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On Existence of Global Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings with Linear Dissipation

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Abstract

We study the existence of global solutions to the initial-boundary value problem for the degenerate nonlinear hyperbolic equation of Kirchhoff type with linear dissipation :

$$\frac{\partial^2 u}{\partial t^2} - \left(\int_a^b \left| \frac{\partial u}{\partial x} \right|^2 dx \right)^\gamma \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + |u|^p u = 0 \,.$$

In the case of $0 < \gamma < 1$, under the conditions that $p > 4\gamma$ and the size of initial data is suitably small, we derive the global existence theorem.

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Introduction

In this paper we consider the initial-boundary value problem for the degenerate hyperbolic equation of Kirchhoff type with linear dissipation :

$$u_{tt} - \|u_x(t)\|^{2\gamma} u_{xx} + u_t + |u|^p u = 0 \qquad \text{in} \quad (a,b) \times (0,\infty) \tag{0.1}$$

with the initial data and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,, \qquad a < x < b\,, \\ &u(a,t) = u(b,t) = 0\,, \qquad 0 < t < \infty \end{aligned}$$

where $u = u(x,t)$ is an unknown real function and $a < b$ and

$$0 < \gamma < 1$$

and $p > 0$ and the symbol $\|\cdot\|$ means the usual norm of $L^2 = L^2(a,b)$.

Equation (0.1) describes small amplitude vibrations of an elastic stretched string and was introduced by Kirchhoff [4].

When $\gamma \geq 1$, in previous paper [9], we have proved the existence of global solutions under the condition that the size of the initial data is small (cf. [5], [8], [10], [12]).

Our interest in this paper is the problem in the case of $0 < \gamma < 1$ (cf. [3], [11] for equations without the nonlinear term $|u|^p u$). In order to get an a-priori estimate for H^2-norm of the solution $u(t)$, we derive the estimate for $\|u_{xx}(t)\|^2/\|u_x(t)\|^2$, and we give the decay estimate $\|u(t)\|^2_{H^2} \leq C(1+t)^{-\frac{1}{\gamma}}$. Our main result is Theorem 3.3 in section 3.

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in L^2 or sometimes duality between the space X and its dual X', and the norm of L^p is often written as $\|\cdot\|_p$ ($\|\cdot\| = \|\cdot\|_2$ for $p = 2$) simplicity. We put $(a)^+ = \max(0,a)$ where $1/(a)^+ = \infty$ if $(a)^+ = 0$. The constant c_* is the Sobolev-Poincaré constant, that is, for $1 \leq p \leq \infty$,

$$\|v\|_p \leq c_* \|v_x\| \,. \tag{0.2}$$

Positive constants will be denoted by C and will change line to line.

1 Preliminaries

By applying the Banach contraction mapping theorem, we get the following local existence theorem (see [1], [2], [9] and the references cited therein).

Proposition 1.1 *Suppose that the initial data* $\{u_0, u_1\}$ *belong to* $H^2 \cap H_0^1 \times H_0^1$ *and* $u_0 \neq 0$. *Then, the problem* (0.1) *admits a unique local solution* $u(t)$ *in the class* $C([0,T); H^2 \cap H_0^1) \cap C^1([0,T); H_0^1) \cap C^2([0,T); L^2)$ *for some* $T \equiv T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$. *Moreover, if* $\|u_x(t)\| > 0$ *for* $0 \leq t < T$, *at least one of the following statements is valid*

$$\begin{aligned} &\text{(i)} \quad T = \infty\,; \\ &\text{(ii)} \quad \|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} \to \infty \quad \text{as } t \to T-\,; \\ &\text{(iii)} \quad \|u_x(t)\| \to 0 \quad \text{as } t \to T-\,. \end{aligned}$$

We define the energy and the potential associated with (0.1) by

$$E(u, u_t) \equiv \frac{1}{2}\|u_t\|^2 + J(u) \tag{1.1}$$

and

$$J(u) \equiv \frac{1}{2(\gamma+1)}\|u_x\|^{2(\gamma+1)} + \frac{1}{p+2}\|u\|_{p+2}^{p+2}, \tag{1.2}$$

respectively.

In what follows, we denote

$$M(t) \equiv \|u_x(t)\|^2 \quad \text{and} \quad E(t) \equiv E(u(t), u_t(t)) \tag{1.3}$$

$(E(0) \equiv E(u_0, u_1)$ for $t = 0)$ and $J(t) \equiv J(u(t))$ for simplicity.

Moreover, we introduce the function $H(t)$ (i.e. modified second energy) by

$$H(t) \equiv \frac{\|u_{xt}(t)\|^2}{M(t)} + \frac{\|u_{xx}(t)\|^2}{M(t)^{1-\gamma}}. \tag{1.4}$$

2 Energy Estimate

The energy $E(t) \equiv E(u(t), u_t(t))$ given by (1.1) has the energy identity

$$\frac{d}{dt}E(t) + \|u_t(t)\|^2 = 0 \tag{2.1}$$

or

$$E(t) + \int_0^t \|u_t(s)\|^2 ds = E(0). \tag{2.2}$$

Indeed, multiplying (0.1) by u_t and integrating it over (a, b) or $(a, b) \times (0, t)$, we have (2.1) or (2.2). Moreover, applying energy method together with the Nakao inequality (see [6], [7]), we have the following the decay estimate of the energy $E(t)$ (see [6], [10], [11] for the proof).

Proposition 2.1 *Let $u(t)$ be a solution of* (0.1).

Then, the energy $E(t)$ satisfies

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(t-1)^+\right)^{-\frac{\gamma+1}{\gamma}}, \tag{2.3}$$

where we define

$$d_1 \equiv \left(3E(0)^{\frac{\gamma}{2(\gamma+1)}} + 20c_*\right)^2. \tag{2.4}$$

Proof. Integrating (2.1) over $[t, t+1]$, we have

$$\int_t^{t+1} \|u_t(s)\|^2\, ds = E(t) - E(t+1) \quad (\equiv D(t)^2). \tag{2.5}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_j)\|^2 \leq 4D(t)^2 \quad \text{for } j = 1, 2. \tag{2.6}$$

Multiplying (0.1) by u and integrating it over (a, b), we have

$$M(t)^{\gamma+1} + \|u\|_{p+2}^{p+2} = \|u_t\|^2 - \frac{d}{dt}(u_t, u) - (u_t, u),$$

and integrating the resulting equality over $[t_1, t_2]$, we observe from (2.5) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \left(M(s)^{\gamma+1} + \|u(s)\|_{p+2}^{p+2}\right) ds \\
&\leq \int_t^{t+1} \|u_t(s)\|^2 \, ds + \sum_{j=1}^{2} \|u_t(t_j)\| \|u(t_j)\| + \int_t^{t+1} \|u_t(s)\| \|u(s)\| \, ds \\
&\leq D(t)^2 + 5c_* D(t) \sup_{t \leq s \leq t+1} M(s)^{\frac{1}{2}} \\
&\leq D(t)^2 + 5c_* D(t)(2(\gamma+1)E(t))^{\frac{1}{2(\gamma+1)}}
\end{aligned} \tag{2.7}$$

where we used the fact that $E(t)$ is a non-increasing function at the last inequality.

Integrating (2.1) over $[t, t_2]$, we have

$$\begin{aligned}
E(t) &= E(t_2) + \int_t^{t_2} \|u_t(s)\|^2 \, ds \\
&\leq 2\int_{t_1}^{t_2} E(s) \, ds + \int_t^{t+1} \|u_t(s)\|^2 \, ds \\
&\leq 2\int_t^{t+1} \|u_t(s)\|^2 \, ds + \int_{t_1}^{t_2} \left(\frac{1}{\gamma+1} M(s)^{\gamma+1} + \frac{2}{p+2} \|u(s)\|_{p+2}^{p+2}\right) ds \\
&\leq 3D(t)^2 + 5c_* D(t) \left(2(\gamma+1)E(t)\right)^{\frac{1}{2(\gamma+1)}},
\end{aligned}$$

where we used (2.7) at the last inequality. Moreover, since $D(t)^2 \leq E(0)^{\frac{\gamma}{\gamma+1}} E(t)^{\frac{1}{\gamma+1}}$, we observe

$$E(t) \leq \left(3E(0)^{\frac{1}{\gamma+1}} + 20c_*\right) D(t) E(t)^{\frac{1}{2(\gamma+1)}}$$

and the Young inequality yields

$$E(t) \leq \left(\left(3E(0)^{\frac{1}{\gamma+1}} + 20c_*\right) D(t)\right)^{\frac{2(\gamma+1)}{2\gamma+1}}$$

or

$$\begin{aligned}
E(t)^{1+\frac{\gamma}{\gamma+1}} = E(t)^{\frac{2\gamma+1}{\gamma+1}} &\leq d_1^2 D(t)^2 \\
&\leq d_1^2 \left(E(t) - E(t+1)\right).
\end{aligned} \tag{2.8}$$

Thus, applying the Nakao inequlaity to (2.8), we obtain the desird estimate (2.3). □

Immediately, we obtain the following estimate as a corollary of Proposition 2.1.

Corollary 2.2 *If $q > \gamma$, under the assumption of Proposition 2.1, it holds that*

$$\int_0^t M(s)^q\, ds \le B_q(0)\,, \tag{2.9}$$

where we define

$$B_q(0) \equiv (2(\gamma+1))^{\frac{q}{\gamma+1}} \left(E(0)^{\frac{q}{\gamma+1}} + \frac{\gamma d_1}{q-\gamma} E(0)^{\frac{q-\gamma}{\gamma+1}} \right). \tag{2.10}$$

Proof. From (1.1) and (1.2), we see

$$M(t)^{\gamma+1} \le 2(\gamma+1)J(t) \le 2(\gamma+1)E(t)\,,$$

and then, we observe from (2.3) that

$$\begin{aligned}\int_0^t M(s)^q ds &\le \int_0^t (2(\gamma+1)E(s))^{\frac{q}{\gamma+1}}\, ds \\ &\le (2(\gamma+1))^{\frac{q}{\gamma+1}} \int_0^t \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(s-1)^+ \right)^{-\frac{q}{\gamma}} ds\,,\end{aligned}$$

which gives the desired estimate (2.9) if $q > \gamma$. □

3 A-priori Estimate

Proposition 3.1 *Let $u(t)$ be a solution of (0.1). In addition to the assumption of Proposition 2.1, suppose that $p > 4\gamma$ and*

$$M(t) > 0 \quad \text{and} \quad (\gamma+2)^2 H(t) \le 1\,. \tag{3.1}$$

Then, it holds that

$$F(t) \equiv \frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t) \le d_2\,, \tag{3.2}$$

where we define

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+1}} \left(M(t)\|u_{xt}(t)\|^2 - \left(\frac{1}{2}M'(t)\right)^2 \right) \ge 0\,, \tag{3.3}$$

$$d_2 \equiv F(0) + 2(p+1)c_*^p B_{\frac{p}{2}-\gamma}(0)\,. \tag{3.4}$$

Proof. Since we observe from (0.1) that

$$M(t)^\gamma \frac{d}{dt}\left(\|u_{xx}\|^2\right) = -2\|u_{xt}\|^2 - 2(u_{xtt}, u_{xt}) + 2(|u|^p u, u_{xxt}),$$
$$M(t)^\gamma \|u_{xx}\|^2 = -\frac{1}{2}M'(t) + \left(\|u_{xt}\|^2 + \frac{1}{2}M''(t)\right) + (|u|^p u, u_{xx}),$$

we have

$$\frac{d}{dt}\left(\frac{\|u_{xx}\|^2}{M(t)}\right) = \frac{1}{M(t)^{\gamma+2}}\left(M(t)^\gamma(\|u_{xx}\|^2)'M(t) - M(t)^\gamma\|u_{xx}\|^2 M'(t)\right)$$
$$= -2Q(t) - R(t) + S(t), \tag{3.5}$$

where $Q(t)$ is defined by (3.3) and

$$R(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(2M(t)(u_{xtt}, u_{xt}) + \left(M'(t)\|u_{xt}\|^2 - \frac{1}{2}M'(t)M''(t)\right)\right),$$
$$S(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(2M(t)(f(u), u_{xxt}) - M'(t)(f(u), u_{xx})\right).$$

On the other hand, we observe

$$\frac{d}{dt}Q(t) = -(\gamma+2)\frac{M'(t)}{M(t)}\frac{1}{M(t)^{\gamma+2}}\left(M(t)\|u_{xt}\|^2 - \left(\frac{1}{2}M'(t)\right)^2\right)$$
$$+ \frac{1}{M(t)^{\gamma+2}}\left(2M(t)(u_{xtt}, u_{xt}) + M'(t)\|u_{xt}\|^2 - \frac{1}{2}M'(t)M''(t)\right)$$
$$= -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) + R(t). \tag{3.6}$$

Adding (3.5) and (3.6), we have

$$\frac{d}{dt}\left(\frac{\|u_{xx}\|^2}{M(t)} + Q(t)\right) = -2\left(1 + \frac{\gamma+2}{2}\frac{M'(t)}{M(t)}\right)Q(t) + S(t). \tag{3.7}$$

Here, we observe from (3.1) that

$$\frac{\gamma+2}{2}\frac{M'(t)}{M(t)} \le (\gamma+2)H(t)^{\frac{1}{2}} \le 1 \tag{3.8}$$

and

$$|S(t)| \le \frac{4(p+1)}{M(t)^{\gamma+2}}\|u\|_\infty^p\|u_x\|^3\|u_{xt}\|$$
$$\le 4(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma}\left(\frac{\|u_{xt}\|^2}{M(t)}\right)^{\frac{1}{2}}$$
$$\le 2(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma} \tag{3.9}$$

where we used (1.4) and (3.1) at the last inequality. Thus, we have from (3.7)–(3.9) that

$$\frac{d}{dt}\left(\frac{\|u_{xx}\|^2}{M(t)} + Q(t)\right) \le 2(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma}\,, \tag{3.10}$$

and then, integrating (3.10) and using (2.9), we obtain the desired estimate (3.2) if $p/2 - \gamma > \gamma$ (i.e. $p > 4\gamma$). □

Proposition 3.2 *Let $u(t)$ be a solution of* (0.1). *Suppose that the assumption of Proposition 3.1 is fulfilled. Then, the function $H(t)$ given by* (1.4) *satisfies*

$$H(t) \le H(0) + I(0)\,, \tag{3.11}$$

where we define

$$I(0) \equiv 2(1-\gamma)^2 d_2^2 B_{2\gamma}(0) + 2c_*^{2p}(p+1)^2 B_p(0)\,. \tag{3.12}$$

Proof. Multiplying (0.1) by $(-2u_{xxt}/M(t))$ and integrating it over (a,b), we have

$$\begin{aligned}\frac{d}{dt}H(t) + \left(2 + \frac{M'(t)}{M(t)}\right)\frac{\|u_{xt}\|^2}{M(t)} &= -(1-\gamma)\frac{M'(t)}{M(t)^{2-\gamma}}\|u_{xx}\|^2 - 2\frac{(|u|^p u, u_{xxt})}{M(t)} \\ &\equiv I_1 + I_2\,.\end{aligned}$$

We observe from (3.2) that

$$I_1 \le 2(1-\gamma)\frac{\|u_{xx}\|^2}{M(t)}M(t)^\gamma\frac{\|u_{xt}\|}{\|u_x\|} \le 2(1-\gamma)d_2 M(t)^\gamma\left(\frac{\|u_{xt}\|^2}{M(t)}\right)^{\frac{1}{2}},$$

$$I_2 \le \frac{2(p+1)}{M(t)}\|u\|_\infty^p\|u_x\|\|u_{xt}\| \le 2c_*^p(p+1)M(t)^{\frac{p}{2}}\left(\frac{\|u_{xt}\|^2}{M(t)}\right)^{\frac{1}{2}},$$

and from (3.1) that

$$2 + \frac{M'(t)}{M(t)} \ge 2 - 2H(t)^{\frac{1}{2}} \ge 2\frac{\gamma+1}{\gamma+2}\,.$$

Thus we have

$$\frac{d}{dt}H(t) + 2\frac{\gamma+1}{\gamma+2}\frac{\|u_{xt}\|^2}{M(t)} \le \left(2(1-\gamma)d_2M(t)^\gamma + 2c_*^p(p+1)M(t)^{\frac{p}{2}}\right)\left(\frac{\|u_{xt}\|^2}{M(t)}\right)^{\frac{1}{2}}$$

and from the Young inequality,

$$\frac{d}{dt}H(t) \le 2(1-\gamma)^2 d_2^2 M(t)^{2\gamma} + 2c_*^{2p}(p+1)^2 M(t)^p\,. \tag{3.13}$$

Therefore, integrating (3.13) and using (2.9), we obtain the desired estimate (3.11). □

Our main result is as follows.

Theorem 3.3 *Let the initial data* $\{u_0, u_1\}$ *belong to* $H^2 \cap H_0^1 \times H_0^1$ *and* $u_0 \neq 0$. *Suppose that* $p > 4\gamma$ *and*

$$(\gamma+1)^2(H(0)+I(0)) < 1\,, \tag{3.14}$$

where c_*, $E(0)$, $H(0)$, $I(0)$ *are defined by* (0.2), (1.3), (1.4), (3.12), *respectively. Then, the problem* (0.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); H^2 \cap H_0^1) \cap C^1([0,\infty); H_0^1) \cap C^2([0,\infty); L^2)$ *satisfying*

$$\|u(t)\|_{H^2}^2 \leq C(1+t)^{-\frac{1}{\gamma}}\,, \tag{3.15}$$

and the energy satisfies

$$E(t) \equiv E(u(t), u_t(t)) \leq C(1+t)^{-1-\frac{1}{\gamma}}\,. \tag{3.16}$$

Proof. Let $u(t)$ be a solution of (0.1) on $[0, T_1)$.

Since $M(0) > 0$ (by $u_0 \neq 0$), putting

$$T_2 \equiv \sup\left\{t \in [0,\infty) \;\middle|\; M(s) > 0 \quad \text{for } 0 \leq s < t\right\}\,,$$

we see that $T_2 > 0$ and $M(t) > 0$ for $0 \leq t < T_2$.

If $T_2 < T_1$, then we see $M(T_2) = 0$.

For $0 \leq t < T_2$, if we assume $(\gamma+2)(H(0)+I(0))^{\frac{1}{2}} < 1$, then there exists $T_3 > 0$ such that

$$(\gamma+2)H(t)^{\frac{1}{2}} \leq 1 \quad \text{for } 0 \leq t \leq T_3\,,$$

and hence, from Proposition 3.2,

$$H(t) \leq H(0) + I(0) \quad \text{for } 0 \leq t \leq T_3\,. \tag{3.17}$$

Thus, we observe from (3.17) that

$$(\gamma+2)H(t)^{\frac{1}{2}} \leq (\gamma+2)(H(0)+I(0))^{\frac{1}{2}} < 1 \quad \text{for } 0 \leq t \leq T_3\,,$$

and hence, we see $T_3 \geq T_2$, that is,

$$H(t) \leq \frac{1}{(\gamma+2)^2} \quad \text{for } 0 \leq t < T_2\,. \tag{3.18}$$

Since $M(T_2) = 0$, we see from (3.18) that $\lim_{t \to T_2} E(t) = \lim_{t \to T_2} E(u(t), u_t(t)) = 0$.

We perform the change of variable $s = T_2 - t$ or $t = T_2 - s$, then the function $U(x,s) = u(x, T_2 - t)$ on $[0, T_2]$ satisfies that

$$U_{ss} - \|U_x(s)\|^{2\gamma} U_{xx} - U_s + |U|^p U = 0\,. \tag{3.19}$$

Multiplying (3.19) by U_s and integrating it over (a,b), we have

$$\frac{d}{dt}E(U,U_s) = \|U_s\|^2 \leq 2E(U,U_s)$$

and from $E(U(0),U_s(0)) = \lim_{t\to T_2} E(u(t),u_t(t)) = 0$, we observe

$$E(U(s),U_s(s)) \leq 2\int_0^s E(U(\tau),U_s(\tau))\,d\tau\,.$$

Applying the Gronwall inequality, we have that $E(U(s),U_s(s)) = 0$ for $0 \leq s \leq T_2$ or $E(u(t),u_t(t)) = 0$ for $0 \leq t \leq T_2$ which contradicts $E(u_0,u_1) \equiv E(0) \geq J(0) \geq \frac{1}{2(\gamma+1)}M(0)^{\gamma+1} > 0$, and hence, we see $T_2 \geq T_1$ and $M(t) > 0$ for $0 \leq t \leq T_1$.

Thus, we conclude from (3.18) that $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} < \infty$ for $t \geq 0$. Therefore, the local solution $u(t)$ of (0.1) in the sense of Proposition 1.1 can be continued globally in time. Also, from Proposition 2.1 and Proposition 3.1 we obtain the decay estimates (3.15) and (3.16). □

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