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Nimstring Values for $2 \times n$ Rectangular Arrays II

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Abstract

In the present paper, succeeding the previous paper [4], we continue to study Nimstring values of $2 \times n$ rectangular arrays.

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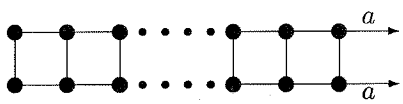
Introduction

In this paper, our main purpose is to obtain the value of an array with two arrows like Figure 1. It has m boxes and two arrows, and it is described as $R_m a^2$. The arrow is denoted by a.

Figure 1

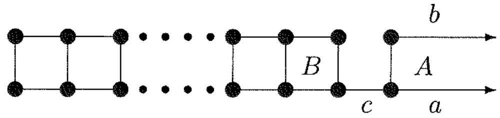
$R_m a^2 =$ 

1 Arrays of form $R_m cA$

Let A be a graph composed of two arrows a, b and a v-edge connecting them. In this section, we study a graph $R_m cA$ composed of m boxes and A which is described in Figure 2 below.

Figure 2

$R_m cA$ =



Let both the sizes of a and of b be x and that of c be y. Let B be the rightmost box with size z. Moreover, let the size of the box next ot B be w. Put $G = R_m cA$

Proposition 1.*(1) In the case $x = y = 1$ and $z = 1$ or $z \geq 4$, the value $|G|$ is 0 (resp $*$) if m is odd (resp. even).*

(2) In the cases $(x = 2,3, y = 2,3, z \geq 1), (x = 1,2,3, y \geq 4, z \geq 1), (x = 3, y = 1, z \geq 1), (x = 2, y = 1, z = 1,2,3)$, the value $|G|$ is $*$ (resp. 0) if m is odd (resp. even), except the case $m = 1, x = 2, y = 1, z = 1,2,3$ in which its value is $*2$.

(3) In the cases $(x = 1, y = 2,3, z \geq 1), (x = y = 1, z = 2,3), (x = 2, y = 1, z \geq 4)$, the value $|G|$ is $*2$ (resp. $*3$) if m is odd (resp. even), except the case $m = 1, x = 1, y = 1, z = 2,3$ in which its value is 0.

(4) In the case $x \geq 4$ and $y = z = 1$ or $y = 1,2,3, z \geq 4$, the value $|G|$ is $*$ (resp. 0) if m is odd (resp, even).

(5) In the case $x \geq 4$ and $y = 2,3, z = 1,2,3$ or $y = 1, z = 2,3$, the value $|G|$ is $*3$ (resp. $*2$) if m is odd (resp. even).

Proof. Let e be the rightmost inner v-edge of R_m. By removing an inner h-edge of R_m, we get a subgraph $H = R_{n_1} d R_{n_2} cA$, where $n_1 + n_2 = m - 1$ and d is a connection of R_{n_1} and $R_{n_2} cA$.

(1) Let m be odd. The value $|c|$ (correctly, the value of a proper edge of c) is $*$. By Proposition 3 in [4], the values $|a|$ and $|b|$ are both $*3$ (resp. $*$) if $z = 1$ (resp. $z \geq 4$). If $z = 1$, the value of a h-edge of B is $*3$ by induction. The value $|e|$ is $*3$ (resp. $*$) if $z = 1, w = 1,2$ (resp. $z = 1, w \geq 3$ or $z \geq 4, w \geq 1$). The values of the other inner v-edges are $*$. We prove the value $|d|$ in H is 0. If both n_1 and n_2 are odd (resp. even), The values $|R_{n_1}|$ and $|R_{n_2} cA|$ are both 0 (resp. $*$). Hence, the value $|H|$ is not 0. Thus, the value $|G|$ is 0, because the values of all its edges are not 0.

Let m be even. The value $|c|$ is 0. The value $|a|$ and $|b|$ are both $*2$ (resp. 0) if $z = 1$ (resp. $z \geq 4$), by Proposition 3 in [4]. If $z = 1$, the value of a h-edge of B is $*2$ by induction. The value $|e|$ is $*2$ (resp. 0) if $z = 1, w = 1,2$ (resp.

$z = 1, w \geq 3$ or $z \geq 4$). The values of the other inner v-edges are 0. We prove the value $|d|$ in H is $*$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*$ (resp. 0). Hence, the value $|H|$ is not $*$. Thus, the value $|G|$ is $*$, because G has some edges with value 0 and the values of all its edges are not $*$.

(2) Let m be odd. The value $|c|$ is 0 (resp. loony) if $x = 2, 3$ and $y = 1, 2, 3$ (resp. $x = 1, 2, 3$ and $y \geq 4$). The value $|a|$ is 0 and $|b|$ is $*3$ (resp. 0) if $x = 2, y = 1$ (resp. $x = 2, 3, y = 2, 3$ or $x = 3, y = 1$). If $z = 1, 2, 3$, the values of outer edges of B are 0. The value $|e|$ is 0 (resp. $*3$) if $(x = 2, 3, y = 2, 3), (x = 1, 2, 3, y \geq 4), (x = 3, y = 1)$ or ($x = 1, y = 2$ and $z = 1, w = 1, 2$ or $z = 2, w = 1$) (resp. $x = 1, y = 2$ and $(z = 1, w \geq 3), (z = 2, w \geq 2)$ or $z = 3$). The values of the other inner v-edges are 0. We prove the value $|d|$ in H is $*$ except the case $n_2 = 1, x = 2, y = 1$ and $z = 1, 2, 3$. If n_1 and n_2 are odd (resp. even), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*$ (resp. 0). When $n_2 = 1, x = 2, y = 1$ and $z = 1, 2, 3$, the value of the lower h-edge of B in $H = R_{m-1}dR_1cA$ is $*$. Hence, the value $|H|$ is not $*$. Thus, the value $|G|$ is $*$, because it has some edges with value 0 and the values of all its edges are not $*$.

Let m be even. The value $|c|$ is $*$ (resp. loony) if $x = 2, 3$ and $y = 1, 2, 3$ (resp. $x = 1, 2, 3$ and $y \geq 4$). The value $|a|$ is $*$, and $|b|$ is $*2$ (resp. $*$) if $x = 2, y = 1$ (resp. $(x = 2, 3, y = 2, 3), (x = 1, 2, 3, y \geq 4)$ or $(x = 3, y = 1)$). If $z = 1, 2, 3$, the values of outer edges of B are $*$. The value $|e|$ is $*$ (resp. $*2$) if $(x = 2, 3, y = 2, 3), (x = 1, 2, 3, y \geq 4), (x = 3, y = 1)$ or ($x = 2, y = 1$ and $z = 1, w = 1, 2$ or $z = 2, w = 1$) (resp. $x = 1, y = 2$ and $(z = 1, w \geq 3), (z = 2, w \geq 2)$ or $(z = 3, w \geq 1)$). The values of the other inner v-edges are $*$. We prove the value $|d|$ in H is 0 except the case $n_2 = 1, x = 2, y = 1$ and $z = 1, 2, 3$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), the values $|R_{n_1}|$ and $|R_{n_2}cA|$ are both 0 (resp. $*$). When $n_2 = 1, x = 2, y = 1$ and $z = 1, 2, 3$, the value of the lower h-edge of B in $H = R_{m-1}dR_1cA$ is 0. Hence, the value $|H|$ is not 0. Thus, the value $|G|$ is 0, because the values of all its edges are not 0.

(3) Let m be odd. The value $|c|$ is $*$ (resp. 0) if $x = 1$ (resp. $x = 2$). The values $|a|$ is 0. The value $|b|$ is $*$, if $z \geq 4$ and $x = 1, y = 2$ or $x = 2, y = 1$. This value is 0 (resp. $*3$) if $x = 1, y = 3, z \geq 1$ (resp. $x = 1, y = 2, z = 1, 2, 3$ or $x = 1, y = 1, z = 2, 3$). If $z = 1, 2, 3$, the values of outer edges of B are 0 or $*3$. The value $|e|$ is $*$ (resp. $*3$) if $(x = 1, y = 1)$ and $z = 2, w \geq 2$ or $z = 3, w \geq 1$ (resp. $(x = 1, y = 2, 3, z \geq 1), (x = 1, y = 1, z = 2, w = 1)$ or $(x = 1, y = 1, z \geq 4)$). The values of the other inner v-edges are $*3$. We prove the value $|d|$ in H is $*2$ except the case $n_2 = 1, x = y = 1, z = 2, 3$. If n_1 and n_2 are odd (resp. even), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*2$ (resp. $*3$). When $n_2 = 1, x = y = 1$ and $z = 2, 3$, the value $|d|$ in $H = R_{m_1}dR_1A$ is $*2$, by Lemma 2 below. Hence, the value $|H|$ is not $*2$. Thus, the value $|G|$ is $*2$, because G has some edges with value 0 and ones with value $*$, and the values of its all edges are not $*2$.

Let m be even. The value $|c|$ is 0 (resp. $*$) if $x = 1$ (resp. $x = 2$). The values $|a|$ is $*$. The value $|b|$ is 0, if $z \geq 4$ and $x = 1, y = 2$ or $x = 2, y = 1$. This value is $*$ (resp. $*2$) if $x = 1, y = 3, z \geq 1$ (resp. $x = 1, y = 2, z = 1, 2, 3$ or $x = 1, y = 1, z = 2, 3$). If $z = 1, 2, 3$, the values of outer edges of B are $*$(resp. $*$ or $*2$), if $x = 1, y = 3, z = 1$ or $z = 2, 3$ (resp. $x = 1, y = 2, z = 1$). The value $|e|$ is $*2$ (resp. 0) if $(x = 1, y = 2, 3, z \geq 1), (x = y = 1, z = 2, w = 1)$ or $(x = 2, y = 1, z \geq 4)$ (resp. $(x = y = 1, z = 2, w \geq 2)$ or $(x = y = 1, z = 3, w \geq 1)$). The values of the other inner v-edges are $*2$. We prove the value $|d|$ in H is $*3$ except the case $n_2 = 1, x = y = 1, z = 2, 3$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*3$ (resp. $*2$). When $n_2 = 1, x = y = 1$ and $z = 2, 3$, the value $|b|$ in $H = R_{m_1}dR_1A$ is $*3$, by Lemma 2 below. Hence, the value $|H|$ is not $*3$. Thus, the value $|G|$ is $*3$, because G has some edges with value 0, ones with value $*$ and ones with value $*2$, and the values of all its edges are not $*3$.

(4) Let m be odd. The value $|c|$ is 0. If $z = 1$, the value of an outer edge of B is 0. The value $|e|$ is 0 or $*2$. The values of the other inner v-edges of R_m are 0. We prove the value $|d|$ in H is $*$. If n_1 and n_2 are odd (resp. even), the value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*$ (resp. 0). Hence, the value $|H|$ is not $*$. Thus, the value $|G|$ is $*$, because G has some edges with value 0 and the values of all its edges are not $*$.

Let m be even. The value $|c|$ is $*$. If $z = 1$, the value of an outer edge of B is $*$. The value $|e|$ is $*$ or $*3$. The values of the other inner v-edges of R_m are $*$. We prove the value $|d|$ in H is 0. If n_1 is odd (resp. even) and n_2 is even (resp. odd), The values $|R_{n_1}|$ and $|R_{n_2}cA|$ is 0 (resp. $*$). Hence, the value $|H|$ is not 0. Thus, the value $|G|$ is 0, because the values of all edges of G are not 0.

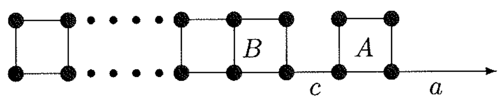
(5) Let m be odd. The value $|c|$ is 0. The value of the lower h-edge of B is $*$ and values of the other outer edges are $0, *$ or $*2$. The value $|e|$ is 0 or $*2$. The values of the other inner v-edges of R_m are $*2$. We prove the value $|d|$ in H is $*3$ except the case $n_2 = 1, y = 1, 2, 3$. If n_1 and n_2 are odd (resp. even), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*3$ (resp. $*2$). When $n_2 = 1, y = 1, 2, 3$, the value of $H = R_{m-1}dR_1cA$ is $*2$, by Lemma 3 below. Hence, the value $|H|$ is not $*3$. Thus, the value $|G|$ is $*3$, because G has some edges with value 0, ones withe value $*$ and ones with value $*2$ and the values of all its edges are not $*3$.

Let m be even. The value $|c|$ is $*$. The value of the lower h-edge of B is 0 and values of the other outer edges are $0, *$ or $*3$. The value $|e|$ is $*$ or $*3$. The values of the other inner v-edges of R_m are $*3$. We prove the value $|d|$ in H is $*2$ except the case $n_2 = 1, y = 1, 2, 3$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}cA|$ is $*2$ (resp. $*3$). When $n_2 = 1, y = 1, 2, 3$, the value of $H = R_{m-1}dR_1cA$ is $*3$, by Lemma 3 below. Hence, the value $|H|$ is not $*2$. Thus, the value $|G|$ is $*2$, because G has some edges with value 0 and ones withe value $*$, and the values of all its edges are not $*2$.

Put $G = R_m cAa$, where Aa is a box with an arrow a and c is a connection of R_m and Aa. Let the size of A be 2 or 3, that of a be 2 or 3 and that of c be $1, 2$ or 3. Let the rightmost box of R_m be B. As in Figure 3, G is described.

Figure 3

$R_m cAa$ =



Lemma 2. The value of $G = R_m cAa$ is $*2$ (resp. $*3$) if m is odd (resp. even).

Proof. Let the size of a be x, that of A be y, that of c be z and that of B be w. By removing an inner h-edge of R_m, we get a subgraph $H = R_{n_1} d R_{n_2} cAa$, where $n_1 + n_2 = m - 1$ and d is a connection of R_{n_1} and $R_{n_2} cAa$.

Let m be odd. The value $|a|$ is $*$ or $*3$ by Proposition 2 in[4], $|c|$ is $*$ and the value of the lower h-edge of A is 0. The values of h-edges of B are $*3$ (resp. 0 or $*3$), if $z = 1, w = 1$ (resp. $z = 2, w = 1$ or $z = 1, w = 2$). These values are 0 if $z = 2, w = 2, 3$ or $z = 3, w = 1, 2, 3$. The values of the inner v-edges of R_m are $*3$. We show the value $|d|$ in H is $*2$ to prove $|H|$ is not $*2$. If n_1 and n_2 are odd (resp. even), the value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2} cAa|$ is $*2$ (resp. $*3$). Thus, the value $|G|$ is $*2$, because G has some edges with value 0 and ones with value $*$, and the value of all its edges are not $*2$.

Let m be even. The value $|a|$ is 0 or $*2$ by Proposition 2 in [4], $|c|$ is 0 and the value of the lower h-edge of A is $*$. The values of h-edges of B are $*2$ (resp. 0 or $*2$), if $z = 1, w = 1$ (resp. $z = 2, w = 1$ or $z = 1, w = 2$). These values are $*$ if $z = 2, w = 2, 3$ or $z = 3, w = 1, 2, 3$. The values of the inner v-edges of R_m are $*2$. We show the value $|d|$ in H is $*3$ to prove $|H|$ is not $*3$. If n_1 is odd(resp. even) and n_2 is even (resp. odd), the value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2} cAa|$ is $*3$ (resp. $*2$). Thus, the value $|G|$ is $*3$, because G has some edges with value 0, ones with value $*$ and ones with value $*2$, and the value of all its edges are not $*3$.

Lemma 3. Let G be a graph $R_m dBcA$, where A is the subgraph, c is the connection given in Proposition 1, B is a box of size z and d is a connection. Assume z is 2 or 3 and the size of d is $1, 2$ or 3. Then, the value $|G|$ is $*2$ (resp. $*3$) if m is odd (resp. even).

Proof. By removing an inner h-edge of R_m, we get a subgraph $H = R_{n_1} e R_{n_2} dBcA$, where $n_1 + n_2 = m - 1$ and e is a connection of R_{n_1} and $R_{n_2} dBcA$.

Let m be odd. The value $|c|$ is $*$ or $*3$ by Proposition 2 in [4] and $|d|$ is $*$. The value of the lower h-edge of B is 0. The values of the inner v-edges of R_m are $*3$. If we can remove an outer edge of the rightmost box of R_m, its value is $*3$ or 0. We prove the value $|e|$ in H is $*2$. If n_1 and n_2 are odd (resp. even), the value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}dBcA|$ is $*2$ (resp. $*3$). Hence, the value $|H|$ is not $*2$. Thus, the value $|G|$ is $*2$, because G has some edges with value 0 and ones withe value $*$, and the values of all its edges are not $*2$.

Let m be even. The value $|c|$ is $0*$ or $*2$ by Proposition 2 in [4] and $|d|$ is 0. The value of the lower h-edge of B is $*$. The values of the inner v-edges of R_m are $*2$. If we can remove an outer edge of the rightmost box of R_m, its value is $*2$ or $*$. We prove the value $|e|$ in H is $*3$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}dBcA|$ is $*3$ (resp. $*2$). Hence, the value of H is not $*3$. Thus, the value $|G|$ is $*3$, because G has some edges with value 0, ones withe value $*$ and ones with value $*2$ and the values of all its edges are not $*3$.

2 Arrays with two arrows

Let $G = R_m a^2$ be an array which has m boxes and two arrows. Let the size of the arrow a be x. Let A be the rightmost box of R_m and B be the box next to A. Let the sizes of A and B be y and z respectively.

Proposition 4.*(1) In the cases $x = y = 1, z = 1, 2, 3$ or $x = 1, 2, 3, y \geq 4, z \geq 1$ except the case $m = 2, x = y = 1, z = 1, 2, 3$, the value $|G|$ is $*2$ (resp. $*3$), if m is odd (resp. even). When $m = 2, x = 1, y = 1$ and $z = 1, 2, 3$, its value is $*$.*

(2) In the case $x = 1, y = 1, z \geq 4, m \geq 2$, the value $|G|$ is 0 (resp. $*$), if m is odd (resp. even).

(3) In the cases $x = 1, y = 2, 3, z \geq 1$ or $x = 2, 3, y = 1, 2, 3, z \geq 1$ except $m = 1$, the value $|G|$ is $*$ (resp. 0) if m is odd (resp. even) except $m = 1$. When $m = 1$, its value is $*2$.

(4) In the case $x \geq 4, y \geq 1, z \geq 1$, the value $|G|$ is $*$ (resp. 0) if m is odd (resp. even)

Proof. When $m = 1$ or $m = 2$, we can get the results directly in any cases. By removing an inner h-edge of R_m, we get a subgraph $H = R_{n_1} c R_{n_2} a^2$, where $n_1 + n_2 = m - 1$ and c is a connection of R_{n_1} and $R_{n_2} a^2$. Let the rightmost v-edge of R_m be denoted by e_1, and the v-edge next to e_1 be denoted by e_2.

(1) Let m be odd. By Proposition 3 in [4], we have $|a| = *$. By induction, we get $|e_1| = 0$, and $|e_2| = 0$ (resp. $|e_2| = *3$) if $x = y = 1, z = 1, 2$ (resp. $x = y = 1, z = 3$ or $x = 1, 2, 3, y \geq 4$). The values of the other v-edges of R_m

are $*3$ (resp. $*$ or $*3$) if $y \geq 4$ (resp. $x = y = 1$). The value of a h-edge of A is $*$ (resp. $*3$), if $x = y = z = 1$ (resp. $x = y = 1, z = 2, 3$), by Proposition 1. We show the value $|c|$ in H is $*2$ except the case $m_2 = 2, x = y = 1, z = 1, 2, 3$. If m_1 and m_2 are odd (resp. even), we have $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a_2| = *2$ (resp. $|R_{m_2}a_2| = *3$). When $m_2 = 2, x = y = 1, z = 1, 2, 3$, the value of a h-edge of A in H is also $*2$. Hence, we get $|H| \neq *2$. Thus $|G| = *2$, because G has some edges with value 0 and ones with value $*$, and the values of all its edges are not $*2$.

Let m be even. By Proposition 3 in [4], we have $|a| = 0$. By induction, we get $|e_1| = *$, and $|e_2| = *$ (resp. $|e_2| = *2$) if $x = y = 1, z = 1, 2$ (resp. $x = y = 1, z = 3$ or $x = 1, 2, 3, y \geq 4$). The value of the v-edge next to e_2 is $*2$ (resp. 0 or $*2$) if $y \geq 4$ (resp. $x = y = 1$). The value of the other v-edges of R_m are $*2$. The value of a h-edge of A is 0 (resp. $*2$), if $x = y = z = 1$ (resp. $x = y = 1, z = 2, 3$), by Proposition 1. We show the value $|c|$ in H is $*3$ except the case $m_2 = 2, x = y = 1, z = 1, 2, 3$. If m_1 is odd (resp. even) and m_2 is even (resp. odd), we get $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a^2| = *3$ (resp. $|R_{m_2}a^2| = *2$. When $m_2 = 2, x = y = 1, z = 1, 2, 3$, the value of a h-edge of A in H is also $*3$. Hence, we get $|H| \neq *3$, Thus $|G| = *3$, because G has some edges with value 0, ones with value $*$ and ones with value $*2$, and the values of all its edges are not $*3$

(2) Let m be odd. By Proposition 3 in [4], we have $|a| = *$. By induction, we get $|e_1| = |e_2| = *3$. The values of the other v-edges of R_m are $*$. The value of a h-edge of A is $*$, by Proposition 1. We show the value $|c|$ in H is 0. If m_1 and m_2 are odd (resp. even),we have $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a^2| = 0$ (resp. $|R_{m_2}a^2| = *$). Hence, we get $|H| \neq 0$. Thus $|G| = 0$, because the values of all edges of G are not 0.

Let m be even. By Proposition 3 in [4], we have $|a| = 0$. By induction, we get $|e_1| = |e_2| = *2$. The values of the other v-edges of R_m are 0. The value of a h-edge of A is 0, by Proposition 1. We show the value $|c|$ in H is $*$. If m_1 is odd (resp. even) and m_2 is even (resp. odd), we have $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a^2| = *$ (resp. $|R_{m_2}a^2| = 0$). Hence, we get $|H| \neq *$. Thus $|G| = *$, because G has some edges with value 0, and the values of all its edges are not $*$.

(3) Let m be odd. By Proposition 3 in [4], we have $|a| = *3$. By induction, we get $|e_1| = 0$ (resp.$|e_1| = *3$), if $(x = 2, 3, y = 2, 3, z \geq 1), (x = 3, y = 1, z \geq 1), (x = 1, y = 3, z \geq 1), (x = 1, y = 2, z = 1, 2, 3) or (x = 2, y = 1, z = 1, 2, 3)$ (resp. $(x = 1, y = 2, z \geq 4)$ or $(x = 2, y = 1, z \geq 4)$). We also have $|e_2| = 0$ (resp. $|e_2| = *3$), if $(x = 2, 3, y = 1, z = 1, 2), (x = 2, 3, y = 2, z = 1)$ or $(x = 1, y = 2, z = 1)$ (resp. $(x = 2, 3, y = 1, 2, 3, y + z \geq 4)$ or $(x = 1, y = 2, 3, y + z \geq 4)$). The values of the other v-edges of R_m are 0. The value of a h-edge of A is 0 (resp. $*3$) if $(x = 2, 3, y = 2, 3, z \geq 1), (x = 3, y = 1, z \geq 1)$ or $x = 2, y = 1, z = 1, 2, 3)$ (resp. $(x = 2, y = 1, z \geq 4)$ or $x = 1, y = 2, 3, z \geq 1)$), by Proposition 1. We show the value $|c|$ in H is $*$ except the case $n_2 = 1$. If m_1 and m_2 are odd (resp. even), $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a^2| = *$

(resp. $|R_{m_2}a^2| = 0$). Hence, we get $|H| \neq *$. When $n_2 = 1$, we can show $H \neq *$ in Lemma 6 below. Thus $|G| = *$, because G has some edges with value 0, and the values of all its edges are not $*$.

Let m be even. By Proposition 3 in [4], we have $|a| = *2$. By induction, we get $|e_1| = *$ (resp.$|e_1| = *2$), if $(x = 2,3, y = 2,3, z \geq 1), (x = 3, y = 1, z \geq 1), (x = 1, y = 3, z \geq 1), (x = 1, y = 2, z = 1,2,3)$ or $(x = 2, y = 1, z = 1,2,3)$ (resp. $(x = 1, y = 2, z \geq 4)$ or $(x = 2, y = 1, z \geq 4)$). We also have $|e_2| = *$ (resp. $|e_2| = *2$), if $(x = 2,3, y = 1, z = 1,2), (x = 2,3, y = 2, z = 1)$ or $(x = 1, y = 2, z = 1)$ (resp. $(x = 2,3, y = 1,2,3, y+z \geq 4)$ or $(x = 1, y = 2,3, y+z \geq 4)$). The values of the other v-edges of R_m are $*$. The value of a h-edge of A is $*$ (resp. $*2$) if $(x = 2,3, y = 2,3, z \geq 1), (x = 3, y = 1, z \geq 1)$ or $x = 2, y = 1, z = 1,2,3)$ (resp. $(x = 2, y = 1, z \geq 4)$ or $(x = 1, y = 2,3, z \geq 1)$), by Proposition 1. We show the value $|c|$ in H is 0 except the case $n_2 = 1$. If m_1 is odd (resp. even) and m_2 is even (resp. odd), we get $|R_m| = 0$ (resp. $|R_m| = *$) and $|R_{m_2}a^2| = 0$ (resp. $|R_{m_2}a^2| = *$). Hence, we get $|H| \neq 0$. When $n_2 = 1$, we can show $H \neq 0$ in Lemma 6 below. Thus $|G| = 0$, because the values of all edges of G are not 0.

(4) Let m be odd. By induction, we get $|e_1| = |e_2| = 0$. The values of the other v-edges of R_m are also 0. If $z = 1,2,3$, the value of a h-edge of A is 0 or $*2$, by Proposition 1. We show the value $|c|$ in H is $*$. If m_1 and m_2 are odd (resp. even), we have $|R_m| = 0$ (resp. $|R_m| = *$ and $|R_{m_2}a^2| = *$ (resp. $|R_{m_2}a^2| = 0$. Hence, we get $|H| \neq *$. Thus $|G| = *$, because G has some edges with value 0, and the values of all its edges are not $*$.

Let m be even. By induction, we get $|e_1| = |e_2| = *$. The values of the other v-edges of R_m are also $*$. If $z = 1,2,3$, the value of a h-edge of A is $*$ or $*3$, by Proposition 1. We show the value $|c|$ in H is 0. If m_1 is odd (resp. even) and m_2 is even (resp. odd), we get $|R_m| = 0$ (resp. $|R_m| = *$ and $|R_{m_2}a^2| = 0$ (resp. $|R_{m_2}a^2| = *$). Hence, we get $|H| \neq 0$. Thus $|G| = 0$, because the values of all edges of G are not 0.

Lemma 5 Let $H = R_{m-3}cR_2a^2$ be the graph given in the proof of Proposition 4(1) for the case $m_2 = 2, x = y = 1, z = 1,2,3$, where $R_2 = AB$. Then, we have $|H| \neq *2$ (resp. $|H| \neq *3$) if m is odd (resp. even).

Proof. Let b_1 (resp. b_2) be the upper (resp. lower) h-edge of A. By removing the edge b_1 (resp. b_2) from H, we get a subgraph $K_1 = R_{m-3}cBb_2a^2$ (resp. $K_2 = R_{m-3}cBb_1a^2$). We prove $|K_1| = |K_2| = *2$ (resp. $|K_1| = |K_2| = *3$), if m is odd (resp. even). This shows our desired result. Let the size of the rightmost box D of R_{m-3} be w. By removing an inner h-edge of R_{m-3}, we get a subgraph $H_1 = R_{n_1}dR_{n_2}cBb_2a^2$, where $n_1+n_2 = m-4$ and d is a connection of R_{n_1} and $R_{n_2}cBb_2a^2$.

Let m be odd. We get $|c| = *$, and $|b| = *$ or $|b| = *3$. The value of the lower (resp. upper) h-edge of B is 0 (resp. 0 or $*3$). If $w = 1,2,3$, then the values of the outer edges of D are 0 or $*3$. The values of the inner v-edges of R_{m-3} is $*3$. We show $|d|$ in H_1 is $*2$. If n_1 is odd (resp. even) and n_2 is even (resp. odd), we have $|R_{n_1}| = 0$ (resp. $|R_{n_1}| = *$) and $|R_{n_2}cBb_2a^2| = *2$ (resp. $|R_{n_2}cBb_2a^2| = *3$). Hence, we get $|H_1| \neq *2$. We will show later that the values of arrows in K_1 are $*3$ (resp. $*2$) if m is odd (resp. even). Thus, when m is odd, we get $|K_1| = *2$, because K_1 has some edges with value 0 and ones with value $*$, and the values of all its edges are not $*2$. Similarly, we can prove $|K_2| = *2$.

Let m be even. We get $|c| = 0$, and $|b| = 0$ or $|b| = *2$. The value of the lower (resp. upper) h-edge of B is $*$ (resp. 0 or $*2$). If $w = 1, 2, 3$, then the values of the outer edges of D are $*$ or $*2$. The values of the inner v-edges of R_{m-3} is $*2$. We show $|d|$ in H_1 is $*3$. If n_1 and n_2 are odd (resp. even), we have $|R_{n_1}| = 0$ (resp. $|R_{n_1}| = *$) and $|R_{n_2}cBb_2a^2| = *3$ (resp. $|R_{n_2}cBb_2a^2| = *2$). Hence, we get $|H_1| \neq *3$. We will show later that the values of arrows in K_1 are $*3$ (resp. $*2$) if m is odd (resp. even). Thus, when m is even, we get $|K_1| = *3$, because K_1 has some edges with value 0, ones with value $*$ and ones with value $*2$, and the values of all its edges are not $*3$. Similarly, we can prove $|K_2| = *3$.

Now, We show that the values of arrows in K_1 are $*3$ (resp. $*2$) if m is odd (resp. even). By removing one of arrows form K_1, we get a subgraph $L = R_{m-3}cBa'$, where a' is an arrow of size 2 or 3. We will show $|L| = *3$ (resp. $|L| = *2$), if m is odd (resp. even).

Let m be odd. We have $|c| = 0$ and the values of h-edges of B are $*$. We also get $|a'| = 0$ or $|a'| = *2$ by Lemma 2. If $w = 1, 2, 3$, the values of the outer edges of D are $*$ or $*2$. The values of the inner v-edges of R_{m-3} are $*2$. By removing an inner h-edge of R_{m-3}, we can show the values of inner h-edges are not $*3$. Thus, we obtain $|L| = *3$.

Let m be even. We have $|c| = *$ and the values of h-edges of B are 0. We also get $|a'| = *$ or $|a'| = *3$ by Lemma 2. If $w = 1, 2, 3$, the values of the outer edges of D are 0 or $*3$. The values of the inner v-edges of R_{m-3} are $*3$. By removing an inner h-edge of R_{m_1}, we can show the values of inner h-edges are not $*2$. Thus, we obtain $|L| = *2$.

Lemma 6 Let $H = R_{m-2}cAa^2$ be the graph given in the proof of Proposition 4(3) for the case $m_2 = 1$ and $x = 1, 2, 3, y = 2, 3, z = 1, 2, 3$ or $x = 2, 3, y = 1, z = 1, 2, 3$, where the size of c is z. Then, we have $|H| \neq *$ (resp. $|H| \neq 0$), when m is odd (resp. even).

Proof. Let m be odd. By Proposition 1, the value of the right v-edge of A is $*$ when $z = 1, 2, 3$ and $x = 1, y = 2$ or $x = 2, y = 1$. The value of the lower h-edge of A is $*$ when $y = 1, 2, 3$ and $x = 1, 2, 3, z = 2, 3$ or $x = 2, 3, z = 1$. When $x = 1, y = 3, z = 1$, the value of the upper h-edge of A is $*$. Hence, in any cases, we have $|H| \neq *$.

Let m be even. By Proposition 1, the value of the right v-edge of A is 0 when $z = 1, 2, 3$ and $x - 1, y = 2$ or $x = 2, y = 1$. The value of the lower h-edge of A is 0 when $y = 1, 2, 3$ and $x = 1, 2, 3, z = 2, 3$ or $x = 2, 3, z = 1$. When $x = 1, y = 3, z = 1$, the value of the upper h-edge of A is 0. Hence, in any cases, we have $|H| \neq 0$.

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