omega))$ by the relation

$$E_S[\Phi(\boldsymbol{r}(\omega))] = \int_S \Phi(\boldsymbol{r}) d\mu_S(\boldsymbol{r}) = \frac{\int_S \Phi(\boldsymbol{r})|\psi(\boldsymbol{r})|^2 d\boldsymbol{r}}{\int_S |\psi(\boldsymbol{r})|^2 d\boldsymbol{r}}.$$

Then, we have the properties of the expectation values as the same as in Theorem 1.3.1, Theorem 1.3.2 and Corollary 1.3.1.

1.5 Dirac measure

In this section, we study the Dirac measure δ as an example of a natural probability. As for the details concerning the Dirac measure, we refer to Ito [9].

When a certain eigenfunction ψ of a certain Schrödinger operator has its Fourier transform $\hat{\psi} = \delta$, how can we understand the natural probability distribution of the momentum variable p which is determined by the Dirac measure $\hat{\psi} = \delta$?

In order to answer this question, we study the new characterization of the Dirac measure δ.

We define the space $\mathcal{K} = C_0(\boldsymbol{R})$ as the space of all continuous functions with compact support in one dimensional space $\boldsymbol{R}$. Then δ is the continuous linear functional on $\mathcal{K}$. Then we say that δ is a **Radon measure**.

In the sequel, we prove that the Dirac measure δ is an orthogonal probability Radon measure. Therefore, we try to characterize it as a natural probability. Thereby, we can understand that the Dirac measure δ determines the law of natural probability distribution of the momentum variable p. Then, it is important to notice the fact that the Dirac measure is not only a Radon measure but also a probability measure as a set theoretical measure.

The Dirac measure δ_p is the measure with the unit mass at the point p which is defined by the relation

$$\delta_p(q) = \delta(q - p).$$

This is a continuous linear functional on $\mathcal{K}$ defined by the relation

$$\delta_p(\varphi) =< \delta(q-p),\ \varphi(q) >=< \delta(q),\ \varphi(q+p) >= \varphi(p)$$

for $\varphi \in \mathcal{K}$.

We assume that $(\boldsymbol{R},\ \mathcal{M},\ \lambda)$ is one dimensional Lebesgue measure space.

Since $\mathcal{K}$ is dense in $L^2 = L^2(\boldsymbol{R})$, we can extend $\delta_p \in \mathcal{K}'$ to the continuous linear functional on L^2.

Therefore, because $\delta_p(f)$ is defined for $f \in L^2$, especially we can define the set function $\delta_p(A)$ on $\mathcal{M}$ by using the relation

$$\delta_p(A) = \delta_p(\chi_A)$$

when $\chi_A \in L^2$ holds for $A \in \mathcal{M}$.

This set function $\delta_p(A)$ satisfies the conditions

$$\delta_p(A) = \begin{cases} 1, & (p \in A), \\ 0, & (p \notin A) \end{cases}$$

for $A \in \mathcal{M}$ such as $\chi_A \in L^2$. Then we prove that the set function $\delta_p(A)$ is an orthogonal probability measure on the probability space $(\boldsymbol{R},\ \mathcal{M},\ \delta_p)$ in the following.

At first we prove the following Proposition 1.5.1 as a preparation.

Proposition 1.5.1 *Assume that $(\boldsymbol{R},\ \mathcal{M},\ \lambda)$ is the Lebesgue measure space. Then we put $L^2 = L^2(\boldsymbol{R})$. We define the inner product in $\boldsymbol{C}$ by the relation $(\alpha,\ \beta) = \overline{\alpha}\beta$.*

Further, we define the norm of $\alpha \in \boldsymbol{C}$ by the relation $\|\alpha\| = \sqrt{(\alpha,\ \alpha)}$. Then $\boldsymbol{C}$ is one dimensional Hilbert space and $\boldsymbol{C}$ is embedded in L^2 as one dimensional subspace.

Proof We fix one φ_0 in L^2 as a unit vector . Then we consider the map

$$T:\ \alpha \in \boldsymbol{C} \to \alpha\varphi_0 \in L^2.$$

Then we have the equality

$$\overline{\alpha}\beta = (\alpha,\ \beta) = (\alpha\varphi_0,\ \beta\varphi_0)_{L^2}.$$

Namely we have the equality

$$(T\alpha,\ T\beta)_{L^2} = (\alpha,\ \beta)_{\boldsymbol{C}}.$$

Therefore T is an isometry from $\boldsymbol{C}$ into L^2. Thereby, $\boldsymbol{C}$ can be embedded in L^2 as one dimensional subspace. //

Proposition 1.5.2 *We assume that* $(\boldsymbol{R}, \mathcal{M}, \delta_p)$ *is the probability space defined by the Dirac measure* δ_p *which has the unit mass at the point* p *in* $\boldsymbol{R}$. *Here we assume that* $\mathcal{M}$ *is the* σ*-algebra of the family of all Lebesgue measurable sets in* $\boldsymbol{R}$. *Here we define the* $\boldsymbol{C}$*-valued set function* $\delta_p(A)$ *for* $A \in \mathcal{M}$ *such as* $\chi_A \in L^2$ *by the following relation:*

$$\delta_p(A) = \begin{cases} 1, & (p \in A), \\ 0, & (p \notin A). \end{cases}$$

Here the set function $\delta_p(A)$, $(A \in \mathcal{M})$ *on* $\boldsymbol{R}$ *is an orthogonal probability measure on the probability space* $(\boldsymbol{R}, \mathcal{M}, \delta_p)$. *Namely we have the following* (1)~(3):

(1) *If every pair of sets* $A_1, A_2, \cdots$ *in* $\mathcal{M}$ *is mutually disjoint, we have the equality*

$$\delta_p(A) = \sum_{n=1}^{\infty} \delta_p(A_n)$$

for the set

$$A = \sum_{n=1}^{\infty} A_n.$$

(2) *If* $A \cap B = \phi$ *holds for* $A, B \in \mathcal{M}$, *we have the relation*

$$\delta_p(A)\delta_p(B) = 0.$$

Namely we have

$$\delta_p(A) \perp \delta_p(B).$$

(3) *For* $A \in \mathcal{M}$ *such as* $\chi_A \in L^2$, *we have the relation*

$$\|\delta_p(A)\|^2 = \delta_p(A).$$

There is a case where the Fourier transform $\hat{\psi}$ of a L^2_{loc}-solution ψ of a certain Schrödinger equation is the Dirac measure δ.

It is the case when a L^2_{loc}-density ψ determines the natural probability distribution of the position variable of a generalized proper state of a system of free particles.

Now we assume that $\mathcal{K} = C_0(\boldsymbol{R})$ is the space of all continuous functions with compact support in one dimensional space $\boldsymbol{R}$. Then δ_{p_0} is an element of $\mathcal{K}'$. Namely δ_{p_0} is a continuous linear functional on $\mathcal{K}$. Then δ_{p_0} is an orthogonal Radon measure.

We assume that A is a Lebesgue measurable set in $\boldsymbol{R}$ and χ_A is the definition function of A. Then the domain of the orthogonal Radon probability measure

δ_{p_0} can be enlarged in order to include χ_A. Here we assume $\chi_A \in L^2$. Then if we define the set function $\delta_{p_0}(A)$ by the relation

$$\delta_{p_0}(A) = \delta_{p_0}(\chi_A),$$

we have the relation

$$\delta_{p_0}(A) = \begin{cases} 1, & (p_0 \in A), \\ 0, & (p_0 \notin A). \end{cases}$$

Therefore we have the relations

$$(\delta_{p_0}(\{p_0\}),\ \delta_{p_0}(\{p_0\})) = \delta_{p_0}(\{p_0\}) = 1,$$

$$(\delta_{p_0}(A),\ \delta_{p_0}(A)) = \delta_{p_0}(A) = 0,\ (p_0 \notin A).$$

Hence, the set function $\delta_{p_0}(A)$ is an orthogonal probability measure in Proposition 1.5.2. Therefore $\delta_{p_0}(A)$ is an example of natural probability.

Thereby, when the Fourier transform of a L^2_{loc}-density ψ which determines the state of natural probability distribution of the position variable of a certain physical system is $\hat{\psi} = \delta_{p_0}$, the state of natural probability distribution of the momentum variable p is ruled by the Dirac measure δ_{p_0}. This means that the state of natural probability distribution of p determined by δ_{p_0} is the state of distribution such that the momentum value p takes the constant value p_0 with probability 1. Thereby, when the Fourier transform of a L^2_{loc}-density ψ is $\hat{\psi} = \delta_{p_0}$, we see that the state of natural probability distribution of the momentum variable p is determined by $\hat{\psi} = \delta_{p_0}$.

In this case, the state of natural distribution of position variable x is ruled by the L^2_{loc}-density ψ. Thus the natural probability distribution of x is the uniform distribution on $\boldsymbol{R}$.

2 Laws of natural statistical physics

The theory of natural statistical physics is the theory for investigating the natural statistical phenomena arising for the system of electrons, atoms or molecules. In this chapter, we present the laws of natural statistical physics which rule the phenomena of these physical systems composed of electrons, atoms or molecules. Corresponding to the phases of the physical systems, we have some different respresentations of the laws of natural statistical physics.

In sections 2.2∼2.4, we give the laws of natural statistical physics for the three types of the physical systems.

In this chapter, we give the laws of natural statistical physics in the following.

When we study the natural statistical phenomena by using the theory of natural statistical physics, we postulate the three concepts in the following:

(1) The physical system.

(2) The state of the physical system.

(3) The motion of the physical system.

We say that these postulates are the laws of natural statistical physics. These laws are the natural laws of the natural statistical phenomena.

We mention the laws of natural statistical physics in the following.

2.1 Fundamental problem of natural statistical physics

In the natural statistical physics, we understand the physical phenomena on the basis of the statistical properties of the physical quantities such as expectation values or means of the physical quantities of a certain physical system.

Then, because the physical quantities of the physical system are functions of the position variables and the momentum variables, we have to know that the natural statistical states of this physical system are described as the states of natural probability distributions of the position variables and the momentum variables in order to understand the natural statistical phenomena of the physical system.

By virtue of the laws of natural statistical physics, if we determine the L^2-density ψ which determines the natural probability distribution of the position variables of this physical system, the natural probability distribution of the momentum variables are determined by its Fourier transform $\hat{\psi}$. By virtue of the laws of natural statistical physics, this L^2-density ψ is determined as a solution of a certain Schrödinger equation. Therefore, in order to investigate the natural statistical phenomena of the physical system, it is known that the Schrödinger equation is the fundamental equation and it is the fundamental problem to solve the Schrödinger equation. Thus we can understand the natural statistical phenomena on the bases of the laws of natural statistical physics.

In the following sections, we establish the laws of natural statistical physics.

2.2 Laws of natural statistical physics

In this section, we establish the laws of natural statistical physics in the case where a Schrödinger operator has only the discrete spectrum.

Law I(physical system) We postulate that the physical system Ω is a probability space $\Omega = \Omega(\mathcal{B}, P)$. Here Ω is the set of systems ρ of microparticles, $\mathcal{B}$ is the σ-algebra composed of subsets of Ω and P is the σ-additive probability measure.

Law II(state of physical system) We postulate that the state of the physical system Ω is the natural probability distribution of the position variable $\boldsymbol{r}(\rho)$ and the momentum variable $\boldsymbol{p}(\rho)$ of a system $\rho \in \Omega$ of micro-particles. Here $\boldsymbol{r}(\rho)$ moves in $\boldsymbol{R}^n$ and $\boldsymbol{p}(\rho)$ moves in its dual space $\boldsymbol{R}_n = \boldsymbol{R}^n$.

Further we put $n = Md$. Here d is the dimension of the physical space and M is the number of micro-particles which compose the elementary event ρ.

(i) We postulate that the natural probability distribution of the position variable $\boldsymbol{r} = \boldsymbol{r}(\rho)$ is ruled by the law of natural probability distribution which is determined by a L^2-density $\psi(\boldsymbol{r})$ defined on $\boldsymbol{R}^n$.

(ii) We postulate that the natural probability distribution of the momentum variable $\boldsymbol{p} = \boldsymbol{p}(\rho)$ is ruled by the law of natural probability distribution determined by its Fourier transform $\hat{\psi}(\boldsymbol{r})$. Here the Fourier transform $\hat{\psi}(\boldsymbol{p})$ of a L^2-density $\psi(\boldsymbol{r})$ is defined by the following:

$$\hat{\psi}(\boldsymbol{p}) = \frac{1}{(\sqrt{2\pi\hbar})^n} \int \psi(\boldsymbol{r}) e^{-i(\boldsymbol{p},\, \boldsymbol{r})/\hbar} d\boldsymbol{r},$$

$$\psi(\boldsymbol{r}) = \frac{1}{(\sqrt{2\pi\hbar})^n} \int \hat{\psi}(\boldsymbol{p}) e^{i(\boldsymbol{p},\, \boldsymbol{r})/\hbar} d\boldsymbol{p}$$

where we put

$$\boldsymbol{r} = {}^t(x_1,\ x_2,\ \cdots,\ x_n),\ \boldsymbol{p} = {}^t(p_1,\ p_2,\ \cdots,\ p_n),$$

$$(\boldsymbol{p},\ \boldsymbol{r}) = p_1x_1 + p_2x_2 + \cdots + p_nx_n.$$

Here we put $\hbar = \dfrac{h}{2\pi}$ and h denotes Planck's constant. The reason why we define the Fourier transformation in Law (II), (ii) in such a form is to derive the Schrödinger equation for the physical system by using the variational principle.

The constant is chosen so that the theoretical results of a certain physical system coincide with the observed data of a certain physical quantities for the natural statistical phenomena.

Law III(motion of physical system) We postulate that the L^2-density $\psi(\boldsymbol{r},\ t)$ which determines the law of natural probability distribution of the position variable $\boldsymbol{r}$ at time t is the solution of the time evolving Schrödinger equation. We say that this time evolution is the motion of the physical system. The law of motion of the physical system is described by the Schrödinger equation. We say that this Schrödinger equation is the equation of motion of the physical system.

The Schrödinger equation is described in the following form:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

We say that the operator H is a Schrödinger operator. H is a self-adjoint operator on a certain Hilbert space. The concrete form of this Schrödinger operator H is determined concretely for every concrete physical system.

Remark 2.2.1 In the theory of natural statistical physics, we study the derivatives or the partial derivatives of L^2-functions or L^2_{loc}- functions respectively. Then these L^2-functions or these L^2_{loc}-functions are not always differentiable in the classical sense. Therefore, we remark that we define the derivatives or the partial derivatives of L^2-functions or L^2_{loc}-functions in the sense of L^2-convergence or L^2_{loc}-convergence respectively in these cases.

Remark 2.2.2 We consider that a L^2-density in the Law III is a L^2-valued function of the time variable t. Here we put $L^2 = L^2(\boldsymbol{R}^n)$. Therefore the function $\psi(\boldsymbol{r},\ t)$ is not imposed the condition that $\psi(\boldsymbol{r},\ t)$ is a L^2-function of t. Namely we consider that the time variable t is a parameter. Then the partial derivative $\dfrac{\partial\psi}{\partial t}$ is the derivative of the vector-valued function $\psi(\boldsymbol{r},\ t)$ with respect to t. This derivative is taken in the sense of strong derivative. Namely, we have

$$\lim_{h\to 0}\|\frac{\psi(\boldsymbol{r},\ t+h)-\psi(\boldsymbol{r},\ t)}{h}-\frac{\partial\psi(\boldsymbol{r},\ t)}{\partial t}\|=0$$

with respect to the norm of $L^2 = L^2(\boldsymbol{R}^n)$. In the sequel, we do not repeat this remark.

2.3 Laws of generalized natural probability distribution

In this section, we establish the laws of generalized natural probability distribution. Here we study the case where the Schrödinger operator has the continuous spectrum.

Law I′((generalized) proper physical subsystem) We postulate that a proper physical subsystem or a generalized proper physical subsystem is a physical subsystem Ω' as a probability subspace of the total probability space $\Omega = \Omega(\mathcal{B},\ P)$. Here Ω is the set of systems ρ of micro-particles, $\mathcal{B}$ is a σ-algebra composed of subsets of Ω and P is a σ- additive probability measure.

Then this satisfies the Law II′ of state of generalized proper physical subsystem and Law III′ of motion in the following.

Law II′(state of (generalized) proper physical subsystem) We postulate that the state of a proper physical subsystem Ω' is the natural probability distribution or the generalized natural probability distribution of the

position variable $\boldsymbol{r}(\rho)$ and the momentum variable $\boldsymbol{p}(\rho)$ of the systems of microparticles $\rho \in \Omega'$. This is determined in the following (1) and (2):

(1) We postulate that, if the Schrödinger operator has the discrete spectrum, the state of the proper physical subsystem Ω' is determined by an eigenfunction ψ of the Schrödinger operator by the similar way to the Law II in section 2.2.

(2) We postulate that, if the Schrödinger operator has the continuous spectrum, the state of the generalized proper physical subsystem Ω' is determined by a generalized eigenfunction ψ of the Schrödinger operator in the following (i)′ and (ii)′:

(i)′ We postulate that the generalized natural probability distribution of the position variables $\boldsymbol{r} = \boldsymbol{r}(\rho)$ is ruled by the law of local natural probability distribution determined by the L^2_{loc}-density $\psi(\boldsymbol{r})$ on $\boldsymbol{R}^n$.

(ii)′ We postulate that the generalized natural probability distribution of the momentum variables $\boldsymbol{p} = \boldsymbol{p}(\rho)$ is ruled by the law of local natural probability distribution determined by the Fourier transform $\hat{\psi}$ of $\psi(\boldsymbol{r})$.

In the above Law II′, (ii)′, $\hat{\psi}$ is the Fourier transform of ψ defined by the relation

$$\hat{\psi}(\boldsymbol{p}) = \lim_S \hat{\psi}_S(\boldsymbol{p}).$$

Here the limit $\lim_S$ means the Moore-Smith limit for the family $\mathbb{S} = \{S\}$ of all compact subsets of $\boldsymbol{R}^n$ in the sense of convergence of generalized functions.

Here the local Fourier transform $\hat{\psi}_S$ of the L^2_{loc}-density $\psi(\boldsymbol{r})$ is defined in the following:

$$\hat{\psi}_S(\boldsymbol{p}) = \frac{1}{(\sqrt{2\pi\hbar})^n} \int \psi_S(\boldsymbol{r}) e^{-i(\boldsymbol{p},\, \boldsymbol{r})/\hbar} d\boldsymbol{r},$$

$$\psi_S(\boldsymbol{r}) = \frac{1}{(\sqrt{2\pi\hbar})^n} \int \hat{\psi}_S(\boldsymbol{p}) e^{i(\boldsymbol{p},\, \boldsymbol{r})/\hbar} d\boldsymbol{p},$$

$$\boldsymbol{r} = {}^t(x_1,\ x_2,\ \cdots,\ x_n),\ \boldsymbol{p} = {}^t(p_1,\ p_2,\ \cdots,\ p_n),$$

$$(\boldsymbol{p},\ \boldsymbol{r}) = p_1x_1 + p_2x_2 + \cdots + p_nx_n.$$

Here, for an arbitrary compact subset $S \in \mathbb{S}$ of $\boldsymbol{R}^n$, ψ_S denotes the section over $S \in \mathbb{S}$. Namely, $\psi_S(\boldsymbol{r})$ is defined by the relation $\psi_S(\boldsymbol{r}) = \psi(\boldsymbol{r})\chi_S(\boldsymbol{r})$. Here $\chi_S(\boldsymbol{r})$ denotes the defining function of the set $S \in \mathbb{S}$. Further we put $\hbar = \dfrac{h}{2\pi}$, where h denotes Planck's constant.

Then, the Fourier transform $\hat{\psi}(\boldsymbol{p})$ is generally defined as a generalized function. Especially, $\hat{\psi}(\boldsymbol{p})$ is equal to a L^2_{loc} -density or a Dirac measure $\delta_{\boldsymbol{p}}$.

The reason why we defined the Fourier transformation as in the form of the Law II′, (ii)′ is that we can derive the Schrödinger equation of the physical system by using the variational principle.

The constants are chosen so that the theoretical results of a certain physical system coincide with the observed data of the physical quantities for the natural statistical phenomena.

Law III′(motion of physical subsystem) We postulate that the L^2-density $\psi(\boldsymbol{r},\ t)$ which determines the law of natural probability distribution of the position variable at time t is determined by the time-evolving Schrödinger equation. We say that this time evolution is the motion of the physical subsystem. The law of motion of the physical subsystem is the Schrödinger equation. We say that this Schrödinger equation is the equation of motion of the physical subsystem.

The Schrödinger equation is described in the following form:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi.$$

We say that the operator H is a Schrödinger operator. H is a self-adjoint operator on a certain Hilbert space. The concrete form of this Schrödinger operator H is determined concretely for every concrete physical subsystem.

2.4 Laws of natural statistical physics for periodical motion

In this section, we establish the laws of natural statistical physics for a physical system which is moving periodically.

Law I(physical system) We postulate that the physical system Ω is a probability space $\Omega = \Omega(\mathcal{B},\ P)$. Here Ω is the set of the systems of micro-particles, $\mathcal{B}$ is the σ-algebra composed of subsets of Ω and P is the σ-additive probability measure. Every micro-particle ρ moves periodically on the interval $D = [-a,\ a]^n$ and its basic period is equal to $2a$ on the direction of each orthogonal axis. Here we assume $a > 0$.

Law II(state of physical system) We postulate that the state of the physical system $\Omega = \Omega(\mathcal{B},\ P)$ is the probability distribution of the position variable $\boldsymbol{r}(\rho)$ and the momentum variable $\boldsymbol{p}(\rho)$ of the system $\rho \in \Omega$ of micro-particles. Here $\boldsymbol{r}(\rho)$ varies periodically on the interval $D = [-a,\ a]^n$ in the n-dimensional space $\boldsymbol{R}^n$ and $\boldsymbol{p}(\rho)$ varies on its dual space $\boldsymbol{P}_n$.

Further we assume $n = Md$. Here d is the dimension of the physical space and M is the number of micro-particles composing an elementary event ρ.

(i) We postulate the natural probability distribution of the position variable $\boldsymbol{r} = \boldsymbol{r}(\rho)$ is ruled by the law of natural probability distribution which is determined by the L^2-density $\psi(\boldsymbol{r})$ defined on D. Here $\psi(\boldsymbol{r})$ satisfies the periodic boundary conditions:

$$\psi(\boldsymbol{r})|_{x_j=-a} = \psi(\boldsymbol{r})|_{x_j=a}, \ (\boldsymbol{r} \in D, \ j = 1, \ 2, \ \cdots, \ n).$$

(ii) We postulate that the natural probability distribution of the momentum variable $\boldsymbol{p} = \boldsymbol{p}(\rho)$ is ruled by the law of natural probability distribution which is determined by the Fourier type coefficients $\hat{\psi}(\boldsymbol{p})$ of $\psi(\boldsymbol{r})$.

Here the Fourier type coefficients $\hat{\psi}(\boldsymbol{p})$ of $\psi(\boldsymbol{r})$ and the Fourier type series of $\psi(\boldsymbol{r})$ are defined in the following:

$$\hat{\psi}(\boldsymbol{p}) = \frac{1}{(\sqrt{2a\hbar})^n} \int_D \psi(\boldsymbol{r}) e^{-i(\boldsymbol{p}, \boldsymbol{r})/\hbar} d\boldsymbol{r},$$

$$\psi(\boldsymbol{r}) = \frac{1}{(\sqrt{2a\hbar})^n} \sum_{\boldsymbol{p} \in \boldsymbol{P}_n} \hat{\psi}(\boldsymbol{p}) e^{i(\boldsymbol{p}, \boldsymbol{r})/\hbar},$$

$$\int_D |\psi(\boldsymbol{r})|^2 d\boldsymbol{r} = \sum_{\boldsymbol{p} \in \boldsymbol{P}_n} |\hat{\psi}(\boldsymbol{p})|^2 = 1,$$

$$\boldsymbol{r} = {}^t(x_1, \ x_2, \ \cdots, \ x_n), \ \boldsymbol{p} = {}^t(p_1, \ p_2, \ \cdots, \ p_n),$$

$$(\boldsymbol{p}, \ \boldsymbol{r}) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

Further we assume $\hbar = \dfrac{h}{2\pi}$. Here h denotes Planck's constant.

The reason why we define the Fourier type coefficients as in the form of the Law II, (ii) is that we can derive the Schrödinger equation of the physical system by using the variational principle.

The constants are chosen so that the theoretical results of a certain physical system coincide with the observed data of the physical quantities for the natural statistical phenomena.

Law III(motion of physical system) We postulate that the L^2-density $\psi(\boldsymbol{r}, \ t)$ which is ruled by the law of natural probability distribution of the position variable $\boldsymbol{r}$ at time t is determined by the time-evolving Schrödinger equation. We say that this time-evolution is the motion of the physical system. The law of motion of the physical system is described by the Schrödinger equation. We say that this Schrödinger equation is the equation of motion of the physical system. The Schrödinger equation has the form in the following:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

We say that the operator H is a Schrödinger operator. H is a self-adjoint operator on a certain Hilbert space. The concrete form of this Schrödinger operator is determined concretely for each concrete physical system.

Here $\psi(\boldsymbol{r},\ t)$ satisfies the following conditions (1) and (2):

(1) (Initial condition)

$$\psi(\boldsymbol{r},\ 0) = \psi(\boldsymbol{r}),\ (\boldsymbol{r} \in D).$$

(2) (Periodic boundary conditions)

$$\psi(\boldsymbol{r})|_{x_j=-a} = \psi(\boldsymbol{r})|_{x_j=a},\ \psi(\boldsymbol{r},\ t)|_{x_j=-a} = \psi(\boldsymbol{r},\ t)|_{x_j=a},$$

$$(\boldsymbol{r} \in D,\ 0 < t < \infty),\ (j = 1,\ 2,\ \cdots,\ n).$$

Here $\psi(\boldsymbol{r})$ is the given L^2-density.

When the Schrödinger operator H includes the potential $V = V(\boldsymbol{r})$, we assume that it satisfies the periodic boundary conditions:

$$V(\boldsymbol{r})|_{x_j=-a} = V(\boldsymbol{r})|_{x_j=a},\ (\boldsymbol{r} \in D,\ j = 1,\ 2,\ \cdots,\ n).$$

2.5 Laws of marginal distribution

In this section, we study the concept of the law of marginal distributions.

When we study the expectation values of the angular momentum of the system of the inner electron of a hydrogen atom, we need the concept of the law of marginal distributions.

At first, we consider the mathematical model for the system of hydrogen atoms. The system of hydrogen atoms is the set of hydrogen atoms, and the inner electron of each hydrogen atom is moving in the Coulomb potential

$$V(r) = -\frac{e^2}{r},\ (r = \|\boldsymbol{r}\|)$$

at the center of its nucleus.

Each electron moves by virtue of Newton's equation of motion on the basis of the law of causality. As the mathematical model, this physical system is the system of electrons moving in the Coulomb potential at the origin as the center.

We denote this system of electrons as $\Omega = \Omega(\mathcal{B},\ P)$. We assume that an elementary event ρ of Ω is an electron, $\mathcal{B}$ is the σ-algebra composed of subsets of Ω, P is the σ-additive probability measure. We consider that this system is

a system of one particle. Here an electron ρ has the position variable $\boldsymbol{r} = \boldsymbol{r}(\rho)$ and the momentum variable $\boldsymbol{p} = \boldsymbol{p}(\rho)$.

Here we postulate that the position variable $\boldsymbol{r} = \boldsymbol{r}(\rho)$ and the momentum variable $\boldsymbol{p} = \boldsymbol{p}(\rho)$ are the vector-valued random variables defined on Ω.

In this case, each electron has the total energy

$$E(\rho) = \frac{1}{2m_e}\boldsymbol{p}(\rho)^2 - \frac{e^2}{r},\ (r = \|\boldsymbol{r}\|).$$

Here m_e and e denote the mass and the electric charge of an electron respectively.

Now we calculate the expectation value of the angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = {}^t(L_x,\ L_y,\ L_z)$$

of the system of electrons.

By virtue of natural probability distribution of the position variable $\boldsymbol{r} = \boldsymbol{r}(\rho)$ is determined by the L^2-density ψ which is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi(\boldsymbol{r},\ \boldsymbol{t})}{\partial t} = (-\frac{\hbar^2}{2m_e}\Delta - \frac{e^2}{r})\psi(\boldsymbol{r},\ t)$$

of the system of hydrogen atoms. Here the partial derivative of the L^2-function ψ of $\boldsymbol{r}$ with respect to $x,\ y,\ z$ are defined in the sense of L^2-convergence. The variable t of $\psi(\boldsymbol{r},\ t)$ is considered to be a parameter. Therefore, the partial derivative of ψ with respect to t denotes the strong derivative of the function $\psi(\boldsymbol{r},\ t)$ with respect to the parameter t.

Here we remark that $\psi(\boldsymbol{r},\ t)$ is not assumed to be a L^2-function as the function of t.

Then the law of natural probability distribution of the momentum variable $\boldsymbol{p} = \boldsymbol{p}(\rho)$ is determined by the Fourier transform $\hat{\psi}$ of ψ.

Here we define the Fourier transformation $\hat{\psi}$ of ψ in the following:

$$\hat{\psi}(\boldsymbol{p}) = \frac{1}{(\sqrt{2\pi\hbar})^3}\int \psi(\boldsymbol{r})e^{-i(\boldsymbol{p},\ \boldsymbol{r})/\hbar}d\boldsymbol{r}.$$

Here we put

$$\boldsymbol{r} = {}^t(x,\ y,\ z),\ \boldsymbol{p} = {}^t(p_x,\ p_y,\ p_z),$$
$$(\boldsymbol{p},\ \boldsymbol{r}) = p_x x + p_y y + p_z z.$$

Here we omit the time variable t.

Now, we give the fundamental statistical formula of the law of natural probability distribution in the following:

$$P\left(\ \{\rho \in \Omega;\ \boldsymbol{r}(\rho) \in A\}\ \right) = \int_A |\psi(\boldsymbol{r})|^2 d\boldsymbol{r},$$

$$P\left(\ \{\rho \in \Omega;\ \boldsymbol{p}(\rho) \in B\}\ \right) = \int_B |\hat{\psi}(\boldsymbol{p})|^2 d\boldsymbol{p},$$

where A and B are two Lebesgue measurable sets of $\boldsymbol{R}^3$.

Further, we postulate that the law of natural probability distribution of the variable ${}^t(x(\rho),\ p_y(\rho))$ is determined by the partial Fourier transform $\hat{\psi}(x,\ p_y,\ z)$ as the marginal distribution of the simultaneous distribution of the variable ${}^t(x(\rho),\ p_y(\rho),\ z(\rho))$.

Here the partial Fourier transform $\hat{\psi}(x,\ p_y,\ z)$ of ψ is defined by the following relation:

$$\hat{\psi}(x,\ p_y,\ z) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,\ y,\ z) e^{-ip_y y/\hbar} dy.$$

The other marginal distributions are defined similarly.

Thereby, by using the law of natural probability distribution of the variable ${}^t(x(\rho),\ p_y(\rho))$ as the marginal distribution, we calculate the expectation value of the z-component $L_z = xp_y - yp_x$ of the angular momentum by the following relation

$$E[L_z] = \int_\Omega L_z(\rho) dP(\rho) = \frac{\hbar}{i} \int \overline{\psi(\boldsymbol{r})} (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\psi(\boldsymbol{r}) d\boldsymbol{r}.$$

In the third side of the above equality, we remark that the operator expression has nothing about the physical meaning.

These expressions are formal and used for the benefit of the mathematical calculations.

For the other angular momenta

$$L_x = yp_z - zp_y,\ L_y = zp_x - xp_z,\ L^2 = L_x^2 + L_y^2 + L_z^2,$$

we calculate these expectation values similarly.

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