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Modified Farey Trees and Pythagorean Triples

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Abstract

In 1963, F. J. M. Barning discovered a ternary tree of primitive Pythagorean triples, where each triple is transformed to other three triples by three distinct 3×3 unimodular matrices. This fact has been rediscovered many times. In this paper, we shall give an elementary explanation of this fact using classical Euclidean parametrization of primitive Pythagorean triples and modified ternary Farey trees.

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Introduction

In his paper [2], Barning found the following interesting parametrization of the primitive Pythagorean triple. The primitive Pythagorean triple is the set of positive integers (a, b, c) which satisfy

$$a^2 + b^2 = c^2, \text{ with } (a, b) = 1.$$

From the condition $(a, b) = 1$, a and b must satisfy $a \not\equiv b \mod 2$. Therefore, without loss of generality, we may assume any primitive Pythagorean triple (a, b, c) with a odd and b even in the following. Barning gave the following 3×3 unimodular matrices

$$M_1 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \ M_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \ M_3 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix},$$

where Barning's original M_1 is the above M_3 and Barning's M_3 is the above M_1.

Proposition 1 (Barning [2]). *Any primitive Pythagorean triple (a, b, c) has the unique representation as the matrix product*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_{\sigma(1)} M_{\sigma(2)} \dots M_{\sigma(r)} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix},$$

for some $r \geq 0$, $(\sigma(1), \sigma(2), \dots, \sigma(r)) \in (1, 2, 3)^r$.

It is well known that Euclid has described a parametrization of primitive Pythagorean triples in his book *Elements* as follows.

Proposition 2 (Theorem 225 of [4]). *Any primitive Pythagorean triple $a^2 + b^2 = c^2$, with $2|b$ can be uniquely represented by*

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2, \text{ with } (m, n) = 1 \text{ and } m > n > 0.$$

Moreover m and n must satisfy the condition $m \not\equiv n \mod 2$.

From these propositions, there exists the following well known bijection.

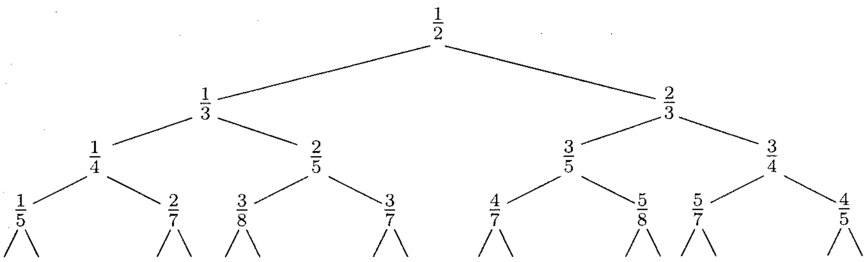
Reduced fractions $\frac{n}{m}$ with $m > n > 0$ and $m \not\equiv n \mod 2$	$\longleftrightarrow$	Primitive Pythagorean triple (a, b, c) with $2\|b$

1 Modified Farey trees

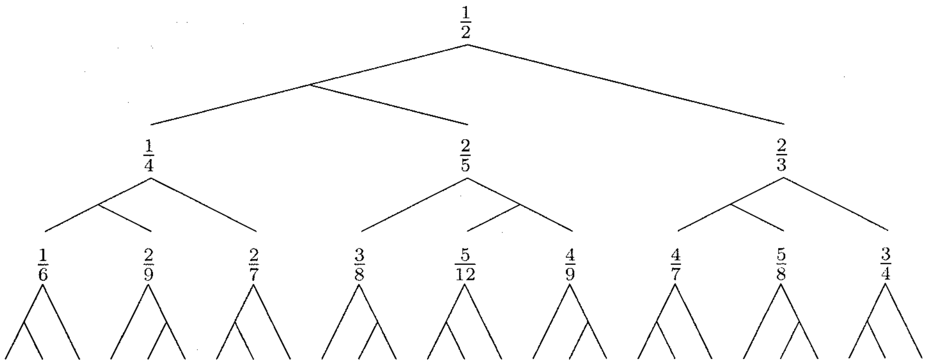
The *Farey series* of order N, denoted by $\mathcal{F}_N$ is the set of all reduced fractions between 0 and 1 whose denominators are N or less, and arranges in increasing order. For example, if $N = 5$, we have

$$\mathcal{F}_5 = \{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\}.$$

The following tree is the usual Farey tree consisting of Farey series $\mathcal{F}_N$ $(N \geq 2)$.

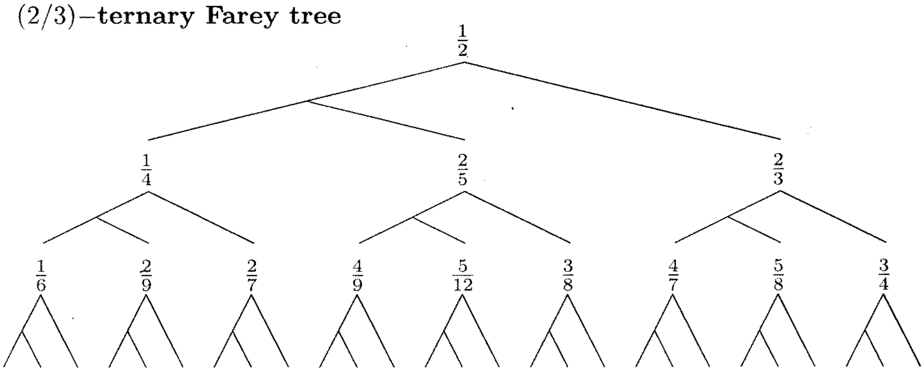


Skipping the reduced fractions $\frac{n}{m}$ with m, n odd in the above Farey tree, we shall obtain the following modified Farey tree of reduced fractions $\frac{n}{m}$, where m, n are of odd parity and $m > n > 0$.



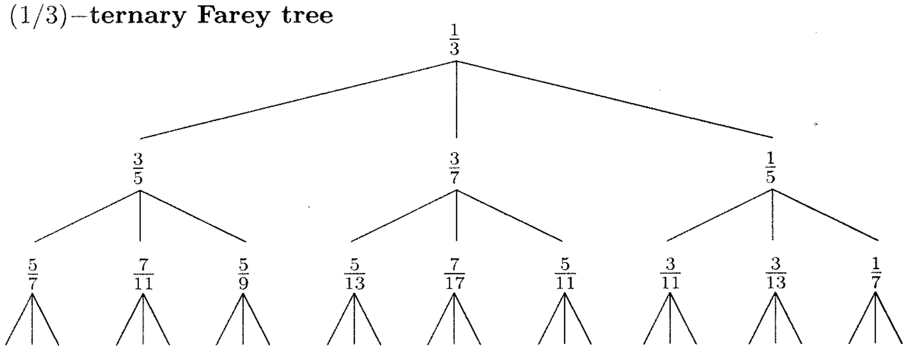
Here the branch between $\frac{1}{2}$ and $\frac{1}{4}$ in this tree means the skipped fraction $\frac{1}{3}$, other branches also represent the skipped fractions $\frac{n}{m}$ with $m \equiv n \equiv 1 \mod 2$. Now, we shall call the fraction $\frac{1}{2}$ *the fraction of level* 1 and the fractions $\frac{1}{4}, \frac{2}{5}, \frac{2}{3}$ *the fractions of level* 2 and so on. Thus, for any $n \geq 1$, there exist 3^{n-1} fractions of level n, 3^n fractions of level $n+1$ and 3^{n-1} branches which correspond to the skipped fractions between level n and level $n+1$. For each $n \geq 1$, we shall replace the fractions of level n each other so as the branches to be placed at the left hand side of the lines between the fractions of level $n-1$ and level n. For example, in the level $n = 3$, $\frac{3}{8}$ is changed place with $\frac{4}{9}$ in the following tree. Here the branch which corresponds to the skipped fraction $\frac{3}{7}$ is placed at the left hand side of the line between the fraction $\frac{2}{5}$ of level 2 and the fraction $\frac{4}{9}$ of level 3.

(2/3)−**ternary Farey tree**



We will call this modified Farey tree by (2/3)−*ternary Farey tree.* Then each fraction in the above (2/3)−ternary Farey tree of level n corresponds to a branch between the fractions of level n and the fractions of level $n+1$, that is, a skipped fraction $\frac{n}{m}$ with m, n odd bijectively. Hence we can construct another modified Farey tree from this (2/3)−ternary Farey tree as follows. We note that we have to transpose the reduced fractions symmetrically with respect to the center line through the fraction $\frac{1}{3}$ so as to commute Barning's Pythagorean tree.

(1/3)−**ternary Farey tree**



We shall call this modified Farey tree by (1/3)−*ternary Farey tree.*

2 Unimodular matrix tree

In this section, we shall relate each reduced fraction $\frac{n}{m}$ in (2/3)−ternary Farey tree to a 2×2 unimodular matrix. By virtue of the extended Euclidean algorithm, we can find unique positive integers $x < m, y < n$ which satisfy the following linear diophantine equation for any reduced fraction $\frac{n}{m}$ $(m > n > 0)$,

$$nx - my = 1.$$

Put $x' = m - x, y' = n - y$. Then $\frac{y}{x}, \frac{n}{m}, \frac{y'}{x'}$ are successive Farey series of order m. Since m and n are of odd parity, exact one of x, x', y, y' is even. We denote by $\frac{d}{c}$ one of $\frac{x}{y}$ and $\frac{x'}{y'}$ which satisfies $c \equiv d \equiv 1 \mod 2$. We denote another remaining fraction by $\frac{b}{a}$. Then we know the following linear fractional transformation

$$\begin{pmatrix} b & d \\ a & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{n}{m}.$$

Thus the reduced fraction $\frac{n}{m}$ corresponds to the matrix $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$ one to one. Hence we have verified that there exists the following bijection;

Reduced fractions $\frac{n}{m}$ with $m > n > 0$ and $m \not\equiv n \mod 2$	$\longleftrightarrow$	Unimodular matrices $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$

More precisely, we have a bijection from the following essential part of (2/3)−ternary Farey tree

$$\frac{n}{m}$$

$$\frac{n+b}{m+a} \qquad \frac{n+d}{m+c}$$

$$\frac{n+2b}{m+2a} \qquad \frac{2n+b}{2m+a}$$

to the following corresponding part of the tree of unimodular matrices;

$$\begin{pmatrix} b & d \\ a & c \end{pmatrix}$$

$$\begin{pmatrix} b & 2b+d \\ a & 2a+c \end{pmatrix} \quad \begin{pmatrix} b+d & 2b+d \\ a+c & 2a+c \end{pmatrix} \quad \begin{pmatrix} b+d & d \\ a+c & c \end{pmatrix}$$

Let F_1 be the matrix such that

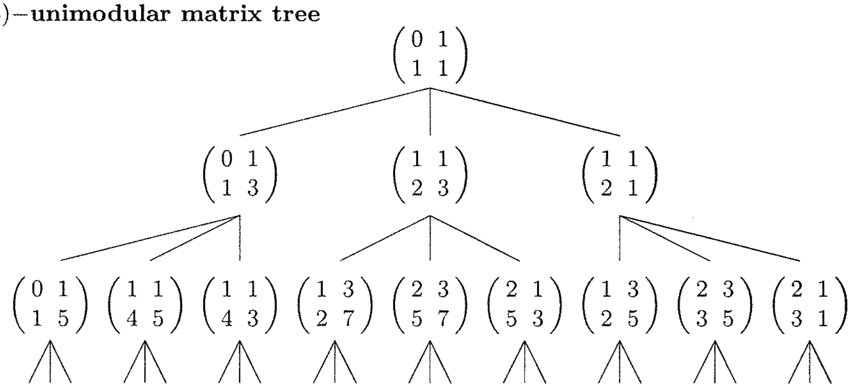
$$\begin{pmatrix} b & d \\ a & c \end{pmatrix} F_1 = \begin{pmatrix} b+d & d \\ a+c & c \end{pmatrix}.$$

Then we have $F_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Similarly let F_2, F_3 be the matrices which satisfy

$$\begin{pmatrix} b & d \\ a & c \end{pmatrix} F_2 = \begin{pmatrix} b+d & 2b+d \\ a+c & 2a+c \end{pmatrix}, \begin{pmatrix} b & d \\ a & c \end{pmatrix} F_3 = \begin{pmatrix} b & 2b+d \\ a & 2a+c \end{pmatrix}.$$

Then we have $F_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $F_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Now we have obtained the following 2×2 unimodular matrix tree from $(2/3)-$ternary Farey tree. Since we shall construct another 2×2 unimodular matrix tree from $(1/3)-$ternary Farey tree, we will call the following unimodular matrix tree $(2/3)-$*unimodular matrix tree.*

$(2/3)-$**unimodular matrix tree**



Hence the matrix A in the above tree corresponds $1:1$ to the reduced fraction $\frac{n}{m}$ in $(2/3)-$ternary Farey tree by the linear fractional transformation $\frac{n}{m} = A\left(\frac{1}{1}\right)$. Since $\frac{1}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\left(\frac{1}{1}\right)$, we get the following proposition inductively.

Proposition 3. *For any* $r \geq 0$, *any reduced fraction* $\frac{n}{m}$ *of level* $r+1$, *where* $m > n > 0$ *and* $m \not\equiv n \mod 2$ *has the unique liear fractional transformation represented as the matrix product*

$$\frac{n}{m} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} F_{\sigma(1)} F_{\sigma(2)} \cdots F_{\sigma(r)} \left(\frac{1}{1}\right),$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(r)) \in (1,2,3)^r$.

Let A_i be the unimodular matrix which satisfies

$$A_i = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} F_i \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}, \text{ for } 1 \leq i \leq 3.$$

Then one knows that

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Now we have

$$\frac{n}{m} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} F_{\sigma(1)} F_{\sigma(2)} \dots F_{\sigma(r)} \left(\frac{1}{1}\right)$$

$$= \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} F_{\sigma(1)} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\right) \cdots \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} F_{\sigma(r)} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\right) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \left(\frac{1}{1}\right)$$

$$= A_{\sigma(1)} \cdots A_{\sigma(r)} \left(\frac{1}{2}\right).$$

Theorem 1. *For any $r \geq 0$, any reduced fraction $\frac{n}{m}$ of level $r+1$, where $m > n > 0$ and $m \not\equiv n \mod 2$ has the unique linear fractional transformation represented as the matrix product*

$$\frac{n}{m} = A_{\sigma(1)} A_{\sigma(2)} \dots A_{\sigma(r)} \left(\frac{1}{2}\right),$$

where $(\sigma(1), \sigma(2), \dots, \sigma(r)) \in (1,2,3)^r$.

3 Relation to Barning's matrices

From the above theorem, we know each reduced fraction in $(2/3)-$ternary tree is transformed to other three reduced fractions by the following three distinct matrices A_1, A_2 and A_3 as follows.

$$\begin{cases} \frac{n}{m} \longmapsto \frac{n'}{m'} = A_1 \left(\frac{n}{m}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \left(\frac{n}{m}\right) = \frac{m}{2m-n}, \\ \frac{n}{m} \longmapsto \frac{n'}{m'} = A_2 \left(\frac{n}{m}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \left(\frac{n}{m}\right) = \frac{m}{2m+n}, \\ \frac{n}{m} \longmapsto \frac{n'}{m'} = A_3 \left(\frac{n}{m}\right) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \left(\frac{n}{m}\right) = \frac{n}{m+2n}. \end{cases}$$

From the classical Euclidean parametrization of primitive Pythagorean triples, the reduced fraction $\frac{1}{2}$ corresponds to the primitive Pythagorean triple $(3, 4, 5)$ and each transformation $A_i (1 \leq i \leq 3)$ implies a transformation of the primitive Pythagorean triple

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} \longmapsto \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} m'^2 - n'^2 \\ 2m'n' \\ m'^2 + n'^2 \end{pmatrix}$$

In the case A_1, we have

$$a' = (2m-n)^2 - m^2 = 3m^2 - 4mn + n^2 = (m^2 - n^2) - 2(2mn) + 2(m^2 + n^2) = a - 2b + 2c,$$

$$b' = 2(2m-n)m = 4m^2 - 2mn = 2(m^2 - n^2) - (2mn) + 2(m^2 + n^2) = 2a - b + 2c,$$

$$c' = (2m-n)^2 + m^2 = 5m^2 - 4mn + n^2 = 2(m^2 - n^2) - 2(2mn) + 3(m^2 + n^2) = 2a - 2b + 3c.$$

Thus the transformation of the primitive Pythagorean triple M_1 induced from A_1 is defined by

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a - 2b + c \\ 2a - b + 2c \\ 2a - 2b + 3c \end{pmatrix} = M_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Hence we have obtained $M_1 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$, which was denoted by M_3 in Barning [2].

In the case A_2, we have

$$a' = (2m+n)^2 - m^2 = 3m^2 + 4mn + n^2 = (m^2 - n^2) + 2(2mn) + 2(m^2 + n^2) = a + 2b + 2c,$$

$$b' = 2(2m+n)m = 4m^2 + 2mn = 2(m^2 - n^2) + (2mn) + 2(m^2 + n^2) = 2a + b + 2c,$$

$$c' = (2m+n)^2 + m^2 = 5m^2 + 4mn + n^2 = 2(m^2 - n^2) + 2(2mn) + 3(m^2 + n^2) = 2a + 2b + 3c.$$

Thus the transformation of the primitive Pythagorean triple M_2 induced from A_2 is defined by

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a + 2b + c \\ 2a + b + 2c \\ 2a + 2b + 3c \end{pmatrix} = M_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Hence we have obtained $M_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, which was denoted by M_2 in Barning [2].

In the case A_3, we have

$$a' = (m+2n)^2 - n^2 = m^2 + 4mn + 3n^2 = -(m^2 - n^2) + 2(2mn) + 2(m^2 + n^2) = -a + 2b + 2c,$$

$b' = 2(m+2n)n = 2mn+4n^2 = -2(m^2-n^2)+(2mn)+2(m^2+n^2) = -2a+b+2c,$
$c' = (m+2n)^2+n^2 = m^2+4mn+5n^2 = -2(m^2-n^2)+2(2mn)+3(m^2+n^2) = -2a+2b+3c.$

Thus the transformation of the primitive Pythagorean triple M_3 induced from A_3 is defined by

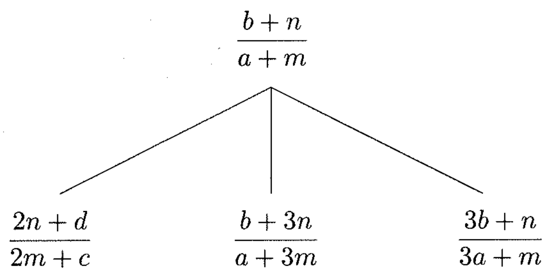
$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} -a+2b+c \\ -2a+b+2c \\ -2a+2b+3c \end{pmatrix} = M_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Hence we have obtained $M_3 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$, which is denoted by M_1 in Barning [2].

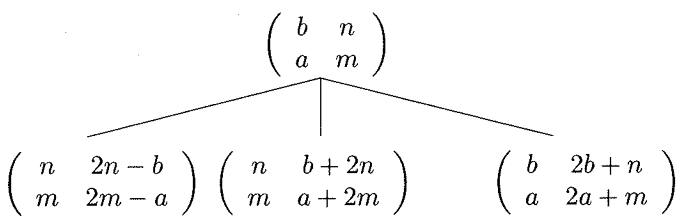
Thus we have given a very elementary explanation of the reason why Barning's three unimodular matrices generate all the primitive Pythagorean triples.

4 The case of (1/3)−ternary tree

In this section, using (1/3)−ternary Farey tree, we shall show another Euclidean parametrization induces the same representation of Pytagorean triples of Barning. Using the notations in section 2, we have the following essential part of (1/3)−ternary Farey tree and corresponding unompdular matrices.



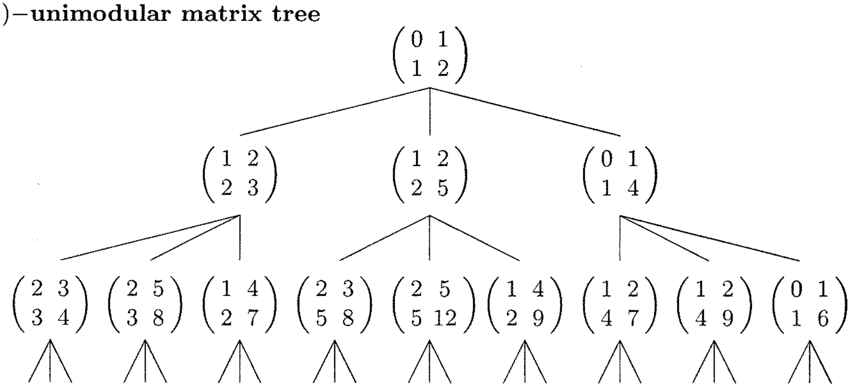
where $m = a + c, n = b + d$ with $c \equiv d \equiv 1 \mod 2$.



Now we will call the following 2×2 unimodular matrix tree corresponding to

(1/3)−ternary Farey tree, (1/3)−*unimodular matrix tree.*

(1/3)−**unimodular matrix tree**



Let G_1 be the matrix such that

$$\begin{pmatrix} b & n \\ a & m \end{pmatrix} G_1 = \begin{pmatrix} b & 2b+n \\ a & 2a+m \end{pmatrix}.$$

Then we have $G_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Similarly let G_2, G_3 be the matrices which satisfy

$$\begin{pmatrix} b & n \\ a & m \end{pmatrix} G_2 = \begin{pmatrix} n & 2n+b \\ m & 2m+a \end{pmatrix}, \begin{pmatrix} b & n \\ a & m \end{pmatrix} G_3 = \begin{pmatrix} n & 2n-b \\ m & 2m-a \end{pmatrix}.$$

Thus we have $G_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ and $G_3 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, respectively. In the same way as (2/3)−ternary Farey tree, we know that $\frac{1}{3} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \left(\frac{1}{1}\right)$ and get the following proposition inductively.

Proposition 4. *Any reduced fraction* $\frac{n_1}{m_1}$ *with* $m_1 > n_1 > 0$ *and* $m_1 \equiv n_1 \equiv 1 \mod 2$ *has the unique linear fractional transformation represented as the matrix product*

$$\frac{n}{m} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} G_{\sigma(1)} G_{\sigma(2)} \dots G_{\sigma(r)} \left(\frac{1}{1}\right),$$

for some $r \geq 0$, $(\sigma(1), \sigma(2), \dots, \sigma(r)) \in (1, 2, 3)^r$.

Let B_i be the unimodular matrices which satisfy

$$B_i = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} G_i \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^{-1}, \text{ where } 1 \le i \le 3.$$

Then one knows that

$$B_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = A_3, B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = A_2, B_3 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = A_1.$$

Hence we have

$$\frac{n_1}{m_1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} G_{\sigma(1)} G_{\sigma(2)} \dots G_{\sigma(r)} \begin{pmatrix} \frac{1}{1} \end{pmatrix}$$

$$= \left(\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} G_{\sigma(1)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \right) \cdots \left(\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} G_{\sigma(r)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \right) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{1} \end{pmatrix}$$

$$= A_{4-\sigma(1)} \cdots A_{4-\sigma(r)} \begin{pmatrix} \frac{1}{3} \end{pmatrix}.$$

Theorem 2. *Any reduced fraction* $\frac{n_1}{m_1}$ *with* $m_1 > n_1 > 0$ *and* $m_1 \equiv n_1 \equiv 1$ mod 2 *has the unique linear fractional transformation represented as the matrix product*

$$\frac{n_1}{m_1} = A_{\sigma(1)} A_{\sigma(2)} \dots A_{\sigma(r)} \begin{pmatrix} \frac{1}{3} \end{pmatrix},$$

for some $r \ge 0$, $(\sigma(1), \sigma(2), \dots, \sigma(r)) \in (1, 2, 3)^r$.

5 Concluding remarks

There exist two parametrizations of the Pythagorean triple by $(2/3)-$ternary Farey tree and $(1/3)-$ternary Farey tree. In the following, we shall show these two parametrizations induce the same Barning's tree and the same 3×3 unimoduler matrices. The bijection between the reduced fraction $\frac{n}{m}$ in $(2/3)-$ternary Farey tree and the reduced fraction $\frac{n_1}{m_1}$ in $(1/3)-$ternary Farey tree is given by the matrix tranformation

$$\frac{n_1}{m_1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{n}{m} \end{pmatrix}, \text{ and conversely } \frac{n}{m} = \left(\frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} \frac{n_1}{m_1} \end{pmatrix}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \dfrac{n}{m} & \longleftrightarrow & \dfrac{n_1}{m_1} \\ \Big\downarrow A_i & & \Big\downarrow A_{4-i} \\ \dfrac{n'}{m'} = A_i\left(\dfrac{n}{m}\right) & \longleftrightarrow & \dfrac{n'_1}{m'_1} = A_{4-i}\left(\dfrac{n_1}{m_1}\right) \end{array}$$

Here we have used the facts

$$\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} A_i \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = A_{4-i}, \quad \text{for any } 1 \leq i \leq 3.$$

We also have the correspondence of two Euclid's parametrization as follows;

$$\begin{array}{ccccc} \dfrac{n}{m} & & \longleftrightarrow & & \dfrac{n_1}{m_1} \\ \Big\updownarrow & & & & \Big\updownarrow \\ \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} & = & \begin{pmatrix} a \\ b \\ c \end{pmatrix} & = & \begin{pmatrix} m_1 n_1 \\ \dfrac{m_1^2 - n_1^2}{2} \\ \dfrac{m_1^2 + n_1^2}{2} \end{pmatrix} \end{array}$$

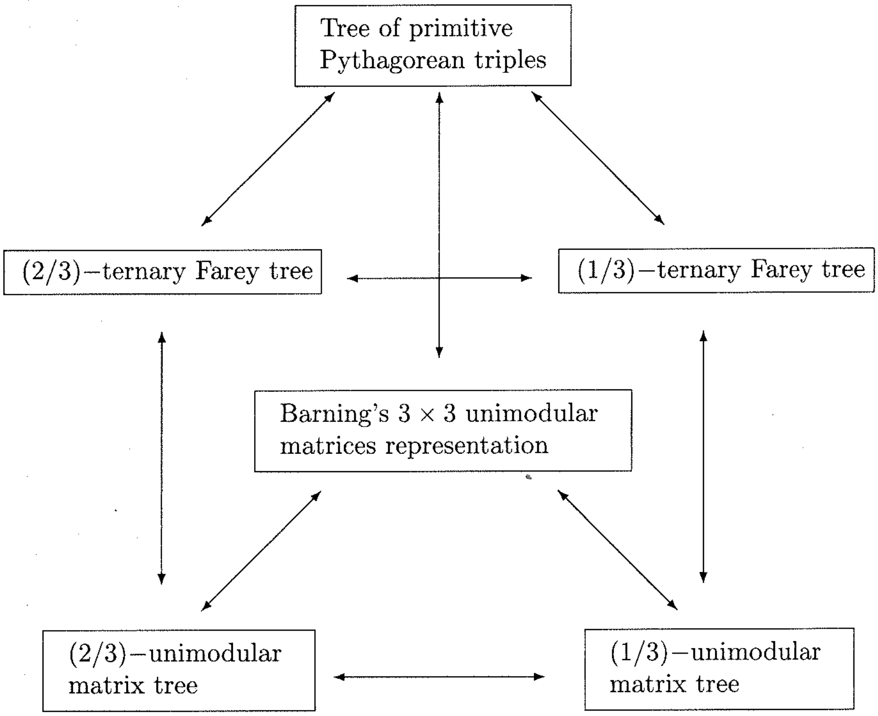
Now it is easily verified that there exists the following commutative diagram of three representations;

$$\begin{array}{ccc} & M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(r)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} & \\ \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \\ A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(r)} \left(\frac{1}{2}\right) & \longleftrightarrow & A_{4-\sigma(1)} A_{4-\sigma(2)} \cdots A_{4-\sigma(r)} \left(\frac{1}{3}\right) \end{array}$$

Remark. Since the above explanations are very elementary and straightforward, these results must be already known to the specialists. But, to the best of my knowledge, I have never seen any literature which write down these facts explicitly. Thus it will be of some worth for writing these facts explicitly in this note.

Finally, we shall summarize above results in the following diagram.

Bijective relations of 5 trees and representation



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