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Differential Calculus of L^p-functions and L^p_{loc}-functions Revisited

By

Yoshifumi Ito

Professor Emeritus, The University of Tokushima
Home Address : 209-15 *Kamifukuman Hachiman-cho*
Tokushima 770-8073, *JAPAN*
e-mail address : *yoshifumi@md.pikara.ne.jp*

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Abstract

I studied the concepts of differentiability, derivatives and partial derivatives as the fundamental concepts of differential calculus in Ito [4], [5].

In this paper, we study the fundamental properties of derivatives and partial derivatives of classical functions such as L^p-functions and L^p_{loc}-functions in the sense of L^p-convergence and L^p_{loc}-convergence respectively.

Here we assume that p is a real number such that $1 \leq p < \infty$ holds.

In the calculation of such derivatives and partial derivatives, we do not need the theory of distributions except the case $p = 1$.

Thereby, I give the new characterization of Sobolev spaces and give the new meaning of Stone's Theorem.

Especially, in the cases of L^2-functions and L^2_{loc}-functions, these results have the essential role in the study of Schrödinger equations.

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Introduction

Every solution of Schrödinger equation which is the basic equation of natural statistical physics should be an L^2-density. Further, the solutions of an eigenvalue problem of a Schrödinger operator can be obtained as L^2-densities and L^2_{loc}-densities.

For these functions, these derivatives in the sense of distributions are the most general one at present.

But, when we consider the derivatives of such classical functions as L^2-functions and L^2_{loc}-functions, it is extreme to consider the derivatives in the sense of distributions.

It is not easy to judge that the derivatives or the partial derivatives of L^2-functions in the sense of distributions are L^2-functions.

Therefore, it is hard to see the concrete meanings of the concept of derivatives in the sense of distributions.

Against the above, if we calculate the derivatives or the partial derivatives in the sense of L^2-convergence, the results are L^2-functions at once.

Similarly, if we calculate the derivatives or the partial derivatives of L^2_{loc}-functions in the sense of L^2_{loc}-convergence, the results are L^2_{loc}-functions at once. Therefore, the differential calculus in the sense of L^2-convergence or L^2_{loc}-convergence is considered to be very concrete and useful.

Therefore, for the differential equations such as Schrödinger equations, the concepts of differential calculus considered in ito [4], [5] are very useful.

In order to solve a Schrödinger equation, we need not use the differential calculus in the sense of distributions. Namely, here, it is enough to use the differential calculus in the sense of L^2-convergence or L^2_{loc}- convergence.

In this paper, in the more, we consider the derivatives or the partial derivatives of L^2-functions in the sense of L^2-convergence or those of L^2_{loc}-functions in the sense of L^2_{loc}- convergence, and we also consider these fundamental properties.

Here we assume $1 \leq p < \infty$.

In the special case, we consider the case $p = 2$.

Further, thereby, we obtain the new characterizations of Sobolev spaces and the new meaning of Stone's theorem.

1 Differential calculus

1.1 Definition of differentiability

As for the results in this subsection, we refer to Ito [1], Chapter 4.

Assume $d \geq 1$.

We consider the d-dimensional Euclidean space $\boldsymbol{R}^d$ and a point x in $\boldsymbol{R}^d$ is considered to be a numerical vector $x = {}^t(x_1, x_2, \cdots, x_d)$.

Now, we consider a scalar function $y = f(x) = f(x_1, x_2, \cdots, x_d)$ defined in a certain neighborhood of a point $x = {}^t(x_1, x_2, \cdots, x_d)$.

Now we put

$$\Delta x = {}^t(\Delta x_1, \Delta x_2, \cdots, \Delta x_d),$$
$$x + \Delta x = {}^t(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_d + \Delta x_d).$$

Further, ρ denotes the distance of x and $x + \Delta x$, which is equal to

$$\rho = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + \cdots + (\Delta x_d)^2}.$$

Definition 1.1.1 In the above notation, we put the increment Δy at a point x as

$$\Delta y = f(x + \Delta x) - f(x) = \sum_{i=1}^{d} A_i \Delta x_i + \varepsilon(x, \Delta x)\rho.$$

Here, A_i, $(i = 1, 2, \cdots, d)$ depend only on x, but not on Δx. $\varepsilon = \varepsilon(x, \Delta x)$ is a scalar function of x and Δx.

Now, if, at a fixed point x, the condition

$$\rho \longrightarrow 0 \Longrightarrow \varepsilon(x, \Delta x) \longrightarrow 0$$

is satisfied, we say that $y = f(x)$ is **differentiable** at the point x.

Here we extend the definition of $\varepsilon(x, \Delta x)$ as $\varepsilon(x, 0) = 0$.

Remark 1.1.1 In the definition 1.1.1, $\varepsilon(x, \Delta x)$ is determined by the relation

$$\varepsilon(x, \Delta x) = \begin{cases} \dfrac{\Delta y - \sum_{i=1}^{d} A_i \Delta x_i}{\rho}, & (\text{if } \Delta x \neq 0), \\ 0, & (\text{if } \Delta x = 0). \end{cases}$$

Further, A_i will be known to be a partial derivative of the scalar function $y = f(x)$ with respect to x_i at a point x, $(i = 1, 2, \cdots, d)$.

In the case $d = 1$, we only consider the derivative of the scalar function $y = f(x)$.

By virtue of Definition 1.1.1, the fact that a scalar function $y = f(x)$ is differentiable at x means that the increment Δy of y for the increment Δx of x

can be approximated well enough by a homogeneous function of the components of Δx of degree 1, namely by 1-form of Δx.

Theorem 1.1.1 *If a scalar function $y = f(x)$ is differentiable at a point x, $f(x)$ is continuous at the point x.*

Definition 1.1.2 Let D be a domain in $\boldsymbol{R}^d$. If $y = f(x)$ is differentiable at every point in D, we say that this function is differentiable on D.

Theorem 1.1.2 *Let D be a domain in $\boldsymbol{R}^d$. If a function $y = f(x)$ is differentiable on D, $y = f(x)$ is continuous on D.*

It is important that the definitions and the results in the theorems in the above hold for the cases $d \geq 1$.

Now we consider the vector function $y = \Phi(x)$ defined by the system of scalar functions

$$y_i = g_i(x_1, x_2, \cdots, x_d), \ (1 \leq i \leq n).$$

Then we give the definition of differentiability of a vector function in the following.

Definition 1.1.3 In the above notation, we put the increment Δy of y at a point x corresponding to the increment Δx of x as follows:

$$\Delta y = \Phi(x + \Delta x) - \Phi(x) = \sum_{i=1}^{d} A_i \Delta x_i + \varepsilon(x, \Delta x)\rho.$$

Here the column vectors A_i, $(i = 1, 2, \cdots, d)$ depend on only x, but not on Δx. $\varepsilon = \varepsilon(x, \Delta x)$ is a vector function of x and Δx.

Now, if, at a fixed point x, the condition

$$\rho \to 0 \Longrightarrow \varepsilon(x, \Delta x) \to 0$$

is satisfied, we say that $y = \Phi(x)$ is differentiable at the point x.

Here we extend the definition of $\varepsilon(x, \Delta x)$ as $\varepsilon(x, 0) = 0$.

Remark 1.1.2 In the definition 1.1.3, the vector function $\varepsilon(x, \Delta x)$ is determined by the relation

$$\varepsilon(x, \Delta x) = \begin{cases} \dfrac{\Delta y - \sum_{i=1}^{d} A_i \Delta x_i}{s} \rho, & (\text{if } \Delta x \neq 0), \\ 0, & (\text{if } \Delta x = 0). \end{cases}$$

Further, the column vector A_i will be known to be the partial derivative of the vector function $y = \Phi(x)$ with respect to x_i at the point x, where we put $i = 1, 2, \cdots, d$.

Then, the fact that the vector function $y = \Phi(x)$ is differentiable means that the increment Δy of y corresponding to the increment Δx of x is approximated well enough by the image of Δx by the linear map

$$\frac{\partial y}{\partial x} = (A_1\, A_2\, \cdots\, A_d).$$

Here the symbols of column vectors $A_1, A_2, \cdots, A_d$ are the same as in Definition 1.1.3.

Then this linear map is equal to the Jacobi matrix

$$\frac{\partial y}{\partial x} = (A_1\, A_2\, \cdots\, A_d)$$
$$= \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_d} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_d} \end{pmatrix}.$$

This is the differential coefficient of the vector function $y = \Phi(x)$.

When x is varing, this also means that the function $\dfrac{\partial y}{\partial x}$ of x is the derivative.

Corollary 1.1.1 *The vector function $y = \Phi(x)$ is differentiable at a point x if and only if each coordinate function $g_i(x)$ is differentiable at the point x, $(1 \le i \le n)$.*

Theorem 1.1.3 *If the vector function $y = \Phi(x)$ is differentiable at a point x, $y = \Phi(x)$ is continuous at the point.*

Definition 1.1.4 If the vector function $y = \Phi(x)$ is differentiable at each point in a domain D, we say that this vector function $y = \Phi(x)$ is **differentiable** on D.

Theorem 1.1.4 *If the vector function $y = \Phi(x)$ is differentiable on the domain D, $y = \Phi(x)$ is continuous on D.*

1.2 Some aspects of differentiability

Here we assume $d \geq 1$.

In Definition 1.1.1 and Definition 1.1.2, we give the differentiability of a scalar function $y = f(x)$ defined on a domain D in $\boldsymbol{R}^d$.

Then, considering the convergence

$$\varepsilon(x, \Delta x) \longrightarrow 0 \tag{1.2.1}$$

in Definition 1.1.1 according several notions of convergence, we have several notions of differentiability.

This means that we consider the notions of differentiability of scalar functions of several classes of functions.

In the sequel, especially, we consider the differentiability of L^p-functions and L^p_{loc}-functions.

We remark that these results are new ones.

(1) L^p-differentiability

We denote the space of all p-integrable scalar functions on $\boldsymbol{R}^d$ as $L^p = L^p(\boldsymbol{R}^d)$.

Then, we give the notion of L^p-differentiability. Namely, we consider the notion of differentiability of scalar L^p-functions in the sense of the convergence of L^p-norm.

Here, L^p-norm is defined by the relation

$$||f||_p = \Big(\int |f(x)|^p dx \Big)^{1/p}, \quad (f \in L^p).$$

Here the integration denotes the Lebesgue integral on $\boldsymbol{R}^d$.

Now we give the following Definition 1.2.1.

Definition 1.2.1(L^p-differentiability) Let a scalar function $y = f(x)$ be an L^p-function defined on $\boldsymbol{R}^d$.

Then we put the increment Δy of the scalar function $y = f(x)$ according to the increment Δx of the independent variable x as follows:

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= \sum_{i=1}^{d} A_i(x)\Delta x_i + \varepsilon(x, \Delta x)\rho. \end{aligned} \tag{1.2.2}$$

Here we assume that $A_i(x)$, $(i = 1, 2, \cdots, d)$ are the scalar functions of x which do not depend on Δx. $\varepsilon = \varepsilon(x, \Delta x)$ is a scalar function of x and Δx.

Then we say that the scalar function $y = f(x)$ is differentiable in the sense of L^p-convergence on $\boldsymbol{R}^d$ if the condition

$$\Delta x \longrightarrow 0 \Longrightarrow \varepsilon(x, \Delta x) \longrightarrow 0 \tag{1.2.3}$$

is satisfied in the sense of L^p-convergence on $\boldsymbol{R}^d$ as an L^p-function of x.

Namely, this is equivalent to the condition that the relation

$$\lim_{\Delta x \to 0} \|\varepsilon(x, \Delta x)\|_p = 0$$

is satisfied.

Here, instead of saying that a scalar function $y = f(x)$ is differentiable in the sense of L^p-convergence, we say that it is L^p-differentiable simply.

Now we denote the space of all p-integrable scalar functions on a general domain D in $\boldsymbol{R}^d$ as $L^p = L^p(D)$. We may consider that $L^p(D)$ is a subspace of $L^p(\boldsymbol{R}^d)$.

Therefore, we say a function $f \in L^p(D)$ to be L^p-differentiable if the function f is L^p-differentiable considering that this function f is a function of $L^p(\boldsymbol{R}^d)$.

(2) L^p_{loc}-differentiability

We denote the space of all locally p-integrable scalar functions defined on a domain D as $L^p_{\text{loc}} = L^p_{\text{loc}}(D)$.

Then, we give the notion of L^p_{loc}-differentiability. Namely, we consider the notion of differentiability of scalar L^p_{loc}-functions in the sense of the convergence of the space L^p_{loc}.

Here, in L^p_{loc}, we use the notion of convergence with respect to the system of semi-norms $\{q_K(f); K \text{ is a compact subset of } D\}$, where the semi-norm $q_K(f)$ is defined by the relation

$$q_K(f) = \Big(\int_K |f(x)|^p dx \Big)^{1/p}, \quad (f \in L^p_{\text{loc}},\ K \text{ is a compact subset of } D)$$

Now we give the following Definition 1.2.2.

Definition 1.2.2(L^p_{loc}-differentiability) Let a scalar function $y = f(x)$ be an L^p_{loc}-function defined on a domain D.

Then we put the increment Δy of the scalar function $y = f(x)$ according to the increment Δx of the independent variable x as follows:

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= \sum_{i=1}^{d} A_i(x) \Delta x_i + \varepsilon(x, \Delta x)\rho. \end{aligned} \tag{1.2.4}$$

Here we assume that $A_i(x),\ (i=1,2,\cdots,d)$ are the scalar functions of x which do not depend on Δx. $\varepsilon=\varepsilon(x,\Delta x)$ is a scalar function of x and Δx.

Then we say that the scalar function $y=f(x)$ is differentiable in the sense of L^p_{loc}-convergence on the domain D if the condition

$$\Delta x \longrightarrow 0 \Longrightarrow \varepsilon(x,\Delta x) \longrightarrow 0 \tag{1.2.5}$$

is satisfied in the sense of L^p_{loc}-convergence on D.

Namely, this is equivalent to the condition that the relation

$$\lim_{\Delta x\to 0} q_K(\varepsilon(x,\Delta x)) = \lim_{\Delta x\to 0}\Big(\int_K |\varepsilon(x,\Delta x)|^p dx\Big)^{1/p} = 0$$

is satisfied for an arbitrary compact subset K in the domain D.

Here instead of saying that a scalar function $y=f(x)$ is differentiable in the sense of L^p_{loc}-convergence, we say that it is L^p_{loc}-differentiable simply.

1.3 Partial derivatives

We assume that a scalar function $y=f(x)=f(x_1,x_2,\cdots,x_d)$ is differentiable at a point x.

Then, for $1\le j\le d$, we have the limit

$$\frac{\partial y}{\partial x_j} = \lim_{\Delta x_j\to 0}\frac{f(x_1,\cdots,x_j+\Delta x_j,\cdots,x_d)-f(x_1,\cdots,x_j,\cdots,x_d)}{\Delta x_j} \tag{1.3.1}$$

at the point x by virtue of Definition 1.1.1.

If the limit (1.3.1) exists, we say that $y=f(x)$ is **partially differentiable** with respect to x_j.

We say that the calculation of the limit (1.3.1) is **partially differentiating** $y=f(x_1,x_2,\cdots,x_d)$ with respect to x_j.

This means that we differentiate this function $y=f(x)$ with respect x_j considering that $y=f(x)$ is a function of one variable x_j if one variable x_j is varying and the other variables $x_k,(k\ne j)$ are fixed.

Then, we have the following theorem.

Theorem 1.3.1 *We use the notation in Definition* 1.1.1. *If $y=f(x)$ is differentiable at the point x, it is partially differentiable with respect to each x_i and we have*

$$A_i = f_{x_i}(x),\ (i=1,2,\cdots,d)$$

If $y = f(x)$ is partially differentiable at every point in D with respect to each variable x_j, we say simply that $y = f(x)$ is **partially differentiable** on D, and that $\dfrac{\partial y}{\partial x_j}$, $(1 \leq j \leq d)$ of the formula (1.3.1) are **partial derivatives** with respect to x_j.

We say that the value of the partial derivatives $\dfrac{\partial y}{\partial x_j}$ at the point x is the **partial differential coefficient** at the point x with respect to x_j.

Here we prove the inverse of Theorem 1.3.1.

Namely, we have the sufficient condition in order that a partially differentiable function is differentiable.

We prove this in the following.

Theorem 1.3.2 *If a scalar function $y = f(x)$ is partially differentiable in a neiborhood of x and every $f_{x_i}(x)$ is continuous at x, $f(x)$ is differentiable at x.*

If the partial derivative $f_{x_i}(x_1, x_2, \cdots, x_d)$ of a function $y = f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_d)$ is still partially differentiable with respect to x_j, $(j = 1, 2, \cdots, d)$, we say that this partial derivative is the **partial derivative of second degree** or the **partial derivative of second order** of y, and denote this as follows:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial y}{\partial x_i}\right), \ \frac{\partial^2 y}{\partial x_j \partial x_i} \tag{1.3.2}$$

and so on.

Especially, when $i = j$ holds, we denote this as follows:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right), \ \frac{\partial^2 y}{\partial x_i^2} \tag{1.3.3}$$

and so on.

In general, we define the **partial derivatives of degree** m or the **partial derivatives of order** m in the similar way.

We say that all partial derivatives of order m for $m \geq 2$ are **partial derivatives of higher degree** or **partial derivatives of higher order**.

If all partial derivatives up to order m exist, we say that the function is **m-times partially differentiable**.

Now we assume that a vector function $y = \Phi(x)$ is defined by the family of scalar functions

$$y_i = g_i(x_1, x_2, \cdots, x_d), \ (1 \leq i \leq n). \tag{1.3.4}$$

Then, if the vector function $y = \Phi(x)$ is differentiable, we can calculate its partial derivative in the similar way as the calculation (1.3.1) for a scalar function, Namely, for $1 \leq j \leq d$, this is the calculation as follows:

$$\frac{\partial y}{\partial x_j} = \lim_{\Delta x_j \to 0} \frac{\Phi(x_1, \cdots, x_j + \Delta x_j, \cdots, x_d) - \Phi(x_1, \cdots, x_j, \cdots, x_d)}{\Delta x_j}. \quad (1.3.5)$$

Therefore, the calculation of the partial derivatives of the vector function $y = \Phi(x)$ is carried out by the calculation of the partial derivatives of each coordinate function $y_i = g_i(x)$.

Definition 1.3.1 We use the notation of Definition 1.1.3. Then, if , for $1 \leq j \leq d$, the partial differential coefficients (1.3.5) exist at the point x, we say that the vector function $y = \Phi(x)$ is **partially differentiable**.

We say the calculation of the limit (1.3.5) the **partial differentiation** of $y = \Phi(x)$ with respect to x_j.

Then we have the following theorem.

Teorem 1.3.3 *We use the notation of Definition* 1.1.3. *If* $y = \Phi(x)$ *is differentiable at the point* x, *it is partially differentiable at the point* x *with respect to each* x_i *and we have*

$$A_i = \frac{\partial y}{\partial x_i}, \ (i = 1, 2, \cdots, d).$$

We can also prove the inverse of Theorem 1.3.3.

If $y = \Phi(x)$ is partially differentiable with respect to each variable x_j at every point of a domain D, we say merely that $y = \Phi(x)$ is **partially differentiable** on D and that we say $\frac{\partial y}{\partial x_j}$, $(1 \leq j \leq d)$ in the formula (1.3.5) are the **partial derivatives** with respect to x_j, $(1 \leq j \leq d)$.

We say that the value of the partial derivative $\frac{\partial y}{\partial x_j}$ at the point x is the **partial differential coefficient** at the point x with respect to x_j.

It is known that we can calculate the partial drivatives $\frac{\partial y}{\partial x_j}$ of the scalar function $y = f(x)$, $(1 \leq j \leq d)$ using the formula (1.3.1).

Further, it is known that we can calculate the partial derivatives $\frac{\partial y}{\partial x_j}$ of the vector function $y = \Phi(x)$, $(1 \leq j \leq d)$ using the formula (1.3.5).

Then, we have the concept of L^p-differentiability for the L^p-functions, and L^p_{loc}-differentiability for the L^p_{loc}-functions in the similar way as section 1.2.

Therefore, we can calculate the derivatives of L^p-functions and L^p_{loc}-functions by virtue of the formulas (1.3.1) and (1.3.5) using the concepts of convergence (1), (2) in the section 1.2.

Then, the obtained partial derivatives $\dfrac{\partial y}{\partial x_j}$ and the vector valued partial derivatives $\dfrac{\partial y}{\partial x_j}$, $(1 \leq j \leq d)$ are also an L^p-function or L^p_{loc}-function.

Therefore, we have the concepts of differential calculus adapted to L^p-functions and L^p_{loc}-functions.

1.4 Fundamental properties of partial derivatives

Assume $d \geq 1$. Especially in the case $d = 1$, we only consider the derivatives instead of the partial derivatives in the following arguments.

Further, assume that $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ is the space of all C^∞-functions with compact support defined on $\boldsymbol{R}^d$.

Assume $f(x) \in L^p = L^p(\boldsymbol{R}^d)$.

Then, if $f(x)$ is L^p-differentiable, the partial drivatives of $f(x)$ in the sense of L^p-convergence are considered to be the weak partial derivatives.

But the proof of the inverse assertion is difficult.

Here we give the definition of the weak partial derivative.

Definition 1.4.1 Now assume $f(x) \in L^p$.
Then, for $1 \leq j \leq d$, the weak partial derivative of $f(x)$

$$w\text{-}\frac{\partial f}{\partial x_j} \in L^1_{\text{loc}}$$

is defined to be the function satisfying the condition

$$\left(w\text{-}\frac{\partial f}{\partial x_j}, \varphi \right) = -\left(f, \frac{\partial \varphi}{\partial x_j} \right), \; (\varphi \in \mathcal{D}).$$

Then we have the following theorem.

Theorem 1.4.1 *Now we assume $f(x) \in L^p$.*
Then, if $f(x)$ is partially L^p-differentiable, $f(x)$ is weakly partially differentiable, and its partial derivative $\dfrac{\partial f}{\partial x_j}$ in the sense of L^p-convergence coincides with the weak partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ for $1 \leq j \leq d$. Namely, we have the equality

$$\frac{\partial f}{\partial x_j} = w\text{-}\frac{\partial f}{\partial x_j}, (1 \leq j \leq d),$$

or, the equality

$$\left(\frac{\partial f}{\partial x_j}, \varphi \right) = \left(w\text{-}\frac{\partial f}{\partial x_j}, \varphi \right), \quad (\varphi \in \mathcal{D},\ 1 \le j \le d).$$

Then, the weak partial derivatives $w\text{-}\dfrac{\partial f}{\partial x_j}$, $(1 \le j \le d)$ of $f(x) \in L^p$ are L^p-functions.

This fact is not the property which holds absolutely for any $f(x) \in L^p$.

Theorem 1.4.2 *Now we assume that $1 < p < \infty$ and $f(x) \in L^p$. Then if the weak partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ of $f(x)$ exists and we have $w\text{-}\dfrac{\partial f}{\partial x_j} \in L^p$ for $1 \le j \le d$, $f(x)$ is partially L^p-differentiable and we have*

$$w\text{-}\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_j}, \quad (1 \le j \le d)$$

for the partial derivative $\dfrac{\partial f}{\partial x_j}$ of $f(x)$ in the sense of L^p-convergence.

Further, we assume $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\boldsymbol{R}^d)$.

Then we give the definition of weak partial derivatives.

Definition 1.4.2 Now we assume that $f(x) \in L^p_{\mathrm{loc}}$ hold. Then, for $1 \le j \le d$, we define the weak partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j} \in L^1_{\mathrm{loc}}$ of $f(x)$ is defined to be the function satisfying the condition

$$\left(w\text{-}\frac{\partial f}{\partial x_j}, \varphi \right) = -\left(f, \frac{\partial \varphi}{\partial x_j} \right), \quad (\varphi \in \mathcal{D}).$$

Then we have the following theorem.

Theorem 1.4.3 *Now we assume that $f(x) \in L^p_{\mathrm{loc}}$ hold. If $f(x)$ is partially L^p_{loc}-differentiable, then, $f(x)$ is weakly partially differentiable and its partial derivative $\dfrac{\partial f}{\partial x_j}$ in the sense of L^p_{loc}-convergence coincides with the partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ for $1 \le j \le d$. Namely, we have the equality*

$$\frac{\partial f}{\partial x_j} = w\text{-}\frac{\partial f}{\partial x_j}, \quad (1 \le j \le d),$$

or the equality

$$\left(\frac{\partial f}{\partial x_j},\varphi\right)=\left(w\text{-}\frac{\partial f}{\partial x_j},\varphi\right),\ (\varphi\in\mathcal{D},\ 1\le j\le d).$$

Then the weak partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ of $f(x)\in L^p_{\rm loc}$ is an $L^p_{\rm loc}$-function, $(1\le j\le d)$.

Theorem 1.4.4 *Now we assume that $1<p<\infty$ and $f(x)\in L^p_{\rm loc}$ hold. Then, if the weak partial derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ of $f(x)$ exists and we have $w\text{-}\dfrac{\partial f}{\partial x_j}\in L^p_{\rm loc}$ for $1\le j\le d$, $f(x)$ is partially $L^p_{\rm loc}$-differentiable, and we have*

$$w\text{-}\frac{\partial f}{\partial x_j}=\frac{\partial f}{\partial x_j},\ (1\le j\le d)$$

for the partial derivative $\dfrac{\partial f}{\partial x_j}$ of $f(x)$ in the sense of $L^p_{\rm loc}$-convergence.

Further we have the commutability theorem of the order of partial differentiations in the following.

Theorem 1.4.5 *Assume $d\ge 2$. We assume that $f(x)\in L^p$ hold or $f(x)\in L^p_{\rm loc}$ hold. If, for $1\le i,j\le d$, $(i\ne j)$, there exist $\dfrac{\partial^2 f}{\partial x_i\partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_j\partial x_i}$ in the sense of L^p-convergence or $L^p_{\rm loc}$-convergence respectively, we have the equality*

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{\partial^2 f}{\partial x_j\partial x_i}.$$

Proof For the function $f(x)$, we have $w\text{-}\dfrac{\partial^2 f}{\partial x_i\partial x_j}$ and $w\text{-}\dfrac{\partial^2 f}{\partial x_j\partial x_i}$ in the sense of Schwartz distribution and we have

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=w\text{-}\frac{\partial^2 f}{\partial x_j\partial x_i},$$

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=w\text{-}\frac{\partial^2 f}{\partial x_j\partial x_i}.$$

Then, since we have

$$w\text{-}\frac{\partial^2 f}{\partial x_i\partial x_j}=w\text{-}\frac{\partial^2 f}{\partial x_j\partial x_i},$$

we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.//$$

Theorem 1.4.6 *For a sequence of functions $f_n(x) \in L^p$, $(n = 1, 2, 3, \cdots)$, there exist $f, g \in L^p$ such that*

$$f_n \longrightarrow f,\ (n \longrightarrow \infty),$$

$$\frac{\partial f_n}{\partial x_j} \longrightarrow g,\ (n \longrightarrow \infty),$$

for $1 \leq j \leq d$, there exists $\frac{\partial f}{\partial x_j} \in L^p$ such that we have

$$\frac{\partial f}{\partial x_j} = g.$$

Namely, the partial differential operator $\frac{\partial}{\partial x_j}$ is a closed operator.

Theorem 1.4.7 *For a sequence of functions $f_n(x) \in L^p_{\text{loc}}$, $(n = 1, 2, 3, \cdots)$, there exist $f, g \in L^p_{\text{loc}}$ such that we have*

$$f_n \longrightarrow f,\ (n \longrightarrow \infty),$$

$$\frac{\partial f_n}{\partial x_j} \longrightarrow g,\ (n \longrightarrow \infty),$$

for $1 \leq j \leq d$, then, there exists $\frac{\partial f}{\partial x_j} \in L^p_{\text{loc}}$ such that we have

$$\frac{\partial f}{\partial x_j} = g.$$

Namely the partial differential operator $\frac{\partial}{\partial x_j}$ is a closed operator.

2 New characterization of Sobolev space

In this section, we study the new characterization of Sobolev space.

Now we assume that p is a real number such that $1 \leq p < \infty$, and m is a natural number such that $m \geq 0$. Further, for $d \geq 1$, Ω is an open set in d-dimensional space $\boldsymbol{R}^d$.

Then the Sobolev space $W^{m,p}(\Omega)$ on Ω is defined in the following.

Namely, $u \in L^p(\Omega)$ belongs to $W^{m,p}(\Omega)$ if and only if, for all multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ such that $|\alpha| \leq m$, there exists weak α-derivative of u.

In this definition, the weak α-derivative u_α is an element of $L^p(\Omega)$, but this is calculated as a weak derivative.

In the real, the condition in order that this weak α-derivative u_α is an element of $L^p(\Omega)$ is not evident. It is not so easy to distinguish that this condition is satisfied or not for a concretely given function.

But, then, since, by virtue of Theorem 1.4.2, $u \in W^{m,p}(\Omega)$ is m-times partially differentiable in the sense of L^p-convergence in the case $1 < p < \infty$, we have the following theorem.

Theorem 2.1 *Now we assume that $M^{m,p}(\Omega)$ is the vector space composed of all m-times partially differentiable functions $u \in L^p(\Omega)$ in the sense of L^p-convergence. Then, for α-partial derivative $u^{(\alpha)}$ of u in the sense of L^p-convergence, we have the equality*

$$u^{(\alpha)} = u_\alpha, \quad (|\alpha| \leq m)$$

Here, α denotes a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$.

Especially, when $d = 1$, we consider the derivative instead of the partial derivative.

Then, for $1 < p < \infty$, we have

$$M^{m,p}(\Omega) = W^{m,p}(\Omega).$$

Further, we have

$$M^{m,1}(\Omega) \subsetneqq W^{m,1}(\Omega).$$

Thus, the space $M^{m,1}(\Omega)$ is a new space founded in this paper at the first time.

This gives the new characterization of Sobolev space using the concepts of the derivative and the partial derivative in the sense of L^p-convergence.

Thereby, in order to define the Sobolev space, we need not use the concept of weak derivative or the concept of the partial derivative in the sense of Schwartz distribution except the case $p = 1$.

In Kuroda [7], p.124, the definition of "L^p-derivative" is also given. But this is different from the concepts of L^p-derivative and partial L^p-derivative defined in this paper.

In Kuroda [7], in the calculation of "L^p-derivative", the calculation of the derivative or the partial derivative is not in the sense of L^p-convergence.

Theorem 2.2 *If, for $u \in W^{m,p}(\Omega)$, we denote the partial L^p-derivative $D^\alpha u$ in the following equality*

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}},$$

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d), \; |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d,$$

$W^{m,p}(\Omega)$ is a Banach space by virtue of the norm

$$||u||_{m,p} = \Big(\sum_{|\alpha| \le m} ||D^\alpha u||_p^p \Big)^{1/p}.$$

Here, $|| \cdot ||_p$ denote the norm of $L^p(\Omega)$.

Especially, for $p = 2$, $W^{m,2}(\Omega)$ is a Hilbert space by virtue of this norm. Its inner product $(\cdot, \cdot)_{m,2}$ is given by the equality

$$(u, v)_{m,2} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_2.$$

Here $(\cdot, \cdot)_2$ denote the inner product of $L^2(\Omega)$.

We have the following theorem by virtue of Theorem 1.4.3 and theorem 1.4.4.

Theorem 2.3 *Now we assume that $M_{\mathrm{loc}}^{m,p}(\Omega)$ is the vector space composed of all m-times partially differentiable function $u \in L_{\mathrm{loc}}^p(\Omega)$ in the sense of L_{loc}^p-convergence.*

Especially, when $d = 1$, we consider the derivative instead of the partial derivative.

Then, for $1 < p < \infty$, we have

$$M_{\mathrm{loc}}^{m,p}(\Omega) = W_{\mathrm{loc}}^{m,p}(\Omega).$$

Further, we have

$$M_{\mathrm{loc}}^{m,1}(\Omega) \subsetneqq W_{\mathrm{loc}}^{m,1}(\Omega).$$

Thus, the space $M_{\mathrm{loc}}^{m,1}(\Omega)$ is a new space founded in this paper at the first time.

3 New meaning of Stone's Theorem

By using the concept of derivative in the sense of L^2-convergence, we give the new meaning of Stone's Theorem.

As for Stone's Theorem, we refer to Kuroda [7].

Thus, in the calculation of the generating operator of 1-parameter semigroup or 1-parameter group, we have really used the calculation of L^p-derivative or, especially, L^2-derivative already.

The concept of L^p-derivative in this paper is a generalization of these results of concrete examples.

Theorem 3.1 (1-parameter group) *Put* $\mathcal{H} = L^2(-\infty,\infty)$. *For* $u(s) \in \mathcal{H}$, *we define the unitary operator* $U_t, (-\infty < t < \infty)$ *on* $\mathcal{H}$ *by virtue of the relation*

$$(U_t u)(s) = u(t+s), \ (-\infty < s, t < \infty).$$

Then the system of unitary operators $\{U_t; -\infty < t < \infty\}$ *is 1-parameter group. Namely, this satisfies the following conditions* (1) $\sim$ (3) :

(1) $U_s U_t = U_{s+t}, \ (-\infty < s, t < \infty)$.

(2) $U_0 = I$. *Here the symbol* I *denotes the identity operator of* $\mathcal{H}$.

(3) $\lim_{t \to t_0} U_t = U_{t_0}$*(in the strong sense)*, $(-\infty < t_0 < \infty)$.

Theorem 3.2 (Stone's Theorem) *For the 1-parameter group* $\{U_t; -\infty < t < \infty\}$ *in Theorem* 3.1, *there exists*

$$\lim_{t \to 0} \frac{1}{t} \Big(U_t - U_0 \Big) = A \ (\textit{in the strong sense})$$

such that the following (1), (2) *are satisfied* :

(1) $H = \dfrac{1}{i}\dfrac{d}{ds} = \dfrac{1}{i}A$.

(2) $U_t = \exp\big(tA \big) = \exp\big(itH \big), \ (-\infty < t < \infty)$.

Theorem 3.3 *We use the notation in Theorem* 3.2. *Then, for* $u \in W^{1,2}(-\infty,\infty)$, *we have*

$$\frac{d}{dt} U_t u = U_t A u = A U_t u, \ (-\infty < t < \infty)$$

in the sense of L^2*-convergence.*

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