J. Math. Univ. Tokushima
Vol. 48 (2014), 17–22

On Weakly Exchange Rings

PETER V. DANCHEV
Department of Mathematics, University of Plovdiv
4000 Plovdiv, Bulgaria
e-mails: pvdanchev@yahoo.com, peter.danchev@yahoo.com

(Received June 13, 2014. Revised August 27 and October 14, 2014)

Abstract

We show that an associate unital ring R is weakly exchange (respectively, weakly clean) if $R/J(R)$ is weakly exchange (respectively, weakly clean) and idempotents in R lift modulo $J(R)$. If, in addition, 2 belongs to $J(R)$, then the converse holds too. In particular, if 2 lies in $J(R)$, then any weakly exchange ring is exchange as well as any weakly clean ring is clean. These facts somewhat strengthen some classical results due to Nicholson (Trans. Amer. Math. Soc., 1977).

2010 Mathematics Subject Classification: 16S34.

1 Introduction and Backgrounds

Throughout this note, let R be an associative unital ring with unit group $U(R)$, with Jacobson radical $J(R)$, and with set of idempotents $Id(R)$. All other notations are standard as well as the terminology is classical. For instance, a ring R is said to be *abelian* if all its idempotents are central. Moreover, the most important concepts used in the sequel are recollected below.

In [6] the following fundamental notion was defined.

Definition 1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

Likewise, in [6] it was pointed out that R is clean if, and only if, $R/J(R)$ is clean and all idempotents lift modulo $J(R)$.

The "clean" concept was generalized there to the following one:

Definition 2. A ring R is said to be *exchange* if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

It was obtained in [6] that R is an exchange ring if, and only if, $R/J(R)$ is an exchange ring and all idempotents lift modulo $J(R)$. Also, it was established there that Definitions 1 and 2 are tantamount for abelian rings. However, there is an exchange ring that is not clean.

On the other hand, it was introduced in [1] the notion of *weakly clean* rings but only in a commutative version as follows:

[0]Key words: weakly exchange rings, weakly clean rings, exchange rings, clean rings.

Definition 3. A ring R is called *weakly clean* if each $r \in R$ can be expressed as either $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in Id(R)$.

Certainly, the last definition can be stated in general for arbitrary rings. Besides, it is evident that all clean rings are weakly clean, whereas the converse does not hold even in the commutative aspect (see, e.g., [1]). However, every weakly clean ring of characteristic 2 is clean, and vice versa. One of our motivation in the present paper is to improve this observation by requiring that 2 lies in $J(R)$ that supersedes the condition $2 = 0$.

The following appears in [7].

Definition 4. A ring R is said to be *weakly exchange* if, for any $x \in R$, there exists $e \in Id(R)$ such that $e \in xR$ and either $1-e \in (1-x)R$ or $1-e \in (1+x)R$.

It was established in [3] that the notions of being weakly exchange and weakly clean do coincide for abelian rings, thus extending the aforementioned result due to Nicholson from [6] (see also [7]).

Apparently, all exchange rings are weakly exchange, while the converse does not hold even in the commutative variant. However, every weakly exchange ring of characteristic 2 is exchange, and vice versa. One of our aims in the current article is to enlarge this observation by requiring that 2 lies in $J(R)$ which is weaker than the condition $2 = 0$.

2 Main Results

The following technicality, which extends Proposition 1.1 from [6], was stated in [7] without a proof and with two identical misprints in points (3) and (4), which are presently points (c) and (d), respectively. We however formulate it correctly and provide a transparent proof for the sake of completeness and for the convenience of the reader.

Lemma 2.1 *Let R be a ring. Then the following points are equivalent:*

(a) R is weakly exchange;

(b) For any $x \in R$ there exists $e \in Id(R)$ such that $e - x \in (x - x^2)R$ or $e + x \in (x + x^2)R$;

(c) For any $x \in R$ there exist $e \in Id(R) \cap xR$ and $c \in R$ such that $1 - e - (1-x)c \in J(R)$ or $1 - e - (1+x)c \in J(R)$;

(d) For any $x \in R$ there exists $e \in Id(R) \cap xR$ such that $eR + (1-x)R = R$ or $eR + (1+x)R = R$.

Proof. (a) $\Rightarrow$ (b). Letting for any $x \in R$ there is $e \in Id(R)$ such that $e \in xR$ and either $1-e \in (1-x)R$ or $1-e \in (1+x)R$. Furthermore, for some $r, a \in R$, one sees that $e - x = (1-x)e - x(1-e) = (1-x)xr - x(1-x)a = (x-x^2)r - (x-x^2)a \in (x-x^2)R$. Moreover, $e + x = (1+x)e + x(1-e) = (1+x)xr + x(1+x)a = (x+x^2)r + (x+x^2)a \in (x+x^2)R$, as stated.

(b) $\Rightarrow$ (c). If $e - x = (x-x^2)r$ for some $r \in R$, then $e = x(1+r-xr) \in xR$ and $1 - e = (1-x)(1-xr)$ where we may take $c = 1 - xr$. Analogously, if $e + x = (x+x^2)a$ for some $a \in R$, then $1 - e = (1+x)(1-xa)$ where we may again choose $c = 1 - xa$.

(c) $\Rightarrow$ (d). Observe that either $u = e + (1-x)c$ or $v = e + (1+x)c$ is a unit. Consequently, $1 = eu^{-1} + (1-x)cu^{-1}$ or $1 = eu^{-1} + (1+x)cu^{-1}$ and hence, for any $r \in R$, we deduce that $r = eu^{-1}r + (1-x)cu^{-1}r \in eR + (1-x)R$ or $r = eu^{-1}r + (1+x)cu^{-1}r \in eR + (1+x)R$, as required.

(d) $\Rightarrow$ (a). Writing $1 = et + (1-x)s$, we define $f = e + et(1-e)$. It is not too hard to check that $f^2 = f \in xR$ and $1 - f = (1-e) - et(1-e) = (1-et)(1-e) = (1-x)s(1-e) \in (1-x)R$.

By symmetry, writing $1 = et + (1+x)s$, we as above set $f = e + et(1-e)$. So, $f \in Id(R) \cap xR$ and $1 - f = (1-et)(1-e) = (1+x)s(1-e) \in (1+x)R$, as expected. ∎

Recall that the idempotents of a ring R can be *lifted modulo the ideal* L if, given $x \in R$ with $x - x^2 \in L$, there exists $e \in Id(R)$ such that $e - x \in L$. Replacing x by $-x$ this condition is equivalent to the following: if $x + x^2 \in L$, there exists $e \in Id(R)$ such that $e + x \in L$. Especially, we take into account that $x = -(-x) = -y$ and so $x - x^2 \in L \iff y + y^2 \in L$.

So, we come to our first basic result.

Theorem 2.2 *A ring R is weakly exchange if $R/J(R)$ is weakly exchange and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

Proof. Given an arbitrary $x \in R$, we have $x + J(R) \in R/J(R) = \overline{R}$ and thus $\overline{1} - \overline{a} = (\overline{1} - \overline{x})\overline{c}$ or $\overline{1} - \overline{a} = (\overline{1} + \overline{x})\overline{d}$ for some $\overline{a}^2 = \overline{a} \in \overline{x}\overline{R}$ and $\overline{c}, \overline{d} \in \overline{R}$. Writing $\overline{a} = \overline{xr}$, we derive $a = xr + j = a' + j$ where $a' \in xR$ and $j \in J(R)$. Furthermore, either $1 - a' \in (1-x)c + J(R)$ or $1 - a' \in (1+x)d + J(R)$ for some $c, d \in R$. But $a^2 - a \in J(R)$, hence by assumption there is $f \in Id(R)$ such that $a - f \in J(R)$, so that $a' - f \in J(R)$ and $u = 1 - f + a' \in U(R)$. Writing $a' = f + j_1$ with $j_1 \in J(R)$, one sees that $a^2 - a \in J(R)$ is tantamount to $a'^2 - a' \in J(R)$. Define $e = ufu^{-1}$. It is clear that $e = a'fu^{-1} \in Id(R) \cap xR$. We therefore obtain that either $1 - e - (1-x)c \in J(R)$ or $1 - e - (1+x)d \in J(R)$. In fact, what we need to prove is that $e - a' \in J(R)$. To show this, consider $e - a' = a'fu^{-1} - a' = a'(fu^{-1} - 1) = a'(f-u)u^{-1} = a'(2f - 1 - a')u^{-1} = (f + j_1)(2f - 1 - f - j_1)u^{-1} = (f + j_1)(f - j_1 - 1)u^{-1} = [(f + j_1)(f - j_1) - f - j_1]u^{-1} = [f^2 - j_1^2 - f - j_1]u^{-1} = [-j_1^2 - j_1]u^{-1} \in J(R)$, as wanted. Now Lemma 2.1 (c) applies to deduce the desired claim that R is weakly exchange.

For the second part-half, we observe that a homomorphic image of a weakly exchange ring is also a weakly exchange ring. Next, as for the lifting property, we observe that $x - x^2 = (x + x^2) - 2x^2$, so that $x - x^2 \in J(R)$ is equivalent to $x + x^2 \in J(R)$, provided $2 \in J(R)$. By virtue of Lemma 2.1 (b), there is $e \in J(R)$ such that $e - x \in (x - x^2)R$ or $e + x \in (x + x^2)R$. In the first case, it follows at once that $e - x \in J(R)$. In the second one, it also follows immediately that $e + x \in J(R)$, as required. ∎

Similarly, one can derive the following assertion. Before doing that, we need the following technicality.

Lemma 2.3 *For any ring R the following equality holds:*

$$U(R) + J(R) = U(R).$$

Proof. It is self-evident that the left hand-side contains the right one. To treat the converse, given $x \in J(R) + U(R)$, we may write $x = a + u$ where $a \in J(R)$ and $u \in U(R)$. But it is well known that $J(R) = \{x \in R \mid 1 - rxs \in U(R), \forall r, s \in R\}$. Hence $a + u = u(1 + u^{-1}a) \in U(R)$ taking $r = -u^{-1}$ and $s = 1$, as required. ■

We are now able to proceed by proving the following:

Theorem 2.4 *A ring R is weakly clean if $R/J(R)$ is weakly clean and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

Proof. Choosing an arbitrary $x \in R$, we have $x + J(R) \in R/J(R)$ and so write either $x + J(R) = (u + J(R)) + (e + J(R)) = u + e + J(R)$ or $x + J(R) = (u + J(R)) - (e + J(R)) = u + J(R) - e + J(R) = u - e + J(R)$, where $u + J(R)$ is a unit in $R/J(R)$ and $e + J(R)$ is an idempotent in $R/J(R)$. Consequently, $1 - uv \in J(R)$ and $1 - vu \in J(R)$ for some $v \in R$ as well as $e - e^2 \in J(R)$. Since $1 - (1 - uv) = uv \in U(R)$ and $1 - (1 - vu) = vu \in U(R)$, we deduce that $u \in U(R)$. Moreover, $e - f \in J(R)$ for some $f \in Id(R)$. Therefore, $x - (u + f) \in J(R)$ and $x - (u - f) \in J(R)$. We next refer to Lemma 2.3 to infer that $x = w + f$ or $x = w - f$ where $w \in U(R)$, as required.

As for the second part-half, it is obvious that a homomorphic image of a weakly clean ring is again a weakly clean ring. Since weakly clean rings are necessarily weakly exchange, we just apply Theorem 2.2. ■

So, we are now ready to extend one of our basic results in [4], proved for abelian rings, to the following statement:

Proposition 2.5 *Suppose that R is a ring with $2 \in J(R)$. Then R is weakly exchange if, and only if, R is exchange.*

Proof. One direction being trivial, we consider the other one. So, letting R be weakly exchange, we employ Theorem 2.2 to get that $R/J(R)$ is weakly exchange and all idempotents of R are lifted modulo $J(R)$. Since 2 lies in $J(R)$, the factor-ring $R/J(R)$ is exchange possessing characteristic 2. We further appeal to [6] and conclude that R is exchange, as required. ■

Same type result appears for weakly clean rings. Specifically, the following is valid:

Proposition 2.6 *Suppose that R is a ring with $2 \in J(R)$. Then R is weakly clean if, and only if, R is clean.*

Proof. One direction being elementary, we consider the other one. So, given R is weakly clean, we apply Theorem 2.4 to obtain that $R/J(R)$ is weakly clean and all idempotents of R are lifted modulo $J(R)$. Since 2 is in $J(R)$, the quotient ring $R/J(R)$ is clean having characteristic 2. We furthermore appeal to [6] and infer that R is clean, as expected. ■

It is worthwhile noticing that the above results strengthen these from [4], provided that the ring is abelian.

3 Open Problems

In closing we pose the following challenging questions:

Problem 1. Are Theorems 2.2 and/or 2.4 fulfilled without the assumption that $2 \in J(R)$?

Problem 2. If R is a (commutative) weakly exchange ring or, in particular, a (commutative) weakly clean ring, and $2 \in U(R)$, is R a 2-good ring, that is, each element is the sum of two units?

Problem 3. Characterize weakly exchange rings with prime Artinian factors.

Notice that exchange rings with prime Artinian factors are strongly π-regular – for more details the interested reader may see [8].

Problem 4. If R is a weakly exchange ring with primitive Artinian factors, is then R weakly clean?

Problem 5. Does it follow that weakly exchange rings with bounded index of nilpotency are weakly clean?

It is well known in the literature that a ring R is called *uniquely (weakly) clean* if in Definitions 1 and 3 the existing idempotent is unique. While uniquely clean rings are well-studied (see, for instance, [2]), the same cannot be said of uniquely weakly clean rings. So, the statement of the following has some motivation.

Problem 6. Characterize uniquely weakly clean rings. Are uniquely weakly clean rings R precisely the abelian weakly clean rings R such that for all maximal ideals M of R we have either $R/M \cong \mathbb{Z}_3$ or $R/M \cong B$, where B is Boolean?

We shall call a ring R *uniquely (weakly) exchange* if in Definitions 2 and 4 the existing idempotent is unique. So, we come to the following:

Problem 6. Characterize uniquely (weakly) exchange rings. Are uniquely (weakly) exchange rings exactly the abelian (weakly) exchange rings with some additional conditions on maximal ideals?

References

[1] M-S. Ahn and D. D. Anderson, *Weakly clean rings and almost clean rings,* Rocky Mount. J. Math. **36** (2006), 783–798.

[2] H. Chen, *On uniquely clean rings,* Commun. Algebra **39** (2011), 189–198.

[3] A. Y. M. Chin and K. T. Qua, *A note on weakly clean rings,* Acta Math. Hungar. **132** (2011), 113–116.

[4] P. V. Danchev, *When is an abelian weakly clean ring clean?,* to appear.

[5] C. Y. Hong, N. K. Kim and Y. Lee, *Exchange rings and their extensions,* J. Pure Appl. Algebra **179** (2003), 117–126.

[6] W. K. Nicholson, *Lifting idempotents and exchange rings,* Trans. Amer. Math. Soc. **229** (1977), 269–278.

[7] J.-C. Wei, *Weakly-abel rings and weakly exchange rings,* Acta Math. Hungar. **137** (2012), 254–262.

[8] H.-P. Yu, *On the structure of exchange rings,* Commun. Algebra **25** (1997), 661–670.