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On a Classical-Quantum Correspondence for Mechanics in a Gauge Field

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Abstract

This paper studies the classical and the quantum mechanics in a non-abelian gauge field on the basis of the symplectic geometry and the theory of representation of Lie groups. As a classical-quantum correspondence we present a conjecture on the quasi-mode corresponding to a certain classical energy level.

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Introduction

Let (M, m) be a d dimensional smooth Riemannian manifold without boundary, and let $\pi : P \to M$ be a principal G-bundle, where G is a compact semisimple Lie group with $\dim G = r$. Suppose P is endowed with a connection $\tilde{\nabla}$. The connection $\tilde{\nabla}$ is defined by a $\mathfrak{g}$-valued one form (called the *connection form*) θ on P with certain properties, where $\mathfrak{g}$ is the Lie algebra of G. The $\mathfrak{g}$-valued two form $\Theta := d\theta + \theta \wedge \theta$ on P is called the *curvature form* of $\tilde{\nabla}$. (See [4], for example.)

Take an open covering $\{U_\alpha\}$ of M with $\{\varphi_{\alpha\beta}\}$ being the transition functions of P. Then the curvature form Θ is regarded as a family of $\mathfrak{g}$-valued two forms $\bar{\Theta}_\alpha$ defined on U_α such that

$$\bar{\Theta}_\beta = \mathrm{Ad}(\varphi_{\alpha\beta}^{-1})\bar{\Theta}_\alpha \tag{0.1}$$

on $U_\alpha \cap U_\beta (\neq \phi)$, where $\mathrm{Ad}(\cdot)$ denotes the adjoint action of G on $\mathfrak{g}$. Such a family of $\mathfrak{g}$-valued two forms $\{\bar{\Theta}_\alpha\}$ on M satisfying (0.1) is called a *gauge field*, while the connection form θ is called a *gauge potential.* If G is the abelian group $U(1)$, then $\bar{\Theta}_\alpha = \bar{\Theta}_\beta$ holds, and accordingly we have a two form $\bar{\Theta}$ globally defined on M, which is called a *magnetic field.*

In this paper we study the classical and the quantum mechanics in the non-abelian gauge field $\{\bar{\Theta}_\alpha\}$ on the basis of the symplectic geometry and the theory of representation of Lie groups. Section 1 is devoted to reviewing a geometrical formulation for the classical mechanics in the gauge field, which is essentially the same as that in the previous paper [6] (see also [7]). In Section 2 we introduce the space of quantum states corresponding to the classical system with an integral "charge". (Related arguments are found in [8], [9].) Finally in Section 3 we present a conjecture on the quasi-mode corresponding to a certain classical energy level. This conjecture is a generalization of the eigenvalue theorem given in [5] for the abelian gauge field (the magnetic field).

1 Classical mechanics in a gauge field

1.1 The Kaluza-Klein metric

Let $(\, , \,)_{\mathfrak{g}}$ denote the inner product given by $(-1) \times$ (the Killing form) on the compact semisimple Lie algebra $\mathfrak{g}(= T_eG)$, and let m_G be the metric on the Lie group G induced from $(\, , \,)_{\mathfrak{g}}$. Note that m_G is invariant under left- and right-translations on G.

The connection $\widetilde{\nabla}$ on the principal bundle $\pi : P \to M$ defines the direct decomposition of each tangent space T_pP $(p \in P)$ as

$$T_pP = H_p \oplus V_p, \tag{1.1}$$

where V_p is tangent to the fiber, and H_p is linearly isomorphic with $T_{\pi(p)}M$ through $\pi_*|_{H_p}$. Note that the tangent space V_p to the fiber is linearly isomorphic with $\mathfrak{g}$ by the correspondence $\mathfrak{g} \ni A \mapsto A_p^P := \frac{d}{dt}(p \cdot \exp tA)|_{t=0} \in V_p$. The inner product on $\mathfrak{g}$ induces the inner product $(\, , \,)_{V,p}$ on V_p $(p \in P)$ as $(A^P, B^P)_{V,p} = (A, B)_{\mathfrak{g}}$ $(A, B \in \mathfrak{g})$. On the other hand, we have the inner product $(\, , \,)_{H,p}$ on H_p from the metric m on M such that $\pi_*|_{H_p}$ is an isometry. Finally, we define an inner product $\widetilde{m}$ in each $T_pP (p \in P)$ by defining H_p and V_p to be orthogonal each other. The metric $\widetilde{m}$ on P (which is induced from the metric m on M, the metric m_G on G, and the connection $\widetilde{\nabla}$) is called the *Kaluza-Klein metric* (cf. [3]). Note that $\widetilde{m}$ is invariant under the G-action on P.

Let $\Omega_P = d\omega_P$ be the standard symplectic structure on the cotangent bundleT^*P of P, where ω_P is called the canonical one form on T^*P. We have the natural Hamiltonian function $\widetilde{H}$ on T^*P defined by the Kaluza-Klein metric $\widetilde{m}$, i.e., $\widetilde{H}(q) = \|q\|^2$ $(q \in T^*P)$. Thus, we have the Hamiltonian system $(T^*P, \Omega_P, \widetilde{H})$, which is just the system of geodesic flow on T^*P.

1.2 Reduction of the system (cf. [1, Ch.4])

The action $p \mapsto p \cdot g = R_g(p)$ $(p \in P, \, g \in G)$ of G on P is naturally lifted to the action $R^*_{g^{-1}} := (R_{g^{-1}})^*$ on T^*P (so that $R^*_{g^{-1}} : T^*_pP \to T^*_{p\cdot g}P$ for each $p \in P$), which preserves ω_P (and accordingly Ω_P), i.e., $R^*_{g^{-1}}\omega_P = \omega_P$ holds for every $g \in G$. (We call such action a *symplectic action.*) Moreover, we notice that the Hamiltonian $\widetilde{H}$ is also invariant under the action $R^*_{g^{-1}}$.

A momentum map for the symplectic G-action $R^*_{g^{-1}}$ is a map $J : T^*P \to \mathfrak{g}^*$ (the dual space of $\mathfrak{g}$) given by

$$\langle J(q), A\rangle = \langle q_p, A^P_p\rangle \quad (q \in T^*P,\ q_p \in T^*_pP\,(p \in P)), \tag{1.2}$$

for all $A \in \mathfrak{g}$. The momentum map J is Ad^*-equivariant, i.e.,

$$J \circ R^*_{g^{-1}} = \mathrm{Ad}^*(g^{-1}) \circ J \tag{1.3}$$

holds for $g \in G$, where $\mathrm{Ad}^*(g) := (\mathrm{Ad}(g^{-1}))^*$ (the adjoint of $\mathrm{Ad}(g^{-1})$). Furthermore, J is invariant under the flow of $(T^*P, \Omega_P, \widetilde{H})$.

Note that J is a surjective map with any $\mu \in \mathfrak{g}^*$ to be a regular value, and $J^{-1}(\mu)$ is a submanifold of T^*P. Put $G_\mu := \{g \in G;\ \mathrm{Ad}^*(g)\mu = \mu\}$, which is a closed subgroup of G. Then, $J^{-1}(\mu)$ is G_μ-invariant because of (1.3). The quotient manifold $P_\mu := J^{-1}(\mu)/G_\mu$ is naturally endowed with a symplectic structure Ω_μ induced from Ω_P, and endowed with a Hamiltonian function H_μ induced from $\widetilde{H}$. Thus we have a (reduced) Hamiltonian system $\mathcal{H}_\mu = (P_\mu, \Omega_\mu, H_\mu)$, which we regard as the *dynamical system of classical particle of "charge" μ in the gauge field* given by the connection $\widetilde{\nabla}$ (the gauge potential). We remark that the reduced phase space P_μ is also given as the quotient manifold $J^{-1}(\mathcal{O}_\mu)/G$ for the coadjoint orbit $\mathcal{O}_\mu = \{\mathrm{Ad}^*(g)\mu;\ g \in G\}$ in $\mathfrak{g}^*$.

1.3 A formulation by using the connection form

Suppose $G_\mu \subsetneqq G$. Consider the quotient manifold $M_\mu := P/G_\mu$, and the natural projection $\pi' : M_\mu \to M(= P/G)$ gives a bundle structure with the fiber $G/G_\mu (\cong \mathcal{O}_\mu)$. Let $\pi'_{M_\mu} : M^\#_\mu \to M_\mu$ be the vector bundle obtained by pulling back the cotangent bundle T^*M over M through the map $\pi' : M_\mu \to M$, i,e.,

$$M^\#_\mu = \{(y, \xi) \in M_\mu \times T^*M;\ \pi'(y) = \pi_M(\xi)\}.$$

We note that $M^\#_\mu$ is regarded as a subbundle of T^*M_μ by the immersion $(y, \xi) \mapsto \pi'^*(\xi) \in T^*_yM_\mu$.

Let θ be the connection form (which is a $\mathfrak{g}$-valued one form on P) of $\widetilde{\nabla}$, and put $\theta_\mu = \langle \mu, \theta\rangle$, which is an $\mathbb{R}$-valued one form on P.

Lemma 1 *Let $\mathfrak{g}_\mu$ be the Lie algebra of G_μ. An element A in $\mathfrak{g}$ belongs to $\mathfrak{g}_\mu$ if and only if $d\theta_\mu(A^P, X) = 0$ for any vector field X on P.*

Proof. We have

$$d\theta_\mu(A^P, X) = (i(A^P)d\theta_\mu)(X) = (\mathcal{L}_{A^P}\theta_\mu)(X) - d(i(A^P)\theta_\mu)(X),$$

where $i(A^P)$ and $\mathcal{L}_{A^P}$ denote the interior product and the Lie derivative, respectively. Since $i(A^P)\theta_\mu = \theta_\mu(A^P) = \langle \mu, A\rangle =$ constant, we have $d\theta_\mu(A^P, X) =$

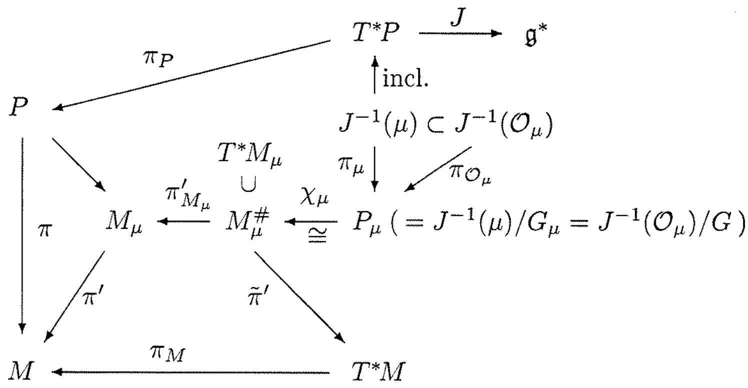

Figure 1: Reduction of the system

$(\mathcal{L}_{A^P}\theta_\mu)(X)$. Note that $R_g^*\theta = \mathrm{Ad}(g^{-1})\theta$ for $g \in G$, and we get

$$(\mathcal{L}_{A^P}\theta_\mu)(X) = \frac{d}{dt}\langle \mu, \mathrm{Ad}(\exp(-tA))(\theta(X))\rangle\big|_{t=0} = \frac{d}{dt}\langle \mathrm{Ad}^*(\exp tA)\mu, \theta(X)\rangle\big|_{t=0}.$$

This formula implies the assertion of the lemma. ■

By virtue of this lemma $d\theta_\mu$ is regarded as a closed two form on M_μ. We introduce a two form

$$\Omega_\mu^\# := (\tilde{\pi}')^*\Omega_M + (\pi'_{M_\mu})^*(d\theta_\mu)$$

on $M_\mu^\#$, where $\tilde{\pi}' : M_\mu^\# \longrightarrow T^*M$ is the natural lift of $\pi' : M_\mu \longrightarrow M$, and Ω_M is the standard symplectic form on T^*M. The two form $\Omega_\mu^\#$ is closed and non-degenerate, and accordingly defines a symplectic structure on $M_\mu^\#$.

Remark The symplectic structure $\Omega_\mu^\#$ is just the restriction of the twisted symplectic form $\Omega_{M_\mu} + (\pi_{M_\mu})^*(d\theta_\mu)$ on T^*M_μ to the subbundle $M_\mu^\#$, where $\pi_{M_\mu} : T^*M_\mu \longrightarrow M_\mu$ is the natural projection.

Let H be the Hamiltonian function on T^*M defined by the Riemannian metric m on M, and put $H_\mu^\# := (\tilde{\pi}')^*H + \|\mu\|^2$, where the norm $\|\mu\|$ is naturally defined by the inner product $m_{\mathfrak{g}}$ on $\mathfrak{g}$. Thus we obtain the Hamiltonian system $(M_\mu^\#, \Omega_\mu^\#, H_\mu^\#)$ (see Figure 1).

Proposition 2 *The Hamiltonian system $\mathcal{H}_\mu$ is isomorphic with $(M_\mu^\#, \Omega_\mu^\#, H_\mu^\#)$, that is, there exists a diffeomorphism $\chi_\mu : P_\mu \longrightarrow M_\mu^\#$ such that*

$$\Omega_\mu = \chi_\mu^*\Omega_\mu^\#, \qquad H_\mu = \chi_\mu^* H_\mu^\#. \qquad (1.4\,a,b)$$

Proof. For each $p \in P$ we put

$$(V^{\perp})_p := \{q \in T^*_p P \mid \langle q, A^P_p \rangle = 0 \text{ for } \forall A \in \mathfrak{g}\} \ (\subset T^*_p P),$$
$$(V^{\perp}_{\mu})_p := \{q \in T^*_p P \mid \langle q, A^P_p \rangle = 0 \text{ for } \forall A \in \mathfrak{g}_{\mu}\} \ (\subset T^*_p P),$$

and define the subbundles $V^{\perp} := \bigcup_{p \in P} (V^{\perp})_p$ and $V^{\perp}_{\mu} := \bigcup_{p \in P} (V^{\perp}_{\mu})_p$ of T^*P, which are invariant under the G_{μ}-action. Moreover we see that

$$M^{\#}_{\mu} \cong V^{\perp}/G_{\mu}, \quad T^*M_{\mu} \cong V^{\perp}_{\mu}/G_{\mu}.$$

For each $q \in T^*_p P$ we define the map

$$\bar{\chi}_{\mu}(q) := q - (\theta_{\mu})_p \in T^*_p P.$$

Then, we see that
(i) $\bar{\chi}_{\mu}(q) \in (V^{\perp})_p$ if $q \in J^{-1}(\mu)$, and that
(ii) $\bar{\chi}_{\mu}(R^*_{g^{-1}}(q)) = R^*_{g^{-1}}(\bar{\chi}_{\mu}(q))$ for $q \in J^{-1}(\mu)$ and $g \in G_{\mu}$.
Indeed, (i) is shown as follows: $\langle q_p, A^P_p \rangle - \langle (\theta_{\mu})_p, A^P_p \rangle = \langle J(q), A \rangle - \langle \mu, A \rangle = 0$ for $\forall A \in \mathfrak{g}$. The assertion (ii) follows from the formula $(\theta_{\mu})_{p \cdot g} = R^*_{g^{-1}}((\theta_{\mu})_p)$ $(g \in G_{\mu})$, that is derived from the property $R^*_{g^{-1}}\theta = \mathrm{Ad}(g)\theta$ $(g \in G)$ for θ and the definition of G_{μ}. Noticing (i) and (ii), we can define the diffeomorphism $\chi_{\mu} : P_{\mu} \to M^{\#}_{\mu}$ from map $\bar{\chi}_{\mu} : T^*P \to T^*P$.

Now, we will prove (1.4 a). A vector $X \in T_q(T^*P)$ $(q \in T^*P, \pi_P(q) = p)$ is written as

$$X(q) = \bar{X}(q) + X^*(q) \quad \text{with } \bar{X}(q) \in T_p P,\ X^*(q) \in T^*_p P (= T_q(T^*_p P)).$$

Then, $X^*(q) \in (V^{\perp})_p$ if $X \in T_q J^{-1}(\mu)$. Let us take two vector fields $X = X(q)$ and $Y = Y(q)$ on $J^{-1}(\mu)$ defined in a neighborhood of $q_0 \in J^{-1}(\mu)$ such that $\bar{X}(q)$ and $\bar{Y}(q)$ are constant along the each fibers of T^*P. Then we have

$$\begin{aligned} \Omega_P(X, Y) &= \frac{1}{2}\{X\langle \omega_P, Y \rangle - Y\langle \omega_P, X \rangle - \langle \omega_P, [X, Y] \rangle\} \\ &= \frac{1}{2}\{X\langle q, \bar{Y} \rangle - Y\langle q, \bar{X} \rangle - \langle q, \overline{[X, Y]} \rangle\}. \end{aligned}$$

Put $q' (= \bar{\chi}_{\mu}(q)) = q - \theta_{\mu} (\in (V^{\perp})_p)$, and we have

$$\begin{aligned} \Omega_P(X, Y) &= \frac{1}{2}\{X\langle q', \bar{Y} \rangle - Y\langle q', \bar{X} \rangle - \langle q', \overline{[X, Y]} \rangle\} \\ &+ \frac{1}{2}\{\bar{X}\langle \theta_{\mu}, \bar{Y} \rangle - \bar{Y}\langle \theta_{\mu}, \bar{X} \rangle - \langle \theta_{\mu}, \overline{[X, Y]} \rangle\}. \end{aligned}$$

Here we notice that $\bar{X}(p') = \bar{X}(p)$ and $\overline{[X, Y]} = [\bar{X}, \bar{Y}]$ hold. Therefore we see that the first term of this formula is regarded as $\Omega_M((\tilde{\pi}' \circ \chi_{\mu})_*([X]), (\tilde{\pi}' \circ \chi_{\mu})_*([Y]))$, and the second is regarded as $d\theta_{\mu}((\pi'_{M_{\mu}} \circ \chi_{\mu})_*([X]), (\pi'_{M_{\mu}} \circ \chi_{\mu})_*([Y]))$.

Finally we prove (1.4 *b*). Take $q \in T_p^*P \cap J^{-1}(\mu)$. Then, we have $q = \bar{\chi}_\mu(q) + (\theta_\mu)_u$ with $\bar{\chi}_\mu(q) \in (V^\perp)_p$, $(\theta_\mu)_p \in (H^\perp)_p$. Since $(V^\perp)_p$ and $(H^\perp)_p$ are orthogonal each other, we have

$$H_\mu([q]) = \|\bar{\chi}_\mu(q)\|^2 + \|(\theta_\mu)_p\|^2 = H(\tilde{\pi}' \circ \chi_\mu([q])) + \|(\theta_\mu)_p\|^2.$$

Here, $(\theta_\mu)_p(A_p^P) = \langle \mu, A\rangle$ for $\forall A \in \mathfrak{g}$, and accordingly $\|(\theta_\mu)_p\| = \|\mu\|$ holds. ■

Wong's equation on M_μ. We represent the flow of the system $(M_\mu^\#, \Omega_\mu^\#, H_\mu^\#)$ using local coordinates. Let $(x,g) = (x^1, \ldots, x^d, g^1, \ldots, g^r)$ be local coordinates of $U \times G \cong \pi^{-1}(U) \subset P$ for $U \subset M$. Note that M_μ is locally diffeomorphic with $U \times (G/G_\mu)$. Suppose the connection form θ of $\widetilde{\nabla}$ is represented as

$$\theta(x,g) = \sum_{j=1}^{d} \theta_j(x,g)dx^j + \sum_{\alpha=1}^{r} \theta_\alpha(x,g)dg^\alpha.$$

Then, the curvature form $\Theta := d\theta + \theta \wedge \theta$ of $\widetilde{\nabla}$ is locally written as

$$\begin{aligned}\Theta(x,g) &= \frac{1}{2}\sum_{i,j} \Theta_{ij}(x,g)dx^i \wedge dx^j \\ &= \frac{1}{2}\sum_{i,j}\Big\{\Big(\frac{\partial\theta_j}{\partial x^i} - \frac{\partial\theta_i}{\partial x^j}\Big) + [\theta_i, \theta_j]\Big\}dx^i \wedge dx^j.\end{aligned}$$

Put $\Theta_\mu := \langle \mu, \Theta\rangle$, and it is shown similarly to $d\theta_\mu$ that Θ_μ is an $\mathbb{R}$-valued two form globally defined on M_μ. We get the following by straightforward calculations.

Proposition 3 *The motion of the particle in the system $(M_\mu^\#, \Omega_\mu^\#, H_\mu^\#)$ is governed by the equation (called Wong's equation [7]) on M_μ locally expressed as*

$$\left.\begin{aligned}&\ddot{x}^i + \sum_{j,k}\Gamma_{jk}^i(x)\dot{x}^j\dot{x}^k - 2\sum_{j,k} m^{ij}(x)\Theta_{jk}^{(\mu)}(x,g)\dot{x}^k = 0 \\ &\dot{g} + L_{g*}\Big(\sum_j \theta_j(x,g)\dot{x}^j\Big) = 0\end{aligned}\right\}$$

where $\Theta_{jk}^{(\mu)}(x,g) := \langle \mu, \Theta_{jk}(x,g)\rangle$, $\Gamma_{jk}^i(x)$ denotes Christoffel's symbol on the Riemannian manifold (M,m), and $L_{g*} : \mathfrak{g}(= T_eG) \to T_gG$ is the left translation. (Note that $\Theta_{jk}^{(\mu)}(x,g)$ and the second equation is invariant under G_μ-action, namely they depend only on the equivalent class $[g] \in G/G_\mu$.)

2 Quantum systems in a gauge field

2.1 Unitary representations of G and the quantum states

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and let R be the root system for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Put $\mathfrak{h}_{\mathbb{R}} := \{H \in \mathfrak{h};\ \alpha(H) \in \mathbb{R} \text{ for } \forall\alpha \in R\}$. Then, $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}} = \mathfrak{h}$ holds for a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. We notice that $\mathfrak{h}_{\mathbb{R}}$ is a $l(= \operatorname{rank} G)$ dimensional real vector space with the inner product $(iH, iH')_K = -(H, H')_K = (H, H')_{\mathfrak{g}}$ $(H, H' \in \mathfrak{t})$, where $(\cdot,\cdot)_K$ denotes the Killing form on $\mathfrak{g}_{\mathbb{C}}$ (or $\mathfrak{g}$). By identifying $\mathfrak{g}$ to $\mathfrak{g}^*$ with respect to the inner product $(\cdot,\cdot)_{\mathfrak{g}}$ we have $\mathfrak{h}_{\mathbb{R}}^* = i\mathfrak{t}^* \subset i\mathfrak{g}^*$. Put $\Gamma := \mathfrak{t} \cap \exp^{-1}(e)$ for $\exp : \mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the simply connected Lie group whose Lie algebra is $\mathfrak{g}_{\mathbb{C}}$. Then, Γ is a lattice in $\mathfrak{t} \cong \mathbb{R}^l$. Let Γ^* be the dual lattice of Γ, namely

$$\Gamma^* = \{\tau \in \mathfrak{t}^* \mid \langle \tau, H\rangle \in 2\pi\mathbb{Z} \text{ for } \forall H \in \Gamma\}.$$

Then, $i\Gamma^*$ is a lattice in $i\mathfrak{t}^* = \mathfrak{h}_{\mathbb{R}}^*$, whose element is called an integral form. Let C be a Weyl chamber in $\mathfrak{h}_{\mathbb{R}}^*$. Then C defines the set R^+ of positive roots and the ordering in $\mathfrak{h}_{\mathbb{R}}^*$. The set $\hat{G}$ of irreducible unitary representations is labeled by the set $C \cap i\Gamma^*$ (whose element is called a dominant integral form).

For a "charge" $\mu \in \mathfrak{g}^*$ the coadjoint orbit $\mathcal{O}_\mu$ in $\mathfrak{g}^*$ intersects the set iC in exactly one point $i\lambda$ $(\lambda \in C)$. We assume that λ lies on $i\Gamma^*\backslash\{0\}$, i.e., λ is integral. We call such μ a *quantized* charge. Let $(\rho_\lambda, V_\lambda)$ be the irreducible unitary representation of G with highest weight λ. We introduce the associated vector bundle $\mathcal{E}_\lambda = P \times_{\rho_\lambda} V_\lambda \to M$ of P through the representation $(\rho_\lambda, V_\lambda)$. We regard the Hilbert space $L^2(M, \mathcal{E}_\lambda)$ of L^2-sections of $\mathcal{E}_\lambda$ as the *space of quantum states* corresponding to the classical system $\mathcal{H}_\mu$ for the quantized charge μ.

The connection $\widetilde{\nabla}$ on P induces the covariant derivative $\widetilde{\nabla}^{(\lambda)} : C^\infty(M, \mathcal{E}_\lambda) \to C^\infty(M, T^*M \otimes \mathcal{E}_\lambda)$ on $\mathcal{E}_\lambda$, and we obtain the Laplacian $\Delta^{(\lambda)} := (\widetilde{\nabla}^{(\lambda)})^* \widetilde{\nabla}^{(\lambda)} : L^2(M, \mathcal{E}_\lambda) \to L^2(M, \mathcal{E}_\lambda)$, which is a non-negative, (formally) self-adjoint, second order elliptic differential operator.

Let $s : U(\subset M) \to P$ be a local section of P, and set $\theta_U := s^*\theta$ for the connection form θ of $\widetilde{\nabla}$. Suppose θ_U is expressed as $\sum A_j(x)dx^j$ $(A_j(x) \in \mathfrak{g})$. Then, the covariant derivative $\widetilde{\nabla}^{(\lambda)}$ is given by

$$\widetilde{\nabla}_j^{(\lambda)} f = \nabla_j f + A_j^{(\lambda)}(x) f \quad (f \in C^\infty(U, V_\lambda))$$

with $A_j^{(\lambda)}(x) = (\rho_\lambda)_*(A_j(x)) \in \mathfrak{u}(V_\lambda)$, and

$$\Delta^{(\lambda)} = -\sum_{j,k} m^{jk}(x)\big(\nabla_j + A_j^{(\lambda)}(x)\big)\big(\nabla_k + A_k^{(\lambda)}(x)\big)$$

where ∇ is the Levi-Civita connection on (M, m).

2.2 Spaces of L^2 functions on P and L^2 sections of $\mathcal{E}_\lambda$

Let $L^2_\lambda(P, V_\lambda)$ be the space of V_λ-valued L^2 functions $\boldsymbol{f}$'s on P satisfying

$$\boldsymbol{f}(p \cdot g) = \rho_\lambda(g^{-1})\boldsymbol{f}(p) \quad (p \in P)$$

for any $g \in G$. Then, we have the natural unitary isomorphism (by taking suitable inner products):

$$L^2(M, \mathcal{E}_\lambda) \cong L^2_\lambda(P, V_\lambda).$$

Let χ_λ denote the character of the representation ρ_λ, and define the map $\mathcal{P}_\lambda : L^2(P) \to L^2(P)$; $f \mapsto f_\lambda$ by

$$f_\lambda(p) := d_\lambda \int_G \chi_\lambda(g^{-1})\overline{f(p \cdot g)}\, dg \quad (p \in P),$$

where $d_\lambda := \dim V_\lambda$, and dg is the Haar measure on G. Let $L^2_\lambda(P)$ be the image of $\mathcal{P}_\lambda$. Using local coordinates, $P \supset \pi^{-1}(U) \ni p = (x, g) \in U \times G$, we can see that $L^2_\lambda(P)$ consists of functions locally expressed as

$$f_\lambda(p) = f_\lambda(x, g) = \sum_{j,k} [\rho_\lambda(g)]^j_k \, f_0(x)^j_k \tag{2.1}$$

for some functions $f_0(x)^j_k$ on U, where $[\rho_\lambda(g)]^j_k$ denotes the matrix-components of the representation ρ_λ. By virtue of the Peter-Weyl theorem we have

$$L^2(P) = \sum_{\rho_\lambda \in \hat{G}} {}^{\oplus} L^2_\lambda(P).$$

Define the map $\mathcal{F}_\lambda : L^2(P) \to L^2(P, V^*_\lambda \otimes V_\lambda)$; $f \mapsto F_\lambda$ by

$$F_\lambda(p) := d_\lambda \int_G f(p \cdot g)\rho_\lambda(g) dg \quad (p \in P).$$

Here $\rho_\lambda(g)$ is regarded as a element of $V^*_\lambda \otimes V_\lambda = \mathrm{End}_{\mathbb{C}}(V_\lambda)$, and we have a local expression

$$F_\lambda(p) = F_\lambda(x, g) = \rho_\lambda(g^{-1})F_0(x)$$

for a matrix-valued function $F_0(x)$ on U. We denote by $L^2_\lambda(P, V^*_\lambda \otimes V_\lambda)$ the image of the map $\mathcal{F}_\lambda$. Then, we have the following.

Lemma 4 *The function $F \in L^2(P, V^*_\lambda \otimes V_\lambda)$ belongs to $L^2_\lambda(P, V^*_\lambda \otimes V_\lambda)$ if and only if*

$$F(p \cdot g) = \rho_\lambda(g^{-1})F(p) \quad (p \in P) \tag{2.2}$$

holds for any $g \in G$.

Proof. The "only if"-part of the statement is shown by directly checking (2.2). Suppose F satisfies (2.2). Then, F is locally expressed as $F(x, g) = \rho_\lambda(g^{-1})K(x)$ for some matrix-valued function $K(x)$. Take the L^2 function f on P (locally) defined by

$$f(x, g) = \mathrm{Trace}\big[\rho_\lambda(g^{-1})\, {}^tK(x)\big].$$

Then, we have $\mathcal{F}_\lambda(f) = F$. ■

Let $\{\boldsymbol{v}_j\}_{j=1}^{d_\lambda}$ be a orthonormal basis of V_λ. It follow from the above lemma that the V_λ-valued functions $\boldsymbol{f}_\lambda^j(p) := F_\lambda(p)\boldsymbol{v}_j$ $(j = 1, \ldots, d_\lambda)$ belong to $L^2_\lambda(P, V_\lambda)$. As a result we have the following isomorphism:

$$L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) \cong \overbrace{L^2_\lambda(P, V_\lambda) \oplus \cdots \oplus L^2_\lambda(P, V_\lambda)}^{d_\lambda \text{ times}}.$$

Finally, for $F_\lambda \in L^2_\lambda(P, V_\lambda^* \otimes V_\lambda)$ (which is a matrix-valued function) we define

$$[\Phi_\lambda(F_\lambda)](p) := \mathrm{Trace}\big[{}^t\overline{F_\lambda(p)}\big] \quad (p \in P).$$

Then, $\mathcal{P}_\lambda = \Phi_\lambda \circ \mathcal{F}_\lambda$ holds, and Φ_λ is a bijection from $L^2_\lambda(P, V_\lambda^* \otimes V_\lambda)$ onto $L^2_\lambda(P)$. In fact, for $f(x, g) = \sum_{j,k} [\rho_\lambda(g)]^j_k f(x)^j_k \in L^2_\lambda(P)$ (locally), we have

$$[\Phi_\lambda^{-1} f](x, g) = \rho_\lambda(g^{-1})\,\overline{F(x)}$$

for the $(d_\lambda \times d_\lambda)$ matrix $F(x) := [f(x)^j_k]$.

As a consequence, we get the following one-to-one correspondences:

$$\begin{aligned} L^2_\lambda(P) &\cong L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) \\ &\cong L^2_\lambda(P, V_\lambda) \oplus \cdots \oplus L^2_\lambda(P, V_\lambda) \\ &\cong L^2(M, \mathcal{E}_\lambda) \oplus \cdots \oplus L^2(M, \mathcal{E}_\lambda), \end{aligned}$$

that is, more explicitly

$$\begin{array}{ccccccc} \sum^\oplus L^2(M, \mathcal{E}_\lambda) & & \sum^\oplus L^2_\lambda(P, V_\lambda) & & L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) & & L^2_\lambda(P) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid \\ (\psi_1, \ldots, \psi_{d_\lambda}) & \leftrightarrow & (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{d_\lambda}) & \leftrightarrow & \Psi = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{d_\lambda}) & \leftrightarrow & \psi_P = \mathrm{Trace}\big[{}^t\overline{\Psi}\,\big]. \end{array}$$

Let Δ_P be the Laplace-Beltrami operator on $(P, \widetilde{m})$. Then, Δ_P leaves $L^2_\lambda(P)$ invariant. Notice that the Laplace-Beltrami operator Δ_G on (G, m_G) satisfies

$$\Delta_G[\rho_\lambda(g)]^j_k = (\|\lambda + \delta\|^2_K - \|\delta\|^2_K)[\rho_\lambda(g)]^j_k,$$

where $\delta = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in \mathfrak{h}^*_{\mathbb{R}}$, and the norm $\|\cdot\|_K$ (and the inner product $(\cdot,\cdot)_K$) on $\mathfrak{h}^*_{\mathbb{R}}$ is naturally induced one from that on $\mathfrak{h}_{\mathbb{R}}$, and we have the following lemma by the formula (2.1).

Lemma 5 *Suppose* $L^2_\lambda(P) \ni \psi_P \mapsto \psi_j \in L^2(M, \mathcal{E}_\lambda)\,(j = 1, \ldots, d_\lambda)$ *is the above correspondence. Then, we have*

$$(\Delta_P \psi_P)_j = \Delta^{(\lambda)}\psi_j + (\|\lambda + \delta\|^2_K - \|\delta\|^2_K)\psi_j.$$

We assume that M is compact. Then, the spectrum of $\Delta^{(\lambda)}$ consists of non-negative eigenvalues

$$\nu_1^{(\lambda)} \leq \nu_2^{(\lambda)} \leq \cdots \leq \nu_k^{(\lambda)} \leq \cdots \uparrow +\infty.$$

If $\psi_P \in L^2_\lambda(P)$ satisfies $\Delta_P \psi_P = \kappa \psi_P$, then

$$\Delta^{(\lambda)} \psi_j = (\kappa - \|\lambda + \delta\|_K^2 + \|\delta\|_K^2)\psi \quad (j = 1, \ldots, d_\lambda).$$

Conversely, suppose $\psi \in L^2(M, \mathcal{E}_\lambda)$ satisfies $\Delta^{(\lambda)}\psi = \nu\psi$. Put

$$\Psi^{(j)} = (0, \ldots, 0, \overset{(j)}{\psi}, 0, \ldots, 0) \quad (j = 1, \ldots, d_\lambda).$$

Then, $\psi_P^{(j)} = \mathrm{Trace}[{}^t\overline{\Psi^{(j)}}] \in L^2_\lambda(P)$ satisfies

$$\Delta_P \psi_P^{(j)} = (\nu + \|\lambda + \delta\|_K^2 - \|\delta\|_K^2)\psi_P^{(j)}.$$

Thus, we have the following for the spectrum $\{\nu_j^{(\lambda)}\}$ of $\Delta^{(\lambda)}$ and that of Δ_P.

Proposition 6 *The spectrum of Δ_P is the set of eigenvalues given by*

$$\bigcup_{\lambda \in \hat{G}} d_\lambda \cdot \left\{ \nu_j^{(\lambda)} + \|\lambda + \delta\|_K^2 - \|\delta\|_K^2 \mid j \in \mathbb{N} \right\},$$

where $d_\lambda \cdot \{\,\}$ denotes the set of d_λ copies of $\{\,\}$, and

$$d_\lambda = \prod_{\alpha \in R_+} \frac{(\lambda + \alpha, \alpha)_K}{(\delta, \alpha)_K}.$$

3 Quasi-mode for the mechanics in a gauge field

3.1 Quantum energies associated to a Lagrangian manifold

Suppose $\mu \in \mathfrak{g}^*$ is a quantized charge, namely, $i\lambda = \mathcal{O}_\mu \cap iC$ belongs to $i\Gamma^* \backslash \{0\}$. We have a quantum system associated to $\mathcal{H}_\mu = (M_\mu^\#, \Omega_\mu^\#, H_\mu^\#)$, that is a quantum Hamiltonian given by

$$\begin{aligned} \hat{H}_\lambda &= \Delta^{(\lambda)} + \|\lambda + \delta\|_K^2 \\ &= -\sum_{j,k} m^{jk}(x)\big(\nabla_j + A_j^{(\lambda)}(x)\big)\big(\nabla_k + A_k^{(\lambda)}(x)\big) + \|\lambda + \delta\|_K^2 \end{aligned}$$

acting on $L^2(M, \mathcal{E}_\lambda)$. For the element $\lambda \in C \cap \Gamma^*$ let us consider the "ladder" of representations with the highest weights $\{n\lambda;\ n \in \mathbb{N}\}$ and the associated family of quantum systems $(\hat{H}_{n\lambda}, L^2(M, \mathcal{E}_{n\lambda}))$.

In the case of abelian gauge group $U(1)$ we established in [5] a eigenvalue theorem for the magnetic Schrödinger operator, which asserts the existence of an approximate quantum energy associated to a certain classical energy level. We here present the following conjecture which is a generalization of the eigenvalue theorem to the case of non-abelian gauge group G.

Conjecture *Suppose there exists a compact Lagrangian submanifold L_P of (T^*P, Ω_P) contained in $J^{-1}(\mathcal{O}_\mu)$. Let $L = \chi_\mu \circ \pi_{\mathcal{O}_\mu}(L_P)$, which is a submanifold of $M_\mu^\#$. Assume the following conditions:*

(i) $H_\mu^\# \equiv e$ on L for a real constant e,

(ii) L_P is invariant under the Hamiltonian flow φ_t on $(T^*P, \Omega_P, \widetilde{H})$, and the restricted flow $\varphi|_{L_P}$ leaves invariant a non-zero half-density on L_P, and

(iii)(quantization condition) for every closed curve γ on L_P,

$$\frac{1}{2\pi}\int_\gamma \omega_P - \frac{1}{4} m_{L_P}([\gamma]) \in \mathbb{Z} \tag{3.1}$$

holds, where $m_{L_P} \in H^1(L_P, \mathbb{Z})$ is the Maslov class of L_P.

Let d be the smallest element of the set $\{1, 2, 4\}$ for which $d \cdot m_{L_P}([\gamma]) \equiv 0 \pmod 4$ for all $[\gamma] \in \pi_1(L_P)$, and set

$$n_k := dk + 1, \quad \tilde{n}_k := \frac{1}{2}\left(n_k + \frac{\|n_k\lambda + \delta\|_K}{\|\lambda\|_K}\right)$$

for $k \in \mathbb{N} \cup \{0\}$. (Note that $\tilde{n}_k \sim n_k$ as $k \to \infty$.)

Then, there is a sequence $\{E_{j_k}^{(n_k\lambda)}\}_{k=0}^\infty$ of eigenvalues of $\hat{H}_{n_k\lambda}$ such that

$$E_{j_k}^{(n_k\lambda)} = e\tilde{n}_k^2 + O(1) \quad (k \to \infty). \tag{3.2}$$

Observation Put $\hbar = 1/\tilde{n}_k$, and consider the Schrödinger operator

$$\hat{H}(\hbar) := \frac{1}{\tilde{n}_k^2}\hat{H}_{n_k\lambda}$$

depending on the Planck constant $\hbar$. Then, $E(\hbar) := E_{j_k}^{(n_k\lambda)}/\tilde{n}_k^2$ is an eigenvalue of $\hat{H}(\hbar)$, and the formula (3.2) means that

$$E(\hbar) = e + O(\hbar^2)$$

as $\hbar \to 0$. Thus, we see that the classical energy e obtained by the quantization condition gives an approximation of a quantum energy of order $\hbar^2$ in a semi-classical sense.

3.2 Plan to prove the conjecture

Let
$$\widetilde{G} := S^1 \times G = \{(e^{it}, g);\ 0 \le t < 2\pi, g \in G\}.$$
The strategy to prove the conjecture is to construct a suitable operator $A : \mathcal{D}'(\widetilde{G}) \to \mathcal{D}'(P)$ (where $\mathcal{D}'(\cdot)$ denotes the space of distributions). The idea is essentially due to [11] by Weinstein, and applied in [5] in the case of magnetic flow, i.e., $G = U(1)$.

By virtue of the Peter-Weyl each element $u(t, g)$ in $L^2(\widetilde{G})$ is written as
$$u(t,g) = \sum_{\ell\in\mathbb{Z}} \sum_{\rho\in\hat{G}} \sum_{j,k} \hat{u}^{jk}_{\ell\rho}\, e^{i\ell t}[\rho(g)]^j_k. \tag{3.3}$$

For the sequence $\{n_k\}_{k=0}^{\infty}$ ($n_k = dk+1$) we define the subspace $L^2(\widetilde{G}; \{n_k\lambda\})$ of $L^2(\widetilde{G})$ as follows: A function $u \in L^2(\widetilde{G})$ written as (3.3) belongs to $L^2(\widetilde{G}; \{n_k\lambda\})$ if and only if $\hat{u}^{\cdot\cdot}_{\ell\rho} = 0$ holds for every $(\ell, \rho) \notin \{(n_k, n_k\lambda)\}_{k=0}^{\infty}$.

Put $D_G := (\Delta_G + \|\delta\|_K)^{1/2}$, which is a first order pseudodifferential operator satisfying
$$D_G[\rho_{n\lambda}(g)]^j_k = (\|n\lambda + \delta\|_K)[\rho_{n\lambda}(g)]^j_k \quad (n \in \mathbb{N}).$$
Let us consider a continuous linear operator $A : \mathcal{D}'(\widetilde{G}) \to \mathcal{D}'(P)$ which satisfies the following conditions:

(A-i) $e^{-1}\Delta_P A - AD_{\widetilde{G}}$ induces a bounded operator from $L^2(\widetilde{G})$ to $L^2(P)$, where
$$D_{\widetilde{G}} := -\frac{1}{4}\Big(\frac{\partial}{\partial t} + \frac{i}{\|\lambda\|_K} D_G\Big)^2.$$

(A-ii) $A : L^2(\widetilde{G}; \{n_k\lambda\}) \to L^2(P)$ is an isometry.

(A-iii) Take
$$(u_k)^j_l(t,g) := \sqrt{\frac{d_k}{2\pi}}\, e^{in_k t}\, [\rho_{n_k\lambda}(g)]^j_l \quad (d_k := \dim V_{n_k\lambda})$$
in $L^2(\widetilde{G}; \{n_k\lambda\})$. Then, $\psi_k = (\psi_k)^j_l := A[(u_k)^j_l]$ belongs to $L^2_{n_k\lambda}(P) \cong L^2_{n_k\lambda}(P, V^*_{n_k\lambda} \otimes V_{n_k\lambda})$.

Suppose we have the above operator A. Note that
$$D_{\widetilde{G}} u_k = \tilde{n}_k^2 u_k.$$

By virtue of (A-i) we have
$$\begin{aligned} \|(e^{-1}\Delta_P - \tilde{n}_k^2)\psi_k\|_{L^2(P)} &= \|(e^{-1}\Delta_P A - AD_{\widetilde{G}})u_k\|_{L^2(P)} \\ &\le M\|u_k\|_{L^2(\widetilde{G})} = M, \end{aligned} \tag{3.4}$$

M being a constant. Let $\{\varphi_j^{(k)}\}$ be the orthonormal basis of eigenfunction of $\Delta_P|_{L^2_{n_k\lambda}(P)}$. By means of Lemma 5 we have

$$\Delta_P \varphi_j^{(k)} = \tilde{E}_j^{(n_k\lambda)} \varphi_j^{(k)}$$

with

$$\tilde{E}_j^{(n_k\lambda)} = E_j^{(n_k\lambda)} - \|\delta\|_K^2. \tag{3.5}$$

Using the expansion: $\psi_k = \sum_j \hat{\psi}_j \varphi_j^{(k)}$, we have

$$\begin{aligned}
\|(e^{-1}\Delta_P - \tilde{n}_k^2)\psi_k\|^2_{L^2(P)} &= \|e^{-1}\sum_j \hat{\psi}_j \tilde{E}_j^{(n_k\lambda)} \varphi_j^{(k)} - \sum_j \tilde{n}_k^2 \hat{\psi}_j \varphi_j^{(k)}\|^2_{L^2(P)} \\
&= \frac{1}{e^2}\sum_j \{\tilde{E}_j^{(n_k\lambda)} - e\tilde{n}_k^2\}^2 |\hat{\psi}_j|^2 \\
&\geq \frac{1}{e^2}\min_j \{\tilde{E}_j^{(n_k\lambda)} - e\tilde{n}_k^2\}^2 \sum_j |\hat{\psi}_j|^2 \\
&= \frac{1}{e^2}\min_j \{\tilde{E}_j^{(n_k\lambda)} - e\tilde{n}_k^2\}^2 .
\end{aligned}$$

Note $\sum_j |\hat{\psi}_j|^2 = 1$ by means of (A-ii). Combining this inequality with (3.4), we have

$$\min_j \{\tilde{E}_j^{(n_k\lambda)} - e\tilde{n}_k^2\}^2 \leq e^2 M,$$

that is

$$|\tilde{E}_{j_k}^{(n_k\lambda)} - e\tilde{n}_k^2| = \min_j |\tilde{E}_j^{(n_k\lambda)} - e\tilde{n}_k^2| \leq \text{Const.} \tag{3.6}$$

We obtain the formula (3.2) from (3.5) and (3.6). The sequence $\{(\psi_k, e\tilde{n}_k^2)\}_{k=0}^{\infty}$ in this argument is called a quasi-mode of Δ_P (cf. [2]).

Thus, a proof of the conjecture is carried out if we can construct the operator A and check the properties (A-i)-(A-iii). We expect that this procedure will be similarly performed as [5] (see also [10], [11]) by constructing the operator A as a Fourier integral operator under the quantization condition (3.1).

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