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Definition and Existence Theorem of the Concept of Ordinal Numbers

By

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Abstract

In this paper, we give the new definition of the concept of ordinal numbers and prove its existence theorem on the basis of the ZFC set theory. This is the generalization of Peano's system of axioms of finite ordinal numbers.

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Introduction

In this paper, we give the new definition of the concept of ordinal numbers and prove its existence theorem. This is the main result obtained in Ito[4].

As for the related result, we refer to Matumura [6], Takeuti [7], Chapter 3, and Iwanami Dictionary of Mathematics, 4th edition edited by Mathematical Society of Japan.

As the result, we can prove the existence theorem of the concept of natural numbers by using the concept of ordinal numbers. As for this result, we refer to Ito [1] $\sim$ [4].

We assume that the set considered in this paper is the set defined in ZFC set theory. As for ZFC set theory, we refer to Ito [4]. As for the related results, we refer to Matumura [6], Takeuti [7], Chapter 3, and Iwanami Dictionary of Mathematics, 4th edition edited by Mathematical Society of Japan.

All mathematical concepts considered in this paper are those which are constructed by the sets of ZFC set theory. The concept of ordinal numbers and the concept of natural numbers are constructed by such a way.

By naming the each mathematical concept constructed by the set of ZFC set theory as, for example, the numbers 0, 1, 2, $\cdots$ or the function f, etc. , we can study the mathematics and express the mathematics by using the usual mathematical words.

Here we show my heartfelt appreciation to my wife Mutuko for her help of typesetting the TEX-file of this manuscript.

1 Definition of the concept of ordinal numbers and its existence theorem

In this section, we study the definition of the concept of ordinal numbers and its existence theorem. As for the related results, we refer to Ito [1] $\sim$ [4], Matumura [6], Takeuti [7], Chapter 3, and Iwanami Dictionary of Mathematics, 4th edition edited by Mathematical Society of Japan.

We assume that $A(a)$ denotes the formula of condition which holds for a set a. Then we denote the family of all x which satisfies $A(x)$ by $\{x;\ A(x)\}$. Especially, when $A(x)$ denotes the condition $x \notin x$, this family is $\{x;\ x \notin x\}$. $\{x;\ x \notin x\}$ is not a set by virtue of Russell's paradox. Therefore, generally, we say that $\{x;\ A(x)\}$ is a **class**.

Definition 1.1 Assume that a class A is $A = \{x;\ A(x)\}$. Then we define that A is a set if the condition

$$\exists a(a = A)$$

holds. Here a denotes a certain set.

Therefore, we say that A is a **proper class** if A is a class and A is not a set.

In the sequel, if a class is a set, we say simply that this class is a **set.** Against this, we say simply that a proper class is a **class**.

Then we have the following proposition.

Proposition 1.1 *We use the notation in Definition* 1.1. *Then the following* (1) $\sim$ (3) *are equivalent:*

(1) $a = A$.

(2) $\forall x(x \in a \longleftrightarrow x \in A)$.

(3) $\forall x(x \in A \longleftrightarrow A(x))$.

Here we give the definition of ordinal numbers.

Definition 1.2 We say that the class On whose elements are sets is the **class of ordinal numbers** if the following conditions (I) $\sim$ (IV) are satisfied:

(I) There exists the only one element ϕ in On which is said to be the empty set.

(II) If $\alpha \in \mathrm{On}$ is not the empty set ϕ, we have $\alpha = \{\beta;\ \beta < \alpha\}$ which satisfies one of the following conditions (i) and (ii):

(i) We have the relation

$$\alpha = \beta \cup \{\beta\}$$

if α contains the maximum β.

(ii) We have the relation

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta<\alpha} \{\beta\}$$

if α does not contain the maximum. Here the symbol $\beta < \alpha$ means that the condition $\beta \in \alpha$ holds for two sets α, β in On.

(III) We have the condition

$$\forall\alpha(\forall\beta < \alpha\, \beta \in \mathrm{On} \to \alpha \in \mathrm{On}) \to \forall\alpha \in \mathrm{On}.$$

(IV) On is the smallest class which satisfies the conditions (I) $\sim$ (III).

We define that an element of On is an **ordinal number**.

We say that the condition (IV) in Definition 1.2 is the **axiom of transcendental induction**.

Here we say that the ordinal number α is an **isolated ordinal number** if α contains the maximum β and we have the relation

$$\alpha = \beta \cup \{\beta\}.$$

Further we say that the ordinal number α is a **limit ordinal number** if α does not contain the maximum and we have the relation

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta<\alpha} \{\beta\}.$$

If the ordinal number α is a limit ordinal number, we denote it as $\lim(\alpha)$.

Then, On is the class of all ordinal numbers but not a set.

Here we formulate the system of Peano's axioms which gives the definition of finite ordinal numbers in the following.

Definition 1.3(System of Peano's axioms) We define that the set $\boldsymbol{N}$ whose elements are all sets is the **set of all finite ordinal numbers** if it satisfies the following conditions (i) $\sim$ (iii):

(i) We have $\phi \in \boldsymbol{N}$.

(ii) If $\alpha \in \boldsymbol{N}$ holds, we have $\alpha' = \alpha \cup \{\alpha\} \in \boldsymbol{N}$.

(iii) $\boldsymbol{N}$ is the maximum set which satisfies the conditions (i) and (ii). **(Axiom of mathematical induction).**

It is known that the axiom (IV) of ordinal numbers in Definition 1.2 is the new axiom which is obtained by changing the axiom of mathematical induction into the axiom of transcendental induction. By using the axiom of transcendental induction, we can prove that the proposition $A(\alpha)$ concerning an ordinal number α holds for an arbitrary ordinal number $\alpha \in \mathrm{On}$. Thus, we say that the method of proof by using the axiom of transcendental induction is the **principle of transcendental induction**. As the formula of transcendental induction usually used, we have the following theorem.

Theorem 1.1(Transcendental induction) *We use the notation in the above. Then, we have the relation*

$$\forall\alpha(\forall\beta < \alpha\, A(\beta) \to A(\alpha)) \to A(\alpha)) \to \forall\alpha\, A(\alpha).$$

Here we prove the existence of the model of the ZFC set theory by using the principle of transcendental induction.

Now we define a class $P(a)$ by using the relation

$$P(a) = \{x;\ x \subset a\}.$$

Then we study the definition of a function $R(\alpha)$ by virtue of the transcendental induction.

We define the function $R(\alpha)$ by using the transcendental induction in the following:

(A) We define $R(0) = 0$.

(B) Assume that $R(\beta)$ is defined for an arbitrary $\beta < \alpha$ when the ordinary number α is constructed. Then we define $R(\alpha)$ as in the following (B.1) and (B.2):

(B.1) If α is not a limit ordinal number, there exists β such that $\alpha = \beta \cup \{\beta\} = \beta + 1$ holds. Then we define

$$R(\alpha) = P(R(\beta)).$$

(B.2) When α is a limit ordinal number, we define

$$R(\alpha) = \bigcup_{\beta<\alpha} R(\beta).$$

(C) According to the construction of the ordinal numbers, we continue the process of definition (B) without limit.

In this definition, when we construct an ordinal number successfully by starting from 0, we define the value of $R(\alpha)$ according to the processes (A), (B), (C) on the stage of construction of the ordinal number α. Thereby, we complete the definition of the function $R(\alpha)$ with the completion of the construction of the ordinal number α.

Therefore it means that we have the unique definition of the function $R(\alpha)$ for all ordinal numbers.

For the function $R(\alpha)$ defined in the above, we have the following theorem $1.2 \sim 1.4$.

Theorem 1.2 *We have the following three relations*:

(1) $R(0) = 0$.

(2) $R(\alpha + 1) = R(\alpha) \bigcup P(R(\alpha))$.

(3) *If α is a limit ordinal number, we have*

$$R(\alpha) = \bigcup_{\beta<\alpha} R(\beta).$$

Theorem 1.3 *We have the following three properties*:

(1) $\alpha < \beta \to R(\alpha) \subset R(\beta) \wedge R(\alpha) \in R(\beta)$.

(2) *The set $R(\alpha)$ is transitive. Namely, we have the condition*

$$\forall x,\ y(x \in y \wedge y \in R(\alpha) \to x \in R(\alpha)).$$

(3) $a \in R(a),\ b \subset a \to b \in R(a)$.

Theorem 1.4 *We have the following two properties:*

(1) $R(\alpha+1) = P(R(\alpha))$.

(2) *We have* $\forall x \exists \alpha (x \in R(\alpha))$.

Namely, we have

$$V = \bigcup_{\alpha \in \mathrm{On}} R(\alpha).$$

Especially, when, starting from the assumption $R(0) = \phi$, we define the function $R(\alpha)$ by virtue of the processes (A), (B), (C), the class V constructed in Theorem 1.4 is the model of the ZFC set theory. Therefore, we have the following theorem.

Theorem 1.5 *There exists a model of the* ZFC *set theory.*

Next we study the **definitions of sum, product and power of ordinal numbers**.

For two ordinal numbers α and β, we define the sum $\alpha+\beta$, the product $\alpha\beta$ and the power α^β of the ordinal numbers in the following by using the principle of transcendental induction.

Definition 1.4(definition of sum) We define the sum $\alpha+\beta$ of the ordinal numbers α and β such that the following conditions (i) $\sim$ (iii) are satisfied:

(i) $\alpha + 0 = \alpha$.

(ii) $\alpha + \beta' = (\alpha+\beta)'$.

(iii) $\alpha + \gamma = \sup\{\alpha + \xi;\ \xi < \gamma\}$.

Here we assume that $\beta' = \beta + 1$ and γ is *a* limit ordinal number.

Definition 1.5(definition of product) We define the product $\alpha\beta = \alpha \cdot \beta$ of the ordinal numbers α and β such that the following conditions (i) $\sim$ (iii) are satisfied:

(i) $\alpha \cdot 0 = 0$.

(ii) $\alpha \cdot \beta' = \alpha \cdot \beta + \alpha$.

(iii) $\alpha \cdot \gamma = \sup\{\alpha \cdot \xi;\ \xi < \gamma\}$.

Here we assume that $\beta' = \beta + 1$ and γ is *a* limit ordinal number.

Definition 1.6 (definition of power) For two ordinal numbers α and β where $\alpha > 0$, we define the power α^β such that the following conditions (i) $\sim$ (iii) are satisfied:

(i) $\alpha^0 = 1$.

(ii) $\alpha^{\beta'} = \alpha^\beta \cdot \alpha$.

(iii) $\alpha^\gamma = \sup\{\alpha^\xi;\ \xi < \gamma\}$.

Here we assume that $\beta' = \beta + 1$ and γ is *a* limit ordinal number.

Then we have the laws of calculations of sum, product and power of ordinal numbers in the following.

Theorem 1.6 *Assume that α, β, γ are the ordinal numbers. Then we have the laws of calculations* (1) $\sim$ (5) *of sum, product and power of ordinal numbers in the following*:

(1) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. (*Associative law*).

(2) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$. (*Associative law*).

(3) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. (*Left distributive law*).

(4) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

(5) $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$.

Remark 1.1 In general, the commutative laws $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$, the right associative law $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$ do not hold. Further, the relation $(\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$ for the power does not holds in general.

Then we have the following theorem concerning the order relation of ordinal numbers.

Theorem 1.7 *For the order relation of ordinal numbers, we have the following* (1) $\sim$ (7):

(1) *For every two ordinal numbers α and β, only one of the relations $\alpha < \beta$, $\alpha = \beta$ or $\beta < \gamma$ holds.*

(2) *If $\alpha < \beta$, $\beta < \gamma$ hold, we have $\alpha < \gamma$.* [*Transitive law*].

(3) *If $\alpha < \beta$ holds, we have $\alpha + \gamma < \beta + \gamma$.*

(4) *If* $\alpha < \beta$, $\gamma > 0$ *holds, we have* $\gamma \cdot \alpha < \gamma \cdot \beta$.

(5) *For an arbitrary ordinal number* α, *we have* $\alpha \geq 0$.

(6) *If* $\alpha < \beta$ *holds, we have* $\alpha + 1 \leq \beta$.

(7) *We have* $a \subset \mathrm{On} \to \exists\alpha(a < \alpha)$.

By virtue of Theorem 1.7, (6), we see that $\alpha + 1$ is the next ordinal number of α.

Further, by virtue of Theorem 1.7, (7), we see that there exists a large ordinal number without limit.

Theorem 1.8 *We have the relation*

$$0 \neq A \subset \mathrm{On} \to \exists\alpha \in A \forall\beta \in A(\alpha \leq \beta).$$

We often say that Theorem 1.8 is the **principle of transcendental induction**.

Definition 1.7 If $0 \neq A \subset \mathrm{On}$, there exists the minimum of A. We denote this element as μA. If $0 \neq A \subset \mathrm{On}$ does not hold, we define $\mu A = 0$.

In fact, by virtue of Theorem 1.7, it is proved that there exists the minimum of A if $0 \neq A \subset \mathrm{On}$ holds.

Therefore the minimum of A is uniquely determined if $0 \neq A \subset \mathrm{On}$ holds.

When A is $\{\alpha;\ A(\alpha)\}$, we denote μA as $\mu\alpha A(\alpha)$. We read this as "the minimum α which satisfies $A(\alpha)$".

Therefore we have the following (1) and (2):

(1) $\exists\alpha A(\alpha) \to A(\mu\alpha A(\alpha))$.

(2) $A(\beta) \to \mu\alpha A(\alpha) \leq \beta$.

Further, if $\neg\exists\alpha A(\alpha)$ holds, we define $\mu\alpha A = 0$.

Here the symbol $\cup a$ denotes the minimum upper bound or the **supremum of** a. If there exists the maximum of a, $\cup a$ denotes the maximum. If there does not exist the maximum of a, $\cup a$ denotes the **limit** of a. We denote $\cup a$ as

$$\sup a \quad \text{or} \quad \sup_{\alpha \in a} \alpha.$$

Theorem 1.9 *We have the relation*

$$a \subset \mathrm{On} \to \cup a \in \mathrm{On}.$$

Theorem 1.10 *We have the relation*

$$a \subset \text{On},\ \alpha \in a \to \alpha \leq \cup a.$$

Theorem 1.11 *We have the relation*

$$a \subset \text{On},\ \forall\beta \in a(\beta \leq \alpha) \to \cup a \leq \alpha.$$

We denote the smallest limit ordinal number in On as ω.
Then the following two conditions are equivalent:

(1) $\alpha \in$ On is a **finite ordinal number**.

(2) We have the condition $\alpha < \omega$.

Theorem 1.12 *The ordinal number ω in the above is equal to the set $\omega = \{0,\ 1,\ 2,\ \cdots\}$ of all finite ordinal numbers.*

Next, on the basis of the principle of transcendental induction, we study the construction of ordinal numbers. Namely we study the **construction of the model of ordinal numbers**.

Theorem 1.13(Existence of the model of ordinal numbers) *We construct the class* On *of sets such that it satisfies the following conditions* (1) $\sim$ (3):

(1) *When we construct the empty set ϕ, we define that this empty set is* $0 \in$ On.

(2) *If $\alpha \in$ On is not* 0, *we construct the set $\{\beta;\ \beta < \alpha\}$ of all ordinal numbers $\beta < \alpha$. Then we have the following conditions* (i) *and* (ii):

(i) *When there exists the maximum β of the set $\{\beta;\ \beta < \alpha\}$, we have*

$$\alpha = \beta + 1 = \beta \cup \{\beta\}.$$

(ii) *When there does not exist the maximum of the set $\{\beta,\ \beta < \alpha\}$, α is a limit ordinal number and we have*

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta<\alpha} \{\beta\}.$$

(3) On *is the smallest class which satisfies the conditions* (1) *and* (2).

Then On is the class of all ordinal numbers. Therefore an element of On is an ordinal number.

The class On constructed in the above satisfies the system of axioms in Definition 1.2. Hence we can construct the model of ordinal numbers by using the principle of transcendental induction. Thereby we can prove the existence of the model of ordinal numbers.

Hence we have the existence theorem of ordinal numbers.

Theorem 1.14 *We use the notation in the above. There exists the model of the class of all ordinal numbers. Therefore there exists the class* On *of all ordinal numbers.*

We can show concretely the beginning part of the construction of the model of ordinal numbers in the above.

At first, we construct the empty set, we define that it is 0. Next, because the set constructed until now is only 0, we construct the set $\{0\}$ which is composed of only 0. Then we define that it is 1.

Then, because the set constructed until now is only 0 and 1, we construct the set $\{0, 1\}$ which is composed of only 0 and 1. Then we define that it is 2.

By repeating this construction, we construct all finite ordinal numbers $0, 1, 2, \cdots$. Further we continue to construct ordinal numbers as possible as we can do by using the principle (C). Therefore, when we construct the set $\{0, 1, 2, \cdots\}$ of all finite ordinal numbers, we define that it is ω.

Next, by constructing the set $\{0, 1, 2, \cdots, \omega\}$, we define that it is $\omega + 1$. Further, by constructing the set $\{0, 1, 2, \cdots, \omega, \omega + 1\}$, we define that it is $\omega + 2$. Thus, by continuing the construction of ordinal numbers, we construct the ordinal numbers

$$\omega + \omega, \cdots, \omega^2, \cdots, \omega^\omega, \cdots, \omega^{\omega^{\cdot^{\cdot^{\cdot}}}}, \cdots$$

as possible as we can do.

Thus we can construct the class On of all ordinal numbers.

The beginning part of ordinal numbers is

$$\begin{array}{llllll} 0, & \{0\}, & \{0, 1\}, & \cdots, \{0, 1, 2, \cdots\}, & \{0, 1, 2, \cdots, \omega\}, & \cdots, \\ 0, & 1, & 2, & \cdots, \quad \omega, & \omega + 1, & \cdots. \end{array}$$

At last, we prove that the set $\boldsymbol{N}$ of all finite ordinal numbers satisfies the system of Peano's axioms. Namely we have the following theorem.

Theorem 1.15 *We denote that the set* $\boldsymbol{N} = \{\alpha;\ \alpha < \omega\}$ *of all finite ordinal numbers is* $\boldsymbol{N} = \{0, 1, 2, \cdots\}$.

Then we have the following (1) $\sim$ (3):

(1) *We have* $0 \in \boldsymbol{N}$.

(2) *If* $a \in \boldsymbol{N}$ *holds, we define* $a + 1 = a \cup \{a\}$. *Then we have* $a + 1 \in \boldsymbol{N}$.

(3) $\boldsymbol{N}$ *is the maximum set which satisfies the conditions* (1) *and* (2).

Therefore the set $\boldsymbol{N} = \{0,\ 1,\ 2,\ \cdots\}$ of all finite ordinal numbers satisfies the system of Peano's axioms gives the definition of finite ordinal numbers. By using the set $\boldsymbol{N}$ of all finite ordinal numbers, we can construct the model of natural numbers. Then we can use the definitions of summation and multiplication and the order relation defined for ordinal numbers as those of natural numbers. As for this fact, we refer to Ito[1] $\sim$ [4].

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