J. Math. Univ. Tokushima
Vol. 45 (2011), 1–7

Definition of the Concept of Natural Numbers and its Existence Theorem. Solution of Hilbert's Second Problem

By

Yoshifumi Ito

Professor Emeritus, The University of Tokushima
Home Address : *209-15 Kamifukuman Hachiman-cho*
Tokushima 770-8073, JAPAN
e-mail address : *yoshifumi@md.pikara.ne.jp*

(Received September 30, 2011)

Abstract

In this paper, we give the new complete solution of the definition of the concept of natural numbers and its existence.

This work is the English version of my work in Ito [6], Chapter 2, Section 2.1.

It is important to notice that we do not use Gödel's Incompleteness Theorem.

This gives the consideration and the solution of Hilbert's second problem "the proof of consistency of the axioms of arithmetic" in the other angle by changing the point of view.

Namely, giving the definition of the concept of natural numbers means providing the complete system of axioms which determines the set of all natural numbers as an algebraic system.

The proof of the existence of the concept of natural numbers means constructing a model of natural numbers as the set of all natural numbers as an algebraic system in the concrete manner on the basis of the ZFC set theory.

Thereby, at the same time, we give the new complete solutions of the problem of the foundation of analysis and the problem of the foundation of geometry.

2000 Mathematics Subject Classification. Primary 03B30, 03C62, 03E10.

1 Problem of the foundation of mathematics

In the sequel, we consider the definition of the concept of natural numbers and its existence. Thereby we give the new complete solution of Hilbert's second problem.

Why and how can we conclude that theories of mathematics, theorems of mathematics, proofs of theorems and mathematical calculations are correct?

It is the problem of the foundation of mathematics. Hilbert's second problem is the basic problem concerning this problem.

Hilbert's second problem is **the Problem of "Proof of Consistency of the System of Axioms of Arithmetic"**.

It is very hard to consider this problem directly. The meanings of this problem are not easy to be understood.

Therefore, we consider this problem as the problem of "the definition of the concept of natural numbers and its existence", and study this new problem.

Namely, this means that we give the solution to the simple question "What are natural numbers?"

In the other words, "giving the definition of the concept of natural numbers" is nothing else but "providing the conditions just enough to determine the set of all natural numbers as an algebraic system".

Namely, if we say such a system of axioms to be the complete system of axioms of natural numbers, it is the problem to provide the complete system of axioms of natural numbers in order to define the concept of natural numbers.

The proof of the existence of the concept of natural numbers means constructing a model of the set of all natural numbers as an algebraic system concretely.

Here, we construct models of the set of all natural numbers on the basis of the ZFC set theory. Thereby, we prove the existence of the concept of natural numbers.

2 The system of axioms of natural numbers and the definition of the concept of natural numbers

In this section, we provide the complete system of axioms of natural numbers. This is the completely new result.

In order that, we provide the complete system of axioms of natural numbers by determining the conditions providing the rules of calculations of addition, multiplication and order relation for natural numbers.

Definition 2.1 (Axioms of natural numbers) If, in a set $\boldsymbol{N}$, addition, multiplication and order relation are defined, and the following conditions (I)~(V) are satisfied, we say each element of the set $\boldsymbol{N}$ to be a **natural number.**

(I) For the addition by which two elements $m, n \in \boldsymbol{N}$ correspond to their **sum** $m + n$, the following (1)~(3) are satisfied :

Let $l, m, n \in \boldsymbol{N}$ be three elements.

(1) $m + n = n + m$. [Commutative Law]

(2) $(l + m) + n = l + (m + n)$. [Associative Law]

(3) There exists the element $0 \in \boldsymbol{N}$ such that $0 + n = n$ holds.
We call the element 0 as "zero".

(II) For the multiplication by which two elements $m, n \in \boldsymbol{N}$ correspond to their **product** $m \cdot n = mn$, the following (1)~(3) are satisfied:

Let $l, m, n \in \boldsymbol{N}$ be three elements.

(1) $mn = nm$. [Commutative Law].

(2) $(lm)n = l(mn)$. [Associative Law].

(3) There exists the element $1 \in \boldsymbol{N}$ such that $1n = n$ holds.
We call the element 1 as "one".

(III) For addition and multiplication, the following (1) is satisfied:

Let $l, m, n \in \boldsymbol{N}$ be three elements.

(1) $l(m + n) = lm + ln$. [Distributive Law].

(IV) If, for two elements $m, n \in \boldsymbol{N}$, the relation $m < n$ is defined, we say "m is smaller than n" or "n is larger than m". If $m = n$ or $m < n$ is satisfied, we denote this as $m \leq n$. Then we say "m is not more than n", or "n is not less than m".

As for this **order relation**, the following (1)~(7) are satisfied:

Let $l, m, n \in \boldsymbol{N}$ be three elements.

(1) For two elements m, n, one and only one of the relations $m < n, m = n, n < m$ holds regularly.

(2) If $l < m, m < n$ holds, we have $l < n$. [Transitive Law].

(3) If $m < n$ holds, we have $m + l < n + l$.

(4) If $m < n, l > 0$ holds, we have $lm < ln$.

(5) If $0 < l < m$ holds, there exists an elements $n > 0$ in $\boldsymbol{N}$ such that we have $m < nl$ necessarily.
[**Axiom of Archimedes**].

(6) For an arbitrary element $n \in \boldsymbol{N}$, we have $n \geq 0$.

(7) If $m < n$ holds, we have $m + 1 \leq n$.

(V) We have $\boldsymbol{N} = \{0\} \cup \{n = m + 1; m \in \boldsymbol{N}\}$.

So far Axioms of Peano are known as Axioms of Natural Numbers. These axioms are given in the following.

Axioms of Peano Assume that, in a nonempty set $\boldsymbol{N}$, the following conditions (I), (II) are satisfied:

(I) There is only one element 0 in $\boldsymbol{N}$.

(II) There exists an injection φ from $\boldsymbol{N}$ into $\boldsymbol{N}$ such that the following conditions (i), (ii) are satisfied:

(i) For an arbitrary element n in $\boldsymbol{N}$, we have $\varphi(n) \neq 0$.

(ii) If an subset M of $\boldsymbol{N}$ satisfies the following conditions

(α) M includes 0,

(β) If an element n of $\boldsymbol{N}$ is included in M, $\varphi(n)$ is also included in M,

we have $M = \boldsymbol{N}$.

Here, we put $\varphi(n) = n^+$ and we say n^+ as **the latter** of n. Then we say an element of $\boldsymbol{N}$ as **a natural number**.

Nevertheless, Axioms of Peano give only the definition of the concept of finite ordinal numbers in the exact sense.

Therefore Axioms of Peano cannot be considered as Axioms of Natural Numbers in the exact sense.

Hence, it is the first that I succeeded in determining Axioms of Natural Numbers.

3 Construction of models of natural numbers and proof of its existence theorem

In this section, we construct models of natural numbers on the basis of the ZFC set theory.

Thereby we prove the existence of natural numbers. At the same time, we solve the problem of definition of the concept of natural numbers and its existence.

Theorem 3.1 *In the ZFC set theory, assume that the set* $\boldsymbol{N} = \{0, 1, 2, \cdots\}$ *is* ***the set of all finite ordinal numbers****. Then the set* $\boldsymbol{N}$ *is a model of natural numbers. Namely* $\boldsymbol{N}$ *satisfies the axioms of natural numbers in section 2.*

Here, addition and multiplication are defined as those of finite ordinal numbers. Then, we use the method of induction for the definition of those calculations. The order relation is defined as the order relation of finite ordinal numbers.

Thereby we can prove that $\boldsymbol{N}$ satisfies the axioms of natural numbers in section 2.

In the following Theorem 3.2, we give another construction of a model of natural numbers.

Theorem 3.2 *In the ZFC set theory, assume that the set* $\boldsymbol{N} = \{0, 1, 2, \cdots\}$ *is* ***the set of all finite cardinal numbers****. Then, the set* $\boldsymbol{N}$ *is a model of natural numbers. Namely* $\boldsymbol{N}$ *satisfies the axioms of natural numbers in section 2.*

Here, addition and multiplication are defined as those of finite cardinal numbers. The order relation is defined as the order relation of finite cardinal numbers.

Thereby we can prove that $\boldsymbol{N}$ satisfies the axioms of natural numbers in section 2.

4 Consistency of axioms of arithmetic

From the studies until now, we find out that there exist models of natural numbers in the frame work of the ZFC set theory.

Therefore, by virtue of the existence of the ZFC set theory, we can solve the problem of the definition of the concept of natural numbers and its existence.

These results give the complete solution of Hilbert's second problem.

Namely, the constructions of models of natural numbers which satisfy the axioms of natural numbers give the proof of the consistency of axioms of natural numbers.

Really, there exists the infinite set of natural numbers which is not empty. Namely, this means the consistency of axioms of natural numbers in another sense.

These are the new results.

Here, we study the **consistency of axioms of natural numbers**.

In general, it is hard to prove the consistency of axioms which define a certain mathematical concept directly.

Therefore, we usually understand a certain mathematical concept and the mathematical theory concerning it in the following manner.

Here we consider these situations in the case of natural numbers.

Axioms of natural numbers are the conditions which are satisfied by the concept of natural numbers. Then we give the definition of the concept of natural numbers by using the complete system of axioms such conditions as above.

We prove the existence of the concept of natural numbers which satisfies the axioms of natural numbers by constructing a model of natural numbers.

Then we can practice the calculation of natural numbers by using the rules of calculation given by these axioms.

Thereby the calculations of natural numbers are completely practiced.

We consider axioms providing other mathematical concepts in the same manner. For example, axioms of groups, rings, fields and modules etc. are in the same situation.

In a mathematical theory, it is enough to the mathematical study concerning this mathematical concept if we give the definition of a certain mathematical concept and the proof of its existence.

Hilbert already knew that , in general, giving the axioms of a certain mathematical concept is giving the conditions of the definition of that mathematical concept.

Nevertheless, I was the first to propose the problem of the definition of a certain mathematical concept and the proof of its existence instead of providing axioms of a certain mathematical concept and the proof of their consistency.

Here we consider **the property of fixed form of axioms and the type of axioms**.

Among the axioms of mathematical concepts, there are those of fixed form and typical ones.

The axioms of natural numbers are of fixed form and the concept of natural numbers satisfying these axioms is only one except isomorphism as an algebraic system.

On the other hand, for example, axioms of groups are typical ones and there are infinite numbers of groups which are not mutually isomorph.

By virtue of Hilbert's "Foundation of Geometry" [3], [5], we solve problems of 'Foundation of Analysis" and "Foundation of Geometry" at the same time by using the studies in the above.

Here, the problem of "Foundation of Analysis" is the problem of the definition of the concept of real numbers and its existence.

For the precise consideration, we refer to the references at the end of this paper, and especially to Y. Ito [6], [7], [8].

References

[1] G. Cantor, Transcendental Set Theory, (Japanese translation and Comments by K. Kunugi and T. Murata), Kyôritu, 1979, (in Japanese).

[2] S. Hayashi and M. Yasugi, Gödel's Incompleteness Theorem, Iwanami, 2006, (in Japanese).

[3] D. Hilbert, Grundlagen der Geometrie, Teubner, 1899, 13th ed., 1987. (Japanese translation : Foundation of Geometry and Others, Kyôritu, 1970, (in Japanese)).

[4] ——, Problems of Mathematics—Hilbert's Problems — (Japanese translation and Comments by S. Hitotumatu), Kyôritu, 1979, (in Japanese).

[5] ——, Foundation of Geometry, (Japanese translation by K. Nakamura), Tikuma Shobô, 2005, (in Japanese).

[6] Y. Ito, Axioms of Arithmetic, Sciencehouse, 1999, (in Japanese).

[7] ——, World of Mathematical Sciences, preprint, 2001, (in Japanese).

[8] ——, Study on the New Axiomatic Method Giving the Solutions of Hilbert's 2nd and 6th Problems, *J. Math. Univ. Tokushima*, **44**(2010), 1-12.

[9] Math. Soc. Japan (Ed.), Iwanami Mathematical Dictionary, Third ed., Iwanami, 1985, (in Japanese).

[10] A. Nozaki, Incompleteness Theorem, Tikuma Shobô, 2006, (in Japanese).

[11] G. Takeuti, What are Sets, Kôdansha, 1976, (in Japanese).

[12] ——, Introduction to Modern Set Theory, Nihon Hyôronsha, 1989, (in Japanese).

[13] G. Takeuti and M. Yasugi, Introduction to Proof Theory, Kyôritu, 1988, (in Japanese).