

On Partitions and k -Polygons

By

Takuya DOI and Shin-ichi KATAYAMA

*Department of Mathematical Sciences,
Faculty of Integrated Arts and Sciences
The University of Tokushima, Tokushima 770-8502, JAPAN*

*e-mail address : c100941033@stud.tokushima-u.ac.jp
: katayama@ias.tokushima-u.ac.jp*

(Received September 30, 2010)

Abstract

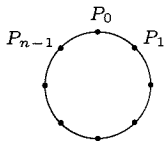
Let C be a circle divided into n parts equally. The set of the ends of these parts on C are denoted by $S = \{P_0, P_1, \dots, P_{n-1}\}$. Let $C_k(n)$ be the number of incongruent k -polygons inscribed in C , where the vertices of k -polygons are chosen from S . In this note, we shall investigate the generating functions of $C_4(n)$ and $C_5(n)$.

2000 Mathematics Subject Classification. Primary 05A17; Secondary 11P81

Introduction

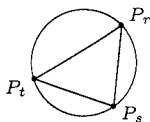
Let C be a circle divided into n parts equally and the ends of parts be labeled $S = \{P_0, P_1, \dots, P_{n-1}\}$ as in the following Figure 1:

Figure 1



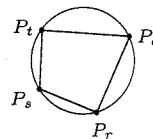
$$S = \{P_0, P_1, \dots, P_{n-1}\}$$

Figure 2



$$P_r, P_s, P_t \in S$$

Figure 3



$$P_q, P_r, P_s, P_t \in S$$

Let $C_3(n)$ be the number of incongruent triangles $\triangle P_r P_s P_t$ with vertices P_r, P_s, P_t chosen from $S = \{P_0, P_1, \dots, P_{n-1}\}$ as in Figure 2. In our previous paper [4], we have proved

$$C_3(n) = p(n - 3, 3),$$

where $p(n, m)$ denotes the number of partitions of n with each part $\leq m$, that is

$$p(n, m) = p(n \mid \text{parts in } \{1, 2, \dots, m\}).$$

It is well known that $p(n, 1) = 1$ and $p(n, 2) = \left\lceil \frac{n}{2} \right\rceil + 1$, where $[x]$ denotes the greatest integer $\leq x$. In the case $m = 3$, it has been known that (see for example [3] Chapter 3)

$$p(n, 3) = \left\{ \frac{(n+3)^2}{12} \right\},$$

where $\{x\}$ denotes the nearest integer to x . One can easily verify that $p(n, m)$ has the following generating function:

$$\sum_{n=0}^{\infty} p(n, m) q^n = \frac{1}{(1-q)(1-q^2) \cdots (1-q^m)}.$$

Let $C_k(n)$ be the number of incongruent k -polygons $P_{i_1} P_{i_2} \dots P_{i_k}$ inscribed in C , where the vertices $\{P_{i_1}, P_{i_2}, \dots, P_{i_k}\}$ ($1 \leq k \leq n$) are chosen from S . In our previous paper [4], we have shown:

$$C_k(n) = p(n - k, k) \quad \text{for } 1 \leq k \leq 3.$$

and

$$C_4(n) \neq p(n - 4, 4), \quad \text{and } C_4 \geq p(n - 4, 4) \quad \text{for small } n.$$

In this paper, we shall show

$$C_4(n) > p(n - 4, 4) \quad \text{for } n \geq 6,$$

and more precisely,

$$C_4(n) = p(n - 4, 4) + p(n - 6, 4) + p(n - 8, 4),$$

where $p(n, m)$ are defined to be 0 for $n < 0$. Moreover, we shall show $C_5(n)$ has much complicated representation in the last section.

Quadratics

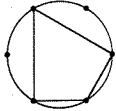
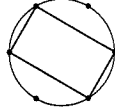
Here we shall introduce several notations. Let $S = \{P_0, P_1, \dots, P_{n-1}\}$ be as above. Let \wp_k be the set of k -polygons where the vertices $P_{i_1}, P_{i_2}, \dots, P_{i_k}$ ($0 \leq$

$i_1 < i_2 < \dots < i_k \leq n-1$) are took from S . Put $a_1 = i_2 - i_1, a_2 = i_3 - i_2, \dots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$. Let O be the center of the given circle C . Then the central angles of k -polygon $P_{i_1}P_{i_2}\dots P_{i_k}$ are $\angle P_{i_j}OP_{i_{j+1}} = \frac{2\pi a_j}{n}$ ($1 \leq j \leq k-1$) and $\angle P_{i_k}OP_{i_1} = \frac{2\pi a_k}{n}$.

Then a_1, a_2, \dots, a_k satisfies $a_1, a_2, \dots, a_k \geq 1$ and $a_1 + a_2 + \dots + a_k = n$. In the following, we shall denote the k -polygon $P_0P_{a_1}P_{a_1+a_2}\dots P_{a_1+a_2+\dots+a_{k-1}}$ by $P(a_1, a_2, \dots, a_k)$. Then any k -polygon $P_{i_1}P_{i_2}\dots P_{i_k}$ is congruent to the k -polygon $P(a_1, a_2, \dots, a_k)$, where $a_1 = i_2 - i_1, \dots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$ as above.

We shall use the notation \cong when two k -polygons $P(a_1, \dots, a_k)$ and $P(b_1, \dots, b_k)$ are congruent.

In our previous paper [4], we noted that $P(a_1, a_2, a_3) \cong P(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation $\sigma \in S_3$. Here we note that these properties do not hold for the cases $k \geq 4$. For example, let us verify the case $n = 6$ and $k = 4$. Then a partition $6 = 2 + 2 + 1 + 1$, corresponds to the following two incongruent quadrangles $P(2, 2, 1, 1)$ and $P(2, 1, 2, 1)$.

 $P(2, 2, 1, 1)$  $P(2, 1, 2, 1)$

Hence, in the case of quadrangles, we must consider the congruent classes of quadrangles which correspond to the same partition $n = a + b + c + d$. Thus we shall consider the following 8 cases of the partitions of n separately.

- (1) $P(n \mid n = a + b + c + d, \text{ with } a > b > c > d \geq 1)$,
- (2) $P(n \mid n = 2a + b + c, \text{ with } a > b > c \geq 1)$,
- (3) $P(n \mid n = a + 2b + c, \text{ with } a > b > c \geq 1)$,
- (4) $P(n \mid n = a + b + 2c, \text{ with } a > b > c \geq 1)$,
- (5) $P(n \mid n = 2a + 2b, \text{ with } a > b \geq 1)$,
- (6) $P(n \mid n = 3a + b, \text{ with } a > b \geq 1)$,
- (7) $P(n \mid n = a + 3b, \text{ with } a > b \geq 1)$,
- (8) $P(n \mid n = 4a, \text{ with } a \geq 1)$.

Firstly, we shall consider the case (1).

It is easy to see that each partition $n = a + b + c + d$ corresponds to exactly 3 incongruent quadrangles $P(a, b, c, d)$, $P(a, c, d, b)$ and $P(a, d, b, c)$. On the other hand, we have

$$\begin{aligned} P(n \mid n = a + b + c + d, \text{ with } a > b > c > d \geq 1) \\ = P(n - 10 \mid n - 10 = (a - 4) + (b - 3) + (c - 2) + (d - 1), \\ \text{with } a - 4 \geq b - 3 \geq c - 2 \geq d - 1 \geq 0) \end{aligned}$$

$$\begin{aligned}
&= P(n-10 \mid n-10 = 4x + 3y + 2z + w, \text{ with } x, y, z, w \geq 0) \\
&= P(n-10 \mid \text{parts in } \{1, 2, 3, 4\}).
\end{aligned}$$

Thus the generating function of the partition $P(n \mid n = a + b + c + d, \text{ with } a > b > c > d \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = a + b + c + d, \text{ with } a > b > c > d \geq 1)q^n = \frac{q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

Now we denote the subset of congruent classes of quadrangles \wp_4 / \cong which correspond to this case (1), by $\overline{\wp}_4(a, b, c, d)$. We also denote the number of congruent classes $\overline{\wp}_4(a, b, c, d)$ by $C_4(a, b, c, d)$. Since $C_4(a, b, c, d) = 3P(n-10 \mid \text{parts in } \{1, 2, 3, 4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a, b, c, d)q^n = \frac{3q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

Now we shall consider the case (2), similarly.

It is easy to see that each partition $n = 2a + b + c$ corresponds to exactly 2 incongruent quadrangles $P(a, a, b, c)$ and $P(a, b, a, c)$. Moreover we have

$$\begin{aligned}
P(n \mid n = 2a + b + c, \text{ with } a > b > c \geq 1) \\
&= P(n-9 \mid n-9 = 2(a-3) + (b-2) + (c-1), \\
&\quad \text{with } a-3 \geq b-2 \geq c-1 \geq 0) \\
&= P(n-9 \mid \text{parts in } \{2, 3, 4\}).
\end{aligned}$$

Thus the generating function of the partition $P(n \mid n = 2a + b + c, \text{ with } a > b > c \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = 2a + b + c, \text{ with } a > b > c \geq 1)q^n = \frac{q^9}{(1-q^2)(1-q^3)(1-q^4)}.$$

We denote the subset of congruent classes of quadrangles \wp_4 / \cong which correspond to this case (2), by $\overline{\wp}_4(a, a, b, c)$. We also denote the number of congruent classes of $\overline{\wp}_4(a, a, b, c)$ by $C_4(a, a, b, c)$. Since $C_4(a, a, b, c) = 2P(n-9 \mid \text{parts in } \{2, 3, 4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a, a, b, c)q^n = \frac{2q^9}{(1-q^2)(1-q^3)(1-q^4)}.$$

Now we shall consider the case (3), similarly and define the symbols $\overline{\wp}_4(a, b, b, c) = C_4(a, a, b, c)$ as above. Then each partition $n = a + 2b + c$ corresponds to exactly 2 incongruent quadrangles $P(a, b, b, c)$ and $P(a, b, c, b)$. The generating function of the partition $P(n \mid n = a + 2b + c, \text{ with } a > b > c \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = a + 2b + c, \text{ with } a > b > c \geq 1)q^n = \frac{q^8}{(1-q)(1-q^3)(1-q^4)}.$$

Since $C_4(a, b, b, c) = 2P(n - 8 \mid \text{parts in } \{1, 3, 4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a, b, b, c)q^n = \frac{2q^8}{(1-q)(1-q^3)(1-q^4)}.$$

In the case (4), we shall use the similar definition of the symbols $\overline{\varphi}_4(a, b, c, c) = C_4(a, b, c, c)$ as above. Since each partition $n = a + b + 2c$ corresponds to exactly 2 incongruent quadrangles $P(a, b, c, c)$ and $P(a, c, b, c)$, we have $C_4(a, b, c, c) = 2P(n - 7 \mid \text{parts in } \{1, 2, 4\})$. Hence

$$\sum_{n=0}^{\infty} C_4(a, b, c, c)q^n = \frac{2q^7}{(1-q)(1-q^2)(1-q^4)}.$$

In the case (5), we shall use the similar definition of the symbols $\overline{\varphi}_4(a, a, b, b) = C_4(a, a, b, b)$ as above. Since each partition $n = 2a + 2b$ corresponds to exactly 2 incongruent quadrangles $P(a, a, b, b)$ and $P(a, b, a, b)$, we have $C_4(a, a, b, b) = 2P(n - 6 \mid \text{parts in } \{2, 4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a, a, b, b)q^n = \frac{2q^6}{(1-q^2)(1-q^4)}.$$

In the case (6), we shall use the similar definition of the symbols $\overline{\varphi}_4(a, a, a, b) = C_4(a, a, a, b)$ as above. Since each partition $n = 3a + b$ corresponds to exactly one quadrangle $P(a, a, a, b)$, we have $C_4(a, a, a, b) = P(n - 7 \mid \text{parts in } \{3, 4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a, a, a, b)q^n = \frac{q^7}{(1-q^3)(1-q^4)}.$$

In the case (7), we shall use the similar definition of the symbols $\overline{\varphi}_4(a, b, b, b) = C_4(a, b, b, b)$ as above. Then each partition $n = a + 3b$ corresponds to exactly one quadrangle $P(a, b, b, b)$ and $C_4(a, b, b, b) = P(n - 5 \mid \text{parts in } \{1, 4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a, b, b, b)q^n = \frac{q^5}{(1-q)(1-q^4)}.$$

In the case (8), we shall use the similar definition of the symbols $\overline{\varphi}_4(a, a, a, a) = C_4(a, a, a, a)$ as above. Then each partition $n = 4a$ corresponds to exactly one quadrangle $P(a, a, a, a)$ and $C_4(a, a, a, a) = P(n - 4 \mid \text{part in } \{4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a, a, a, a)q^n = \frac{q^4}{1-q^4}.$$

From the definition of the symbols, we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_4(n)q^n &= \sum_{n=0}^{\infty} C_4(a, b, c, d)q^n + \sum_{n=0}^{\infty} C_4(a, a, b, c)q^n \\ &\quad + \sum_{n=0}^{\infty} C_4(a, b, b, c)q^n + \sum_{n=0}^{\infty} C_4(a, b, c, c)q^n \\ &\quad + \sum_{n=0}^{\infty} C_4(a, a, b, b)q^n + \sum_{n=0}^{\infty} C_4(a, a, a, b)q^n \\ &\quad + \sum_{n=0}^{\infty} C_4(a, b, b, b)q^n + \sum_{n=0}^{\infty} C_4(a, a, a, a)q^n. \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_4(n)q^n &= \frac{3q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \frac{2q^9}{(1-q^2)(1-q^3)(1-q^4)} \\ &\quad + \frac{2q^8}{(1-q)(1-q^3)(1-q^4)} + \frac{2q^7}{(1-q)(1-q^2)(1-q^4)} \\ &\quad + \frac{2q^6}{(1-q^2)(1-q^4)} + \frac{q^7}{(1-q^3)(1-q^4)} + \frac{q^5}{(1-q)(1-q^4)} + \frac{q^4}{1-q^4} \\ &= \frac{q^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \frac{2q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &\quad + \frac{q^9}{(1-q^2)(1-q^3)(1-q^4)} + \frac{q^8}{(1-q)(1-q^3)(1-q^4)} \\ &\quad + \frac{q^7}{(1-q)(1-q^2)(1-q^4)} + \frac{q^6}{(1-q^2)(1-q^4)} \\ &= \frac{q^4 + q^6 + q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &= \sum_{n=0}^{\infty} (p(n-4, 4) + p(n-6, 4) + p(n-8, 4))q^n, \end{aligned}$$

where $p(m, 4) = 0$ for $m < 0$.

Hence we have shown the following theorem:

Theorem 1. *With the above notation, we have*

$$\begin{aligned} C_4(n) &= p(n-4, 4) + p(n-6, 4) + p(n-8, 4), \\ \sum_{n=0}^{\infty} C_4(n)q^n &= \frac{q^4 + q^6 + q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)}. \end{aligned}$$

Pentagons

In this section we shall briefly show the generating function of $C_5(n)$.

In the case of pentagons, we must also consider the congruent classes of pentagons which correspond to the same partition $n = a + b + c + d + e$. For the sake of simplicity, we shall borrow the name of poker hands. Thus we shall treat the following 16 cases of the partitions of n separately.

(1) High card

$$P(n \mid n = a + b + c + d + e, \text{ with } a > b > c > d > e \geq 1),$$

(2) One pair

$$(2-1) P(n \mid n = 2a + b + c + d, \text{ with } a > b > c > d \geq 1),$$

$$(2-2) P(n \mid n = a + 2b + c + d, \text{ with } a > b > c > d \geq 1),$$

$$(2-3) P(n \mid n = a + b + 2c + d, \text{ with } a > b > c > d \geq 1),$$

$$(2-4) P(n \mid n = a + b + c + 2d, \text{ with } a > b > c > d \geq 1),$$

(3) Two pair

$$(3-1) P(n \mid n = 2a + 2b + c, \text{ with } a > b > c \geq 1),$$

$$(3-2) P(n \mid n = a + 2b + 2c, \text{ with } a > b > c \geq 1),$$

$$(3-3) P(n \mid n = 2a + b + 2c, \text{ with } a > b > c \geq 1),$$

(4) Three card

$$(4-1) P(n \mid n = 3a + b + c, \text{ with } a > b > c \geq 1),$$

$$(4-2) P(n \mid n = a + 3b + c, \text{ with } a > b > c \geq 1),$$

$$(4-3) P(n \mid n = a + b + 3c, \text{ with } a > b > c \geq 1),$$

(5) Full house

$$(5-1) P(n \mid n = 3a + 2b, \text{ with } a > b \geq 1),$$

$$(5-2) P(n \mid n = 2a + 3b, \text{ with } a > b \geq 1),$$

(6) Four card

$$(6-1) P(n \mid n = 4a + b, \text{ with } a > b \geq 1),$$

$$(6-2) P(n \mid n = a + 4b, \text{ with } a > b \geq 1),$$

(7) Five card

$$P(n \mid n = 5a, \text{ with } a \geq 1).$$

Now we shall verify that the number of incongruent pentagons which correspond to the same partition $n = a + b + c + d + e$.

In the case of high card (1), we see there are 12 incongruent pentagons

$$P(a, b, c, d, e), P(a, b, d, c, e), P(a, b, d, e, c), P(a, b, e, d, c),$$

$$P(a, b, e, c, d), P(a, b, c, e, d), P(a, c, b, d, e), P(a, c, d, b, e),$$

$$P(a, c, b, e, d), P(a, c, e, b, d), P(a, d, b, c, e), P(a, d, c, b, e).$$

In the case of one pair (2), we see there are 6 incongruent pentagons for

the same partition. Her we shall list up for the case $(2 - 1)$, i.e., the case $n = 2a + b + c + d$. Then there are 6 incongruent pentagons

$$P(a, a, b, c, d), P(a, a, c, d, b), P(a, a, d, b, c), P(a, b, a, c, d), \\ P(a, c, a, b, d), P(a, d, a, b, c).$$

In the case of two pair (3), we see there are 4 incongruent pentagons for the same partition. Her we shall list up for the case $(3 - 1)$, i.e., the case $n = 2a + 2b + c$. Then there are 4 incongruent pentagons

$$P(c, a, a, b, b), P(c, a, b, a, b), P(c, a, b, b, a), P(c, b, a, a, b).$$

In the case of three card (4), we see there are 2 incongruent pentagons for the same partition. Her we shall list up for the case $(4 - 1)$, that is for the case $n = 3a + b + c$. Then there are 2 incongruent pentagons

$$P(a, a, a, b, c), P(a, a, b, a, c).$$

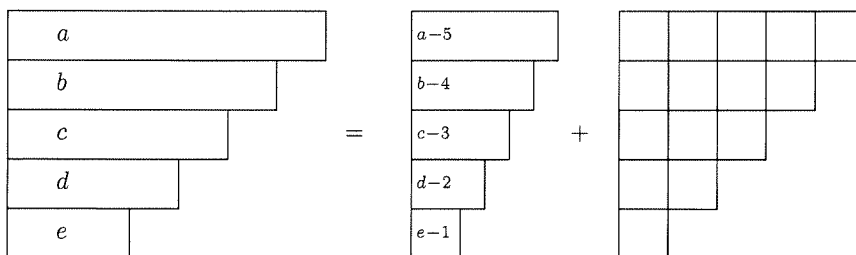
In the case of full house (5), we see there are 2 incongruent pentagons for the same partition. Her we shall list up for the case $(5 - 1)$, that is for the case $n = 3a + 2b$. Then there are 2 incongruent pentagons

$$P(a, a, a, b, b), P(a, a, b, a, b).$$

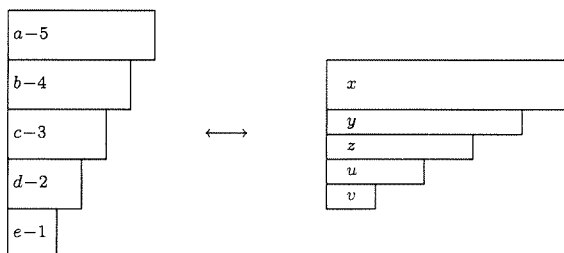
In the cases four card and five card, there is exactly one pentagon which corresponds to each partition.

The case (1)

Firstly we shall consider the case of high card. Then the Young diagram of the partition $n = a + b + c + d + e$ with $a > b > c > d > e \geq 1$ satisfies the following relations.



Put $x = e - 1, y = d - e - 1, z = c - d - 1, u = b - c - 1, v = a - b - 1$. Then $x, y, z, u, v \geq 0$ and the following conjugate of Young diagrams implies:



Thus we know that

$$\begin{aligned} P(n \mid n = a + b + c + d + e, \text{ with } a > b > c > d > e \geq 1) \\ &= P(n - 15 \mid n - 15 = (a - 5) + (b - 4) + (c - 3) + (d - 2) + (e - 1), \\ &\quad \text{with } a - 5 \geq b - 4 \geq c - 3 \geq d - 2 \geq e - 1 \geq 0) \\ &= P(n - 15 \mid n - 15 = 5x + 4y + 3z + 2u + v, \text{ with } x, y, z, u, v \geq 0) \\ &= P(n - 15 \mid \text{parts in } \{1, 2, 3, 4, 5\}). \end{aligned}$$

Hence the generating function of the partition

$P(n \mid n = a + b + c + d + e, \text{ with } a > b > c > d > e \geq 1)$ is

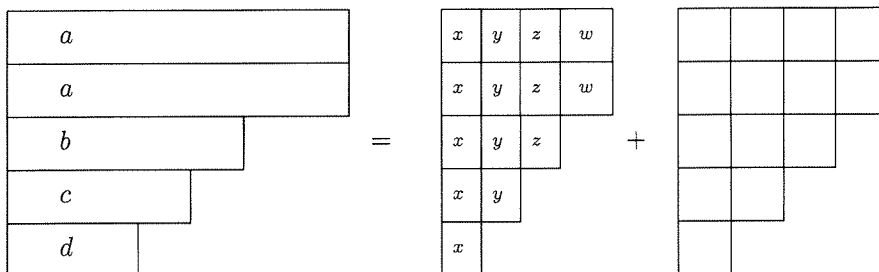
$$\begin{aligned} \sum_{n=0}^{\infty} P(n \mid n = a + b + c + d + e, \text{ with } a > b > c > d > e \geq 1) q^n \\ &= \frac{q^{15}}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)}. \end{aligned}$$

We denote the number of incongruent pentagons which corresponds to the partition $n = a + b + c + d + e$ by $C_5(a, b, c, d, e)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, c, d, e) q^n = \frac{12q^{15}}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)}.$$

The case (2-1)

Now we shall consider the case of one pair (2-1). We know the partition $n = 2a + b + c + d$ with $a > b > c > d \geq 1$ corresponds to



Here $x = d - 1, y = c - d - 1, z = b - c - 1, w = a - b - 1$. Then $x, y, z, w \geq 0$ and the conjugation of Young diagrams imply that $n - 14 = 5x + 4y + 3z + 2w$ with $x, y, z, w \geq 0$.

Thus we know that

$$\begin{aligned} P(n \mid n = 2a + b + c + d, \text{ with } a > b > c > d \geq 1) \\ &= P(n - 14 \mid n - 14 = 2(a - 4) + (b - 3) + (c - 2) + (d - 1), \\ &\quad \text{with } a - 4 \geq b - 3 \geq c - 2 \geq d - 1 \geq 0) \\ &= P(n - 14 \mid n - 14 = 5x + 4y + 3z + 2w, \text{ with } x, y, z, w \geq 0) \\ &= P(n - 14 \mid \text{parts in } \{2, 3, 4, 5\}). \end{aligned}$$

Hence the generating function of the partition $P(n \mid n = 2a + b + c + d, \text{ with } a > b > c > d \geq 1)$ is

$$\begin{aligned} & \sum_{n=0}^{\infty} P(n \mid n = 2a + b + c + d, \text{ with } a > b > c > d \geq 1) q^n \\ &= \frac{q^{14}}{(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)}. \end{aligned}$$

We denote the number of incongruent pentagons which corresponds to the partition $n = 2a + b + c + d$ by $C_5(a, a, b, c, d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, c, d) q^n = \frac{6q^{14}}{(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)}.$$

The case (2-2)

For the case of one pair (2-2), we see that $n = a + 2b + c + d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 2b + c + d$ by $C_5(a, b, b, c, d)$. Then similarly as above, we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, c, d) q^n = \frac{6q^{13}}{(1 - q)(1 - q^3)(1 - q^4)(1 - q^5)}.$$

The case (2-3)

Consider the case of one pair (2-3) similarly. Then $n = a + b + 2c + d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + b + 2c + d$ by $C_5(a, b, c, c, d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, c, c, d) q^n = \frac{6q^{12}}{(1 - q)(1 - q^2)(1 - q^4)(1 - q^5)}.$$

The case (2-4)

Consider the case of one pair (2-4). Then $n = a + b + c + 2d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + b + c + 2d$ by $C_5(a, b, c, d, d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, c, d, d) q^n = \frac{6q^{11}}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^5)}.$$

The case (3-1)

Consider the case of two pair (3-1). Then $n = 2a + 2b + c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a + 2b + c$ by $C_5(a, a, b, b, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, b, c) q^n = \frac{4q^{11}}{(1 - q^2)(1 - q^4)(1 - q^5)}.$$

The case (3-2)

Consider the case of two pair (3-2), similarly. Then $n = 2a + b + 2c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a + b + 2c$ by $C_5(a, a, b, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, c, c)q^n = \frac{4q^{10}}{(1-q^2)(1-q^3)(1-q^5)}.$$

The case (3-3)

Consider the case of two pair (3-3). Then $n = a + 2b + 2c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 2b + 2c$ by $C_5(a, b, b, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, c, c)q^n = \frac{4q^9}{(1-q)(1-q^3)(1-q^5)}.$$

The case (4-1)

Now we shall consider the case of three card (4-1). Then $n = 3a + b + c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 3a + b + c$ by $C_5(a, a, a, b, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, b, c)q^n = \frac{2q^{12}}{(1-q^3)(1-q^4)(1-q^5)}.$$

The case (4-2)

Consider the case of three card (4-2). Then $n = a + 3b + c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 3b + c$ by $C_5(a, b, b, b, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, b, c)q^n = \frac{2q^{10}}{(1-q)(1-q^4)(1-q^5)}.$$

The case (4-3)

Consider the case of three card (4-3). Then $n = a + b + 3c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + b + 3c$ by $C_5(a, b, c, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, c, c, c)q^n = \frac{2q^8}{(1-q)(1-q^2)(1-q^5)}.$$

The case (5-1)

Now we shall consider the case of full house (5-1). Then $n = 3a + 2b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to

the partition $n = 3a + 2b$ by $C_5(a, a, a, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, b, b)q^n = \frac{2q^8}{(1-q^3)(1-q^5)}.$$

The case (5-2)

Consider the case of full house (5-2). Then $n = 2a + 3b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a + 3b$ by $C_5(a, a, b, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, b, b)q^n = \frac{2q^7}{(1-q^2)(1-q^5)}.$$

The case (6-1)

Now we shall consider the case of four card (6-1). Then $n = 4a + b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 4a + b$ by $C_5(a, a, a, a, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, a, b)q^n = \frac{q^9}{(1-q^4)(1-q^5)}.$$

The case (6-2)

Consider the case of four card (6-2). Then $n = a + 4b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 4b$ by $C_5(a, b, b, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, b, b)q^n = \frac{q^6}{(1-q)(1-q^5)}.$$

The case (7)

Now we shall consider the case of five card (7). Then $n = 5a$ with $a \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 5a$ by $C_5(a, a, a, a, a)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, a, a)q^n = \frac{q^5}{(1-q^5)}.$$

From the definition of the symbols, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} C_5(n)q^n &= \frac{12q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} + \frac{6q^{14}}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)} \\
&+ \frac{6q^{13}}{(1-q)(1-q^3)(1-q^4)(1-q^5)} + \frac{6q^{12}}{(1-q)(1-q^2)(1-q^4)(1-q^5)} \\
&+ \frac{6q^{11}}{(1-q)(1-q^2)(1-q^3)(1-q^5)} + \frac{4q^{11}}{(1-q^2)(1-q^4)(1-q^5)} \\
&+ \frac{4q^{10}}{(1-q^2)(1-q^3)(1-q^5)} + \frac{4q^9}{(1-q)(1-q^3)(1-q^5)} \\
&+ \frac{2q^{12}}{(1-q^3)(1-q^4)(1-q^5)} + \frac{2q^{10}}{(1-q)(1-q^4)(1-q^5)} \\
&+ \frac{2q^8}{(1-q)(1-q^2)(1-q^5)} + \frac{2q^8}{(1-q^3)(1-q^5)} + \frac{2q^7}{(1-q^2)(1-q^5)} \\
&+ \frac{q^9}{(1-q^4)(1-q^5)} + \frac{q^6}{(1-q)(1-q^5)} + \frac{q^5}{1-q^5} \\
&= \frac{q^5 + q^7 + q^8 + 2q^9 + 2q^{10} + 2q^{11} + q^{12} + q^{13} + q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}
\end{aligned}$$

Hence we have shown the following theorem:

Theorem 2. *With the above notation, we have the formulae*

$$\sum_{n=0}^{\infty} C_5(n)q^n = \frac{q^5 + q^7 + q^8 + 2q^9 + 2q^{10} + 2q^{11} + q^{12} + q^{13} + q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

References

- [1] G. E. Andrews, A note on partitions and triangles with integer sides, *American Mathematical Monthly*, **86**, (1979), 477-478.
- [2] G. E. Andrews, MacMahon's partition analysis: II, *Annals of Combinatorics*, **4**, (2000), 327-338.
- [3] G. E. Andrews and K. Eriksson, *Integer Partitions*, Cambridge University Press, Cambridge 2004.
- [4] T. Doi and S.-I. Katayama, A remark on partitions and triangles, *Journal of Mathematics, The University of Tokushima*, **43**, (2009), 1-7.