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Energy Decay for a Dissipative Wave Equation with Compactly Supported Data

By

Kosuke ONO

Department of Mathematical Sciences
The University of Tokushima
Tokushima 770-8502, JAPAN
e-mail : ono@ias.tokushima-u.ac.jp

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Abstract

Consider the Cauchy problem for the dissipative wave equation : $u_{tt} - \Delta u + u = 0$, $u = u(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ with $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$. If $\{u_0, u_1\}$ are compactly supported data from the energy space, then there exists a domain X_m in $\mathbb{R}^N$ such that $\{x \in \mathbb{R}^N \mid |x| \geq t^{1/2+\delta}\} \subsetneqq X_m$ for large $t \geq 0$ and $\int_{X_m} (|u_t|^2 + |\nabla u|^2)\, dx \leq C(1+t)^{-m}$ with $m > 0$ for $t \geq 0$, and moreover, if $u_0 + u_1 = 0$, then $\int_{X_m} |u|^2\, dx \leq C(1+t)^{-m}$ for $t \geq 0$.

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1 Introduction

We are concerned with the Cauchy problem for the dissipative wave equation :

$$u_{tt} - \Delta u + u_t = 0\,, \quad u = u(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,\infty) \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x)\,, \tag{1.2}$$

where $\Delta = \nabla \cdot \nabla = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the Laplacian in $\mathbb{R}^N$.

We assume that $\{u_0, u_1\}$ are compactly supported data from the energy space :

$$u_0 \in H^1(\mathbb{R}^N)\,, \quad u_1 \in L^2(\mathbb{R}^N) \tag{1.3}$$

and

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B(K) \tag{1.4}$$

with $K > 0$, where $B(K)$ is an open ball with center 0 and radius K :

$$B(K) \equiv \{x \in \mathbb{R}^N \mid |x| < K\}.$$

Then, it is well known that the problem (1.1)–(1.2) with (1.3)–(1.4) admits a unique global solution $u(t)$ on $[0, \infty)$ such that

$$u(t) \in C([0, \infty); H^1(\mathbb{R}^N)) \cap C^1([0, \infty); L^2(\mathbb{R}^N))$$

(see [2], [5]) and

$$\operatorname{supp} u(t) \subset B(t + K) \qquad \text{for} \quad t \geq 0. \tag{1.5}$$

By the standard energy method, we obtain the following energy estimate :

$$E(t) \leq CE(0)(1 + t)^{-1} \qquad \text{for} \quad t \geq 0$$

where

$$E(t) \equiv \|u_t(t)\|^2 + \|\nabla u(t)\|^2 = \int_{\mathbb{R}^N} \left(|u_t(x, t)|^2 + |\nabla u(x, t)|^2\right) dx$$

and $E(0) = \|u_1\|^2 + \|\nabla u_0\|^2$, and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^N)$ (see [1], [3], [4]).

On the other hand, Todorova and Yordanov [6] have been obtained the following decay estimate :

$$\int_{B(t^{1/2+\delta})^c} \left(|u_t|^2 + |\nabla u|^2\right) dx \leq CE(0) \exp(-t^{2\delta}/2) \tag{1.6}$$

with $\delta > 0$, under the assumptions (1.3) and (1.4). Here, $B(K)^c$ is the complement of $B(K)$, that is,

$$B(K)^c \equiv \mathbb{R}^N \setminus B(K) = \{x \in \mathbb{R}^N \mid |x| \geq K\}.$$

We are interested in the decay estimate for larger domains than $B(t^{1/2+\delta})^c$.

When $m > 0$ and $\delta > 0$, it is easy to see that for large $t > 0$,

$$(t + K)^{1/2} \log(1 + t)^m < t^{1/2+\delta}$$

and hence

$$B(t^{1/2+\delta})^c \subsetneq B((t + K)^{1/2} \log(1 + t)^m)^c.$$

The purpose of this paper is to derive the decay estimate for large domain of integral $B((t + K)^{1/2} \log(1 + t)^m)^c$ than $B(t^{1/2+\delta})^c$ in (1.6).

Our main result is as follows.

Theorem 1.1 *Let* $m > 0$. *Suppose that the initial data* $\{u_0, u_1\}$ *satisfy the conditions* (1.3) *and* (1.4). *Then the solution* u *of* (1.1)–(1.2) *satisfies*

$$\int_{B((t+K)^{1/2}\log(1+t)^m)^c} \left(|u_t|^2 + |\nabla u|^2\right) dx \le e^K E(0)(1+t)^{-m} \tag{1.7}$$

for $t \ge 0$. *Moreover, if* $u_0 + u_1 = 0$, *then*

$$\int_{B((t+K)^{1/2}\log(1+t)^m)^c} |u|^2\, dx \le e^K \|u_0\|^2 (1+t)^{-m} \tag{1.8}$$

for $t \ge 0$.

Theorem 1.1 follows from Theorem 2.2 and Theorem 2.3 in next section.

2 Decay Estimates

The function $\psi(x,t) \equiv \frac{1}{2}\left(t + K - \sqrt{(t+K)^2 - |x|^2}\right)$ given by [6] plays an important role through this paper. It is easy to see that

$$\begin{aligned}
\psi &= \frac{1}{2}\frac{|x|^2}{t + K + \sqrt{(t+K)^2 - |x|^2}}, \\
\psi_t &= \frac{1}{2}\left(1 - \frac{t+K}{\sqrt{(t+K)^2 - |x|^2}}\right) = -\frac{\psi}{\sqrt{(t+K)^2 - |x|^2}}, \\
\psi_t^2 &= \frac{1}{4}\left(1 + \frac{(t+K)^2}{(t+K)^2 - |x|^2} - 2\frac{t+K}{\sqrt{(t+K)^2 - |x|^2}}\right), \\
|\nabla\psi|^2 &= \frac{1}{4}\frac{|x|^2}{(t+K)^2 - |x|^2},
\end{aligned} \tag{2.1}$$

and then we obtain the following.

Lemma 2.1 *The function* $\psi(x,t) \equiv \frac{1}{2}\left(t + K - \sqrt{(t+K)^2 - |x|^2}\right)$ *for* $|x| < t + K$ *satisfies*

$$\psi(x,t) \ge 0\,, \qquad \psi_t(x,t) = \psi_t(x,t)^2 - |\nabla\psi(x,t)|^2 \tag{2.2}$$

and

$$\frac{1}{4}\frac{|x|^2}{t+K} \le \psi(x,t) \le \frac{1}{2}(t+K)\,. \tag{2.3}$$

The following decay estimate means (1.7).

Theorem 2.2 *Let $m > 0$. Suppose that the initial data $\{u_0, u_1\}$ satisfy the conditions* (1.3) *and* (1.4). *Then the solution u of* (1.1)–(1.2) *satisfies*

$$\int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} \left(|u_t|^2 + |\nabla u|^2\right) dx \leq I_1^2(1+t)^{-m} \tag{2.4}$$

for $t \geq 0$, where

$$I_1^2 \equiv \int_{\mathbb{R}^N} e^{2\psi(x,0)} \left(|u_1(x)|^2 + |\nabla u_0(x)|^2\right) dx \leq e^K E(0)\,.$$

Proof. Multiplying (1.1) by $2e^{2\psi}u_t$, we have

$$\begin{aligned} 0 &= e^{2\psi}\left(\frac{d}{dt}(u_t^2 + |\nabla u|^2) - 2\,\mathrm{div}(u_t\nabla u) + 2u_t^2\right) \\ &= \frac{d}{dt}\left(e^{2\psi}(u_t^2 + |\nabla u|^2)\right) - 2\,\mathrm{div}(e^{2\psi}u_t\nabla u) + \frac{2e^{2\psi}}{(-\psi_t)}P(x,t) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} P(x,t) &\equiv (\psi_t^2 - \psi_t)u_t^2 - 2\psi_t u_t \nabla\psi\cdot\nabla u + \psi_t^2|\nabla u|^2\,, \quad \text{by (2.1)} \\ &= u_t^2|\nabla\psi|^2 - 2\psi_t u_t\nabla\psi\cdot\nabla u + \psi_t^2|\nabla u|^2 \\ &= |u_t\nabla\psi - \psi_t\nabla u|^2 \quad (\geq 0)\,. \end{aligned}$$

When $x \neq 0$, we see $\psi_t < 0$ (by (2.1)) and hence $P/(-\psi_t) \geq 0$. When $x = 0$, we see $\psi_t = 0$ and $|\nabla\psi| = 0$ and hence $P/(-\psi_t) = u_t^2 \geq 0$. Moreover, we see from (1.5) that $\mathrm{supp}\,P(\cdot,t) \subset B(t+K)$ for $t \geq 0$.

Integrating (2.5) over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\left(\|e^\psi u_t\|^2 + \|e^\psi\nabla u\|^2\right) \leq 0$$

and hence

$$\|e^\psi u_t\|^2 + \|e^\psi\nabla u\|^2 \leq \|e^{\psi(\cdot,0)}u_1\|^2 + \|e^{\psi(\cdot,0)}\nabla u_0\|^2 \quad (\equiv I_1^2) \tag{2.6}$$

for $t \geq 0$. By (2.3), it is easy to see that $I_1^2 \leq e^K E(0)$.

On the other hand, we observe from (1.5) that for $t > 0$,

$$\begin{aligned} \|e^\psi u_t\|^2 + \|e^\psi\nabla u\|^2 &= \int_{|x|<t+K} e^{2\psi}(|u_t|^2 + |\nabla u|^2)\,dx\,, \quad \text{by (2.3)} \\ &\geq \int_{|x|<t+K} e^{\frac{1}{2}\frac{|x|^2}{t+K}}\left(|u_t|^2 + |\nabla u|^2\right) dx \\ &\geq \int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} e^{\frac{1}{2}\frac{|x|^2}{t+K}}\left(|u_t|^2 + |\nabla u|^2\right) dx \\ &\geq (1+t)^m \int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} \left(|u_t|^2 + |\nabla u|^2\right) dx\,. \end{aligned} \tag{2.7}$$

Therefore, we obtain from (2.6) and (2.7) that

$$\int_{|x|\geq (t+K)^{1/2}\log(1+t)^m} \left(|u_t|^2 + |\nabla u|^2\right) dx \leq I_1^2(1+t)^{-m}$$

for $t \geq 0$, which implies the desired estimate (2.4). □

The following decay estimate means (1.8).

Theorem 2.3 *Let $m > 0$. Suppose that the initial data $\{u_0, u_1\}$ satisfy the conditions* (1.3) *and* (1.4). *Then the solution u of* (1.1)–(1.2) *satisfies*

$$\int_{|x|\geq (t+K)^{1/2}\log(1+t)^m} |u|^2\, dx \leq I_0^2(1+t)^{-m} \tag{2.8}$$

for $t \geq 0$, where

$$I_0^2 \equiv \int_{\mathbb{R}^N} e^{2\psi(x,0)} u_0(x)^2\, dx \leq e^K \|u_0\|^2 .$$

Proof. Putting

$$w(x,t) = \int_0^t u(x,s)\, ds$$

for the solution $u = u(x,t)$ of (1.1)–(1.2), we observe that $w_t = u$, $w(x,0) = 0$ and

$$u_t + u - \Delta w = u_0 + u_1 \qquad \text{in} \quad \mathbb{R}^N \times (0,\infty). \tag{2.9}$$

Multiplying (1.1) by $2e^{2\psi}u$, we have

$$2e^{2\psi}(u_0+u_1)u$$
$$= e^{2\psi}\left(\frac{d}{dt}u^2 + 2u^2 - 2\,\mathrm{div}(u\nabla w) + \frac{d}{dt}|\nabla w|^2\right) \tag{2.10}$$
$$= \frac{d}{dt}\left(e^{2\psi}(u^2 + |\nabla w|^2)\right) - 2\,\mathrm{div}(e^{2\psi}u\nabla w) + \frac{2e^{2\psi}}{(-\psi_t)}Q(x,t) \tag{2.11}$$

where

$$\begin{aligned} Q(x,t) &\equiv (\psi_t^2 - \psi_t)u^2 - 2\psi_t u\nabla\psi\cdot\nabla w + \psi_t^2|\nabla w|^2, \quad \text{by (2.1)} \\ &= u^2|\nabla\psi|^2 - 2\psi_t u\nabla\psi\cdot\nabla w + \psi_t^2|\nabla w|^2 \\ &= |u\nabla\psi - \psi_t\nabla w|^2 \quad (\geq 0). \end{aligned}$$

When $x \neq 0$, we see $\psi_t < 0$ (by (2.1)) and hence $Q/(-\psi_t) \geq 0$. When $x = 0$, we see $\psi_t = 0$ and $|\nabla\psi| = 0$ and hence $Q/(-\psi_t) = u^2 \geq 0$. Moreover, we see from (1.5) that $\mathrm{supp}\, Q(\cdot,t) \subset B(t+K)$ for $t \geq 0$.

Integrating (2.11) over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\left(\|e^{\psi}u\|^2 + \|e^{\psi}\nabla w\|^2\right) \leq 2\int_{\mathbb{R}^N} e^{2\psi}(u_0+u_1)u\,dx$$

If $u_0 + u_1 = 0$, then we observe

$$\|e^{\psi}u\|^2 \leq \|e^{\psi(\cdot,0)}u_0\|^2 \quad (\equiv I_0^2) \tag{2.12}$$

for $t \geq 0$. By (2.3), it is easy to see that $I_0^2 \leq e^K\|u_0\|^2$.

On the other hand, we observe from (1.5) that for $t > 0$,

$$\begin{aligned}
\|e^{\psi}u\|^2 &= \int_{|x|<t+K} e^{2\psi}|u|^2\,dx\,, \quad \text{by (2.3)} \\
&\geq \int_{|x|<t+K} e^{\frac{1}{2}\frac{|x|^2}{t+K}}|u|^2\,dx \\
&\geq \int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} e^{\frac{1}{2}\frac{|x|^2}{t+K}}|u|^2\,dx \\
&\geq (1+t)^m\int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} |u|^2\,dx\,.
\end{aligned} \tag{2.13}$$

Therefore, we obtain from (2.12) and (2.13) that

$$\int_{|x|\geq(t+K)^{1/2}\log(1+t)^m} |u|^2\,dx \leq I_0^2(1+t)^{-m}$$

for $t \geq 0$, which implies the desired estimate (2.8). □

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