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Study on the New Axiomatic Method Giving the Solutions of Hilbert's 2nd and 6th Problems

By

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Abstract

In this paper, we propose the new axiomatic method completely different from old ones.

Thereby we succeeded in giving the definition of the concept of natural number and solving the problem of its existence. This is the complete solution of Hilbert's second problem. As for this, see Ito [4], [24].

Further we give the complete solution of Hilbert's 6th problem concerning the natural statistical physics. As for this, see Ito [1]∼[3], [5]∼[23], [25], [26].

These solutions are obtained by the new axiomatic method completely different from Hilbert's program.

Here we do not need Gödel's incompleteness theorem.

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Introduction

At the starting point of the mathematical studies, we have to answer the question "What is the natural number ?".

In order to answer this question, it is the problem to define the concept of natural number using the system of axioms of natural number and to prove the existence theorem of natural number.

Like this, Hilbert already had a realization that the definition of a certain mathematical concept is given by the system of axioms of that mathematical concept.

In this case, Hilbert request that the system of axioms should be consistent and give the problem of proof of its consistency using Hilbert's program.

Nevertheless, this Hilbert's program failed by the reason of Gödel's discovery of the "incompleteness theorem".

Therefore, in order to define the concept of natural number reasonably, we have to revise the axiomatic method used until now.

Therefore, we propose the new axiomatic method and, thereby, we succeed in giving the definition of the concept of natural number and solving the problem of its existence.

Namely, we completely determine the countable set of all natural numbers as a algebraic system on the basis of the system of axioms defining the concept of natural number.

Thereby, we prove the proposition that the system of axioms of natural number is consistent by using the new axiomatic method completely different from Hilbert's program.

Thus, we give the complete answer of Hilbert's second problem.

We revisit the axiomatic method using in Hilbert's 6th problem in the view point of the new axiomatic methods. Thereby we give the complete answer of Hilbert's 6th problem concerning the natural statistical physics which concerns with the natural statistical phenomena of electrons, atoms and molecules.

As for the more precise facts, we refer to the references.

1 Problems of mathematics

At the student age, I was instructed in "**Problems of Mathematics**" and "**Einstein and Bohr's Argument**". After that, I continued to study these problems in my life.

After all, as the consequences, I contribute the solutions of the second problem "Consistency of the System of Axioms of Natural Number" and 6th problem "Mathematical Treatment of the System of Axioms of Physics" among "Hilbert's 23 Problems of Mathematics".

Here I mention these solutions.

2 Problems of the bases of mathematics

According to my idea, the problems of the bases of mathematics are the problems of "the definition of the concept of natural number and its existence". By virtue of my new axiomatic method, the system of propositions providing the concept is the "system of axioms", and the existence theorem of that concept is proved by constructing the model of that concept satisfying the system of axioms.

Thereby, the problem of "the definition of the concept of natural number and the existence theorem of the concept" was solved.

In the result, the problem of "the definition of the concept of real number and the existence theorem of the concept" was solved, and "the problems of the bases of mathematics" such as the problem of "the basis of analysis" and the problem of "the basis of geometry" are completely solved.

Thereby Hilbert's second problem was solved.

The axiomatic method gives the reason for correctness of the logical reasoning, namely, the demonstration of constructing the mathematical theory deductively by starting from the system of axioms.

Thus, when we solve Hilbert's second problem, we need not use Gödel's "Incompleteness Theorem".

Therefore, when we use the new axiomatic method, it is known that the problem of "the definition of concept and its existence" is important.

3 "The problem of the basis of quantum mechanics"

I discover the fact that the so-called "quantum phenomena" are the natural statistical phenomena which appear for the ensembles of electrons, atoms and molecules. On the basis of this fact, we create the theory of "natural statistical physics".

In the ensemble of electrons, atoms and molecules, each system of micro-particles is moving according Newtonian dynamics. Therefore, each system of micro-particles has its own values of position variables and momentum variables at every time.

Nevertheless, the values of position variables and momentum variables are considered to be determined randomly for every system of micro-particles. Considering such position variables and momentum variables as random variables defined on a probability space composed of the ensemble of micro-particles, we consider the mathematical model of the probability space and the random variables.

Then it is proved that the L^2-density which determines the probability distribution law of position variables and momentum variables considered as random variables is the solution of Schrödinger equation.

Thus the natural statistical phenomena appeared in the ensemble of electrons, atoms and molecules are given representations which have mathematically strict meanings.

Thereby, the problem "What is the physical reality ?" proposed by Einstein is strictly solved in a concrete manner. Therefore, "the problem of the basis of quantum mechanics" was completely solved.

Namely, this problem was solved by the creation of the new theory of "natural statistical physics".

From now on, we do not use the old "quantum mechanics" in order to understand the phenomena of electrons, atoms and molecules.

Thus "natural statistical phenomena" can be completely understood by the causality law.

The propositions which are represented as "the system of axioms" in the Hilbert's 6th problem "mathematical treatment of the system of axioms of physics" is better to consider the "physical laws" followed by physical phenomena.

The understanding and the explanation of a physical phenomenon is considered as rather the understanding and the explanation by virtue of the causality laws than the problem of logical reasoning.

In the mathematical history until now, we have **Euclid's axiomatic method, Hilbert's axiomatic method** and **Bourbaki's axiomatic method**.

Recently I discovered the new axiomatic method beyond them.

4 Axiomatic method

In mathematics or mathematical sciences, the demonstration is a very important problem. Starting from a proposition, we prove another proposition by the logical reasoning.

Then, the main method is an axiomatic method.

In the general point of view of an axiomatic method, we propose the propositions of a system of axioms by virtue of inductive thinkings starting from mathematical phenomena in the real world.

In the result, we determine the system of axioms of the definition of a concept using the propositions of the system of axioms as the conditions of the definition of the concept.

Thereby, we give the definition of that concept and prove the theorem of its existence. Thus we construct the system of the theory concerning that concept deductively.

The system of the theory is constructed by systematizing definitions, theorems, propositions and corollaries etc. systematically.

The mathematical concept used in mathematics is not an empirical concept. It is important to use the concept defined mathematically in a strict manner.

In the result, we have only to demonstrate the propositions concerning a certain mathematical concept or to calculate on the basis of the propositions as the conditions included in the system of axioms.

Until now, even so the definition of "the concept of natural number" which is placed first among all mathematical theorems depends on an empirical definition.

At present, this point is resolved and improved.

Thus "the axiomatic method " is the problem of "the method of logical reasoning".

Generally, the proof of a proposition is carried out by the logical reasoning from the presupposed propositions.

Then we trace back to another supposed proposition of the presupposed proposition.

Then we meet with the fundamental proposition from which we cannot trace back to the other.

This fundamental proposition cannot be proved by logical reasoning on the basis of some presupposed propositions.

Then we call this fundamental proposition an axiom and confirm that this proposition is trivially evident without any proof.

But the proposition confirmed evident here is not the proposition which is proved in the frame of mathematical theory.

Therefore, in a certain mathematical theory, the fundamental proposition as the starting point of this theory cannot be proved in this mathematical theory.

By virtue of the axiomatic method using such an "axiom", it is impossible to construct the completely concluded mathematical theory.

In this point, the axiomatic method until now has the limitation.

By virtue of the new axiomatic method, we can construct the completely concluded mathematical theory.

Here there is no problem of proof of consistency of a system of axioms.

In the result, in the definition of the concept of natural number, we determine the system of axioms as the just enough conditions which is necessary and sufficient in order to determine completely the set of all natural numbers as an algebraic system.

We call this condition of the system of axioms **the completeness condition**.

Next, we prove the existence of the set of all natural numbers as an algebraic system, which is provided by these conditions.

Thereby, the problem of the definition of the concept of natural number and its existence is completely solved.

On the other hand, in order to understand the natural phenomena, it is important to use the causality laws that these natural phenomena come out from any cause and arise as any kind of result.

It is important to understand that, by virtue of the causes of the phenomena, the results of the phenomena arise by the cause of certain principles and laws.

Therefore, when we understand the physical phenomena and explain them, we do not understand and explain them using "the system of axioms of physics" which Hilbert proposed in Hilbert's 6th problem, but we understand and explain that physical phenomena on the basis of the causality laws using "the physical laws".

Therefore, we need revise the methods with which we construct the physical theory for understanding the physical phenomena by using an axiomatic method as until now.

This is a new understanding of the axiomatic methods too.

5 New axiomatic method

In this section, we study the new axiomatic method which I use in my study of mathematics.

The system of axioms for defining a mathematical concept is not a system which is only up to an arbitrary human idea.

It must be constructed so that we can understand all the mathematical phenomena and the natural phenomena, naturally and reasonably , for which we can apply the mathematical theory which is constructed on the basis of that system of axioms of the mathematical concept.

In a certain case, for example, by grasping the mathematical characters of points, lines and planes as the mathematical relations, we need the process of proposing the system of axioms for studying the geometric characters of points, lines and planes. By using the inductive method and the deductive method, we can realize the consistency of the world of the theoretical mathematics and the world of the empirical science.

We cannot construct the geometry by using "tables", "chairs" and "beer glasses" as Hilbert said.

6 Definition of a mathematical concept and its existence theorem

One system of axioms in mathematics is the condition for the definition of a certain mathematical concept. The existence of the mathematical concept defined by such a system of axioms is provided by constructing the model of that mathematical concept which satisfy the system of axioms.

The system of axioms of Euclidean geometry gives the definition of Euclidean geometry. The system of axioms of non-euclidean geometry gives the definition of non-euclidean geometry. Also, the system of axioms of natural number or real number gives the definition of natural number or real number respectively.

If the existence of such a mathematical concept is proved, the proofs of the theorems and calculations concerning that mathematical concept are carried out on the basis of the system of axioms of that mathematical concept. Such a concept of numbers 1, 2, 3, etc. as an empirical knowledge is meaningless in the theoretical calculations in mathematics.

The fact that we know natural numbers empirically is different from the fact that we know these mathematically. The natural numbers known empirically are only those which are read by giving a certain unit for 0, 1, 2, $\cdots$. These units are 一, 十, 百 and so on. For large numbers, there are the units such as 億, 兆, 京 or 恒河沙 and the astronomical units.

Even if we consider so large number as possible, we can only consider the numbers of finite magnitude empirically.

On the other hand, the set of all natural numbers which are the objects of mathematical calculations is an infinite set.

Even if the magnitude of its infinity is the smallest one among all infinities, we cannot reach that set by the way of human experience because it is an infinite set.

In order to define the concept of natural number, we have the difficulty to determine this infinite set uniquely.

The set of all natural numbers is not a mere sequence, but we can define the operations of addition and multiplication and can calculate the order relations for the natural numbers.

The infinite set with such a structure can be determined reasonably by providing the conditions which should be satisfied by natural numbers.

The determination of the infinite set composed of all natural numbers by choosing those conditions reasonably is the problem of the definition of the concept of natural number and its existence.

This problem was already solved by me.

Peano's system of axioms gives only the definition of finite ordinal number. We can construct its model in the ZFC-theory of sets and prove its existence. Therefore, Peano's system of axioms does not give the definition of natural number as an algebraic system in the essential meanings.

Nevertheless, for the finite ordinal numbers defined as above, we can define addition, multiplication and order relation constructively. Thereby we can construct a model of natural number using the finite ordinal numbers and can prove the existence of the concept of natural number. But this is not the definition of the concept of natural number.

Therefore, the definition of the concept of natural number as an algebraic system is now given by me at the first time.

I consider that the system of axioms of a certain concept gives the definition of this concept. Further, I consider that the existence of that concept is proved by constructing its model.

Here, I consider that the problem of the definition of a certain concept and its existence is the mathematically significant problem.

Thus, by changing the point of view, I could solve the problem of the definition of the concept of natural number and its existence.

The definition of a certain mathematical concept is provided by the complete just enough conditions which should be satisfied by that concept. Then that mathematical concept is the mathematical substance provided by these conditions.

For example, the concept of natural number is defined by providing the conditions which should be satisfied by that concept as the complete system of axioms. The concept of Euclidean geometry is also defined by providing the complete system of axioms.

As above, the system of axioms of a mathematical concept is the condition for definition of this concept.

Among the systems of axioms, there are those of fixing form such as the system of axioms of number and the system of axioms of Euclidean geometry, and those of fixing type such as the system of axioms of group. The latter are the systems of axioms of the concept of type such as a mathematical structure.

The former system of axioms gives the definition of the concept, and the latter system of axioms gives the definition of the concept of type such as a mathematical structure.

This is the fact that the definition of the concept is given by providing the system of axioms which should be satisfied by that concept or that concept of type.

The fact that the natural number is a mathematical substance means that the natural number is one mathematical concept.

In general, a mathematical substance is a mathematical concept. It is not a substance which is a material existence as a natural existence.

A theoretical system in mathematics is not only a logical system of operations of logics.

Providing a system of axioms of a certain mathematical concept means giving the definition of the mathematical concept so that a substance satisfying the system of axioms is the mathematical concept.

Then, we can prove the existence of the mathematical concept by constructing a model of the mathematical concept.

Then, the mathematical theory concerning this mathematical concept is the system of theorems and propositions derived from that system of axioms.

As above, it is cleared that the system of axioms of a certain mathematical concept gives the condition of the definition of the mathematical concept.

This is the reformation of the method of understanding the system of axioms used since the age of Greek philosophy.

Until now, we consider the problem of consistency of the system of axioms of a certain mathematical concept leaving the difference between the definition of the mathematical concept and its model dubious.

As considered here, in the point of view of the definition of a certain mathematical concept and the construction of its model., there is no need to consider the problem of consistency.

Rather than the above, the fact that we determine completely the countable set of all natural numbers as an algebraic system on the basis of the system of axioms of the concept of natural number means that we have proved the proposition "the system of axioms of natural number is consistent" by using the another axiomatic method completely different from Hilbert's program.

In a case, the definition means a mere naming. On the other hand, it means "axiom = definition", too.

Although we have two methods of definitions, in a case of the naming or the definition, we give the definition of the concept named by providing the condition as the system of axioms.

7 Proposition of natural laws and physical laws

Hilbert gave the problem "Mathematical treatment of the system of axioms of physics" as the 6th problem of "Problem of mathematics".

Nevertheless, in my study, it is really better to consider "the system of axioms of physics" as "the physical laws" which are followed by the physical phenomena. Therefore, we propose "the physical laws" instead of "the system of axioms of physics" used until now.

8 Laws of natural statistical physics

When we study the natural statistical phenomena using the theory of natural statistical physics, we have to propose the law of natural statistical physics by defining the meanings of the following:

(1) **Physical system.**

(2) **State of physical system.**

(3) **Motion of physical system.**

Here we rephrase "the system of axioms of natural statistical physics" as "the law of natural statistical physics".

The laws of natural statistical physics are formulated at first in Ito [2], [3].

This gives the mathematical expression to the ideas of natural statistical physics explained with words in Ito [5].

In Ito [15], [25], we gives the complete form such as explained in this paper.

9 "Logics" and "causality laws"

How different are "logics" and "causality law" ?

If we cannot find out the true answer of this question, we cannot see what is the truth in mathematics and natural science.

In mathematics, it is the problem that we prove the fact that we have a certain result under certain given prerequisite conditions or we confirm the result by calculation.

Here, in order to derive the result from the prerequisite conditions, we carry out reasoning, demonstration and calculation. In this time, the reasoning is carried out using logical laws as the rules of reasoning. The calculation is carried out using the rules of calculation. In these mathematical reasoning, it is important that they are "logical".

In general, the mathematical thinking is carried out by virtue of "logics", and the understanding of natural phenomena is carried out by virtue of "causality law".

In the understanding of mathematics and natural science, it is required that the conditions are satisfied so that it is natural and reasonable, so that it is integrated and universal, and so that it is understood by virtue of principles and laws.

In order to understand natural phenomena, it is important that we use causality laws such that the natural phenomena arise by virtue of some causes and those phenomena are raised to some results.

It is important to understand that the results of phenomena are raised by virtue of some principles and laws by the cause of the phenomena.

The study of the theory of physics in order to understand the physical phenomena and explain them is not the problem of mere "logics".

We clarify the causality laws of physical phenomena by using physical laws.

In general, "logics" is the rule of reasoning used by us. On the other hand, natural phenomena, especially physical phenomena arise by virtue of causality law. This is not a problem of reasoning, namely it is not the problem of "logics".

Therefore, because "the system of axioms" is the basis of logical reasoning, it is meaningless to consider such "system of axioms" as the basis of physical theory for clarifying physical phenomena.

At present, many studies are carried out by using the axiomatic method in mathematics and physics.

Nevertheless, I wish to notice that, among these studies by using these axiomatic methods, there are many studies which need to be reconsidered in the point of view of the new axiomatic method.

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