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A Note on Symmetric Differential Operators and Binomial Coefficients

By

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Abstract

In this note we derive some identities concerning the binomial coefficients by considering a certain n -th order symmetric differential operator on \mathbb{R}^m associated to the function $p(x, \xi) (x \in \mathbb{R}^m)$ which is a homogeneous polynomial in ξ .

2000 Mathematics Subject Classification. 05A10

Introduction

Let $\binom{n}{k}$ denote the binomial coefficients, namely

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k. \quad (1)$$

Various formulas for the binomial coefficients are well known (see e.g. [1], [2]). For example we have

$$\sum_{k=0}^n \binom{n}{k} = 2^n, \quad (2)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0, \quad (3)$$

$$\sum_{k=r}^n (-1)^{k-r} \binom{k}{r} \binom{n}{k} = 0 \quad (n \geq r), \quad (4)$$

which are easily obtained from (1). (The last one is obtained by differentiating (1) r times relative to x , dividing by $r!$, and putting $x = -1$.)

In this note we consider a certain linear symmetric differential operator, and derive some identities concerning the binomial coefficients (Corollaries 5 and 6).

1. Symmetric differential operators

Let $C_0^\infty(\mathbb{R}^m)$ denote the space of complex-valued C^∞ functions on \mathbb{R}^m with compact support. Suppose the space $C_0^\infty(\mathbb{R}^m)$ is endowed with the inner product (\cdot, \cdot) defined by

$$(f, g) := \int_{\mathbb{R}^m} f(x)\overline{g(x)} dx_1 \cdots dx_m \quad (f, g \in C_0^\infty(\mathbb{R}^m)).$$

Let D_j ($j = 1, \dots, m$) denote the differential operator $\frac{1}{i} \frac{\partial}{\partial x_j}$ ($i := \sqrt{-1}$). Then, D_j is a *symmetric* operator, namely,

$$(D_j f, g) = (f, D_j g) \quad (f, g \in C_0^\infty(\mathbb{R}^m))$$

holds.

Let us consider a function $p(x, \xi)$ of variables $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ which is a polynomial in ξ_j 's :

$$p(x, \xi) = \sum_{p=0}^n \left[\sum_{j_1, j_2, \dots, j_p} a^{j_1 j_2 \cdots j_p}(x) \xi_{j_1} \xi_{j_2} \cdots \xi_{j_p} \right],$$

where $a^{j_1 j_2 \cdots j_p}(x)$'s are symmetric with respect to the indices j_1, j_2, \dots, j_p .

The function $p(x, \xi)$ is regarded as an ‘‘observable’’ in the phase space $T^*\mathbb{R}^m$ of classical mechanics. In theory of quantum mechanics the classical observable $p(x, \xi)$ corresponds to a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^m)$ according to the corresponding rule of variables:

$$\xi_j \mapsto D_j, \quad x_j \mapsto x_j \times .$$

We consider the symmetric (formally self-adjoint) operator corresponding to the homogeneous polynomial of degree n given by

$$p_n(x, \xi) = \sum_{j_1, j_2, \dots, j_n} a^{j_1 j_2 \cdots j_n}(x) \xi_{j_1} \xi_{j_2} \cdots \xi_{j_n}.$$

By applying the corresponding rule directly to $p_n(x, \xi)$ we get the n -th order differential operator

$$P_n = \sum_{j_1, j_2, \dots, j_n} a^{j_1 j_2 \cdots j_n}(x) D_{j_1} D_{j_2} \cdots D_{j_n}.$$

Lemma 1 *The adjoint operator P_n^* of P_n is given by*

$$\begin{aligned} P_n^* &= \sum_{j_1, \dots, j_n} D_{j_1} \cdots D_{j_n} (\bar{a}^{j_1 \cdots j_n}(x) \cdot) \\ &= \sum_{p=0}^n \binom{n}{p} \sum_{j_{p+1}, \dots, j_n} \left(\sum_{j_1, \dots, j_p} D_{j_1} \cdots D_{j_p} \bar{a}^{j_1 \cdots j_n}(x) \right) D_{j_{p+1}} \cdots D_{j_n}, \end{aligned}$$

where $\bar{a}^{j_1 \cdots j_n}(x)$ denotes the complex conjugate of $a^{j_1 \cdots j_n}(x)$.

Remark The property $(P_n^*)^* = P_n$ (formally) derives the formula (4). In fact, by virtue of Lemma 1 we have

$$(P_n^*)^* = \sum_{p=0}^n (-1)^p \binom{n}{p} \left\{ \sum_{q=0}^{n-p} \binom{n-p}{q} \sum_{j_1, \dots, j_n} (D_{j_1} \cdots D_{j_{p+q}} a^{j_1 \cdots j_n}(x)) D_{j_{p+q+1}} \cdots D_{j_n} \right\}.$$

The $(n-r)$ -th order differential term of $(P_n^*)^*$ is given by

$$\sum_{p+q=r} (-1)^p \binom{n}{p} \binom{n-p}{q} \sum_{j_1, \dots, j_n} (D_{j_1, \dots, j_r} a^{j_1 \cdots j_r}(x)) D_{j_{r+1}} \cdots D_{j_n}.$$

Hence, for $1 \leq r \leq n$ we have

$$\begin{aligned} 0 &= \sum_{p+q=r} (-1)^p \binom{n}{p} \binom{n-p}{q} \\ &= \sum_{p=0}^r (-1)^p \binom{n}{n-p} \binom{n-p}{r-p} = \sum_{p=0}^r (-1)^p \binom{n}{n-p} \binom{n-p}{n-r}, \end{aligned}$$

that is nothing but the formula (4). \square

In order to obtain the symmetric operator P corresponding to $p_n(x, \xi)$ we put

$$\begin{aligned} P &= \sum_{j_1, \dots, j_n} a^{j_1 \cdots j_n}(x) D_{j_1} \cdots D_{j_n} \\ &\quad + \sum_{p=1}^n c_{n-p} \left[\sum_{j_1, \dots, j_n} (D_{j_1} \cdots D_{j_p} a^{j_1 \cdots j_n}(x)) D_{j_{p+1}} \cdots D_{j_n} \right], \end{aligned} \quad (5)$$

where $a^{j_1 \cdots j_n}(x)$'s are real-valued functions, and c_{n-p} 's are complex constants.

Proposition 2 *The operator P is symmetric, i.e., $P^* = P$ if and only if the coefficients c_{n-p} ($p = 1, 2, \dots, n$) satisfy*

$$\begin{aligned} c_{n-p} &= (-1)^p \bar{c}_{n-p} + (-1)^{p-1} \binom{n-p+1}{1} \bar{c}_{n-p+1} \\ &\quad + (-1)^{p-2} \binom{n-p+2}{2} \bar{c}_{n-p+2} + \cdots \\ &\quad \cdots - \binom{n-1}{p-1} \bar{c}_{n-1} + \binom{n}{p}. \end{aligned} \quad (6)$$

Proof. The assertion is directly derived by comparing the coefficients of $(n - p)$ -th order differential terms in P and P^* . \square

As examples of symmetric operators of the form (5) we have the following:

$$\begin{aligned} & \sum_j a^j(x) D_j + \frac{1}{2} \sum_j D_j a^j(x), \\ & \sum_{j,k} a^{jk}(x) D_j D_k + \sum_k \left(\sum_j D_j a^{jk}(x) \right) D_k, \\ & \sum_{j,k,l} a^{jkl}(x) D_j D_k D_l + \frac{3}{2} \sum_{k,l} \left(\sum_j D_j a^{jkl}(x) \right) D_k D_l - \frac{1}{4} \sum_{j,k,l} D_j D_k D_l a^{jkl}(x). \end{aligned}$$

Observing these examples we assume the coefficients c_{n-p} ($p = 1, 2, \dots, n$) to be

$$c_{n-p} = \begin{cases} \text{a real number} & (p : \text{odd}) \\ 0 & (p : \text{even}) \end{cases} \quad (7)$$

Theorem 3 For any $n \in \mathbb{N}$, and any real valued functions $a^{j_1 \dots j_n}(x)$ there exists an unique n -th order symmetric differential operator P of the form (5) satisfying the condition (7).

Proof. First we show the existence of P (cf. [3, Lemma 4.2]). Let $Q_0 := \sum a^{j_1 \dots j_n}(x) D_{j_1} \dots D_{j_n} (= P_n)$. Put

$$Q_1 := \frac{1}{2}(Q_0 + Q_0^*).$$

Then, by means of Lemma 1 Q_1 is a symmetric operator with the n -th order term being equal to Q_0 , and the coefficients

$$\frac{1}{2} \binom{n}{p} \sum_{j_1, \dots, j_p} D_{j_1} \dots D_{j_p} a^{j_1 \dots j_p \dots j_n}(x)$$

of the $(n - p)$ -th order term of Q_1 are real if p is even. Let P_{n-2} denote the $(n - 2)$ -th order term of Q_1 , and put

$$Q_2 := Q_1 - \frac{1}{2}(P_{n-2} + P_{n-2}^*).$$

Then, Q_2 is a symmetric operator of the form (5) with c_{n-p} being real and $c_{n-2} = 0$.

Next, let P_{n-4} be the $(n - 4)$ -th order term of Q_2 , and put

$$Q_4 := Q_2 - \frac{1}{2}(P_{n-4} + P_{n-4}^*).$$

Then, Q_4 is a symmetric operator of the form (5) with c_{n-p} being real and $c_{n-2} = c_{n-4} = 0$. Thus by continuing this process we get Q_2, Q_4, Q_6, \dots , and we obtain the required operator P as Q_{n-1} if n is odd, or Q_n if n is even.

Next, we show that the coefficients c_{n-p} is uniquely determined by the condition (6) under the assumption (7).

Suppose n is odd. The condition (6) for $p = 1, 2, \dots$ gives a system of linear equations for $c_{n-1}, c_{n-3}, \dots, c_2, c_0$ as follows:

$$\begin{aligned}
 2c_{n-1} &= \binom{n}{1}, \\
 \binom{n-1}{1}c_{n-1} &= \binom{n}{2}, \\
 2c_{n-3} + \binom{n-1}{2}c_{n-1} &= \binom{n}{3}, \\
 \binom{n-3}{1}c_{n-3} + \binom{n-1}{3}c_{n-1} &= \binom{n}{4}, \\
 &\dots\dots\dots \\
 \binom{2}{1}c_2 + \binom{4}{3}c_4 + \dots\dots\dots + \binom{n-1}{n-2}c_{n-1} &= \binom{n}{n-1}, \\
 2c_0 + \binom{2}{2}c_2 + \binom{4}{4}c_4 + \dots\dots\dots + \binom{n-1}{n-1}c_{n-1} &= \binom{n}{n}.
 \end{aligned}$$

It is easy to see that the rank of the $(n \times (n+1)/2)$ -matrix of the coefficients of the above linear equations is equal to $(n+1)/2$. Hence, the solution (if exists) is unique.

If n is even, the linear equations for $c_{n-1}, c_{n-3}, \dots, c_1$ is the following:

$$\begin{aligned}
 2c_{n-1} &= \binom{n}{1}, \\
 \binom{n-1}{1}c_{n-1} &= \binom{n}{2}, \\
 2c_{n-3} + \binom{n-1}{2}c_{n-1} &= \binom{n}{3}, \\
 \binom{n-3}{1}c_{n-3} + \binom{n-1}{3}c_{n-1} &= \binom{n}{4}, \\
 &\dots\dots\dots \\
 2c_1 + \binom{3}{2}c_3 + \dots\dots\dots + \binom{n-1}{n-2}c_{n-1} &= \binom{n}{n-1}, \\
 \binom{1}{1}c_1 + \binom{3}{3}c_3 + \dots\dots\dots + \binom{n-1}{n-1}c_{n-1} &= \binom{n}{n}.
 \end{aligned}$$

This system similarly derives the uniqueness of the solution. □

2. Properties of binomial coefficients

From the system of linear equations for c_{n-1}, c_{n-3}, \dots in the preceding section we have the following.

Theorem 4 *Let $1 \leq k \leq (n+1)/2$. The following two systems of linear equations for $c_{n-1}, c_{n-3}, \dots, c_{n-2k+1}$ are equivalent each other :*

$$\begin{bmatrix} 2 & & & & & \\ \binom{n-1}{2} & 2 & & & & \\ \binom{n-1}{4} & \binom{n-3}{2} & 2 & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-4} & \binom{n-3}{2k-6} & \cdots & \binom{n-2k+5}{2} & 2 & \\ \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+5}{4} & \binom{n-2k+3}{2} & 2 \end{bmatrix} \begin{bmatrix} c_{n-1} \\ c_{n-3} \\ \vdots \\ \vdots \\ c_{n-2k+3} \\ c_{n-2k+1} \end{bmatrix} = \begin{bmatrix} \binom{n}{1} \\ \binom{n}{3} \\ \vdots \\ \vdots \\ \binom{n}{2k-3} \\ \binom{n}{2k-1} \end{bmatrix}, \quad (8)$$

$$\begin{bmatrix} \binom{n-1}{1} & & & & & \\ \binom{n-1}{3} & \binom{n-3}{1} & & & & \\ \binom{n-1}{5} & \binom{n-3}{3} & \binom{n-5}{1} & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-3} & \binom{n-3}{2k-5} & \cdots & \cdots & \binom{n-2k+3}{1} & \\ \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \cdots & \binom{n-2k+3}{3} & \binom{n-2k+1}{1} \end{bmatrix} \begin{bmatrix} c_{n-1} \\ c_{n-3} \\ \vdots \\ \vdots \\ c_{n-2k+3} \\ c_{n-2k+1} \end{bmatrix} = \begin{bmatrix} \binom{n}{2} \\ \binom{n}{4} \\ \vdots \\ \vdots \\ \binom{n}{2k-2} \\ \binom{n}{2k} \end{bmatrix}. \quad (9)$$

Proof. The system (8) of linear equations is obtained from (6) in Proposition 2 for odd $p = 1, 3, \dots, 2k-1$. On the other hand, the system (9) is obtained from (6) for even $p = 2, 4, \dots, 2k$. These two systems of linear equation have the same solution associated to the unique symmetric differential operator P (Theorem 3). \square

Note Cramer's formulas for the solution c_{n-2k+1} of (8) and (9), and we have the following.

Corollary 5 *For $n, k \in \mathbb{N}$ with $1 \leq k \leq (n+1)/2$ we have the following*

identity, which is equal to $(-1)^{k-1}c_{n-2k+1}$:

$$\frac{1}{2^k} \begin{vmatrix} \binom{n}{1} & 2 & & & \\ \binom{n}{3} & \binom{n-1}{2} & 2 & & \\ \binom{n}{5} & \binom{n-1}{4} & \binom{n-3}{2} & \cdots & \\ \vdots & \vdots & \vdots & & 2 \\ \binom{n}{2k-1} & \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+3}{2} \end{vmatrix} = \frac{(n-2k-1)!!}{(n-1)!!} \begin{vmatrix} \binom{n}{2} & \binom{n-1}{1} & & & \\ \binom{n}{4} & \binom{n-1}{3} & \binom{n-3}{1} & & \\ \binom{n}{6} & \binom{n-1}{5} & \binom{n-3}{3} & \cdots & \\ \vdots & \vdots & \vdots & & \binom{n-2k+3}{1} \\ \binom{n}{2k} & \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \binom{n-2k+3}{3} \end{vmatrix} \quad (10)$$

Remark If $k = 1$, (10) means

$$\frac{1}{2} \binom{n}{1} = \frac{1}{n-1} \binom{n}{2} (= c_{n-1}).$$

Table for c_{n-p}

n	c_{n-1}	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	c_{n-6}	c_{n-7}	c_{n-8}	c_{n-9}	c_{n-10}
1	$\frac{1}{2}$									
2	1	0								
3	$\frac{3}{2}$	0	$-\frac{1}{4}$							
4	2	0	-1	0						
5	$\frac{5}{2}$	0	$-\frac{5}{2}$	0	$\frac{1}{2}$					
6	3	0	-5	0	3	0				
7	$\frac{7}{2}$	0	$-\frac{35}{4}$	0	$\frac{21}{2}$	0	$-\frac{17}{8}$			
8	4	0	-14	0	28	0	-17	0		
9	$\frac{9}{2}$	0	-21	0	63	0	$-\frac{153}{2}$	0	$\frac{31}{2}$	
10	5	0	-30	0	126	0	-255	0	155	0
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
n	$\frac{1}{2} \binom{n}{1}$	0	$-\frac{1}{4} \binom{n}{3}$	0	$\frac{1}{2} \binom{n}{5}$	0	$-\frac{17}{8} \binom{n}{7}$	0	$\frac{31}{2} \binom{n}{9}$	0

Finally, by considering the case $n = 2k$ we have the following from the last equation in (9).

Corollary 6 *For even $n (\in \mathbb{N})$ we have*

$$\begin{aligned}
& \sum_{k=1}^{n/2} c_{n-2k+1} \\
&= \sum_{k=1}^{n/2} \frac{(-1)^{k-1}}{2^k} \begin{vmatrix} \binom{n}{1} & 2 & & & \\ \binom{n}{3} & \binom{n-1}{2} & 2 & & 0 \\ \binom{n}{5} & \binom{n-1}{4} & \binom{n-3}{2} & \cdots & \\ \vdots & \vdots & \vdots & & 2 \\ \binom{n}{2k-1} & \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+3}{2} \end{vmatrix} \\
&= \sum_{k=1}^{n/2} (-1)^{k-1} \frac{(n-2k-1)!!}{(n-1)!!} \begin{vmatrix} \binom{n}{2} & \binom{n-1}{1} & & & \\ \binom{n}{4} & \binom{n-1}{3} & \binom{n-3}{1} & & 0 \\ \binom{n}{6} & \binom{n-1}{5} & \binom{n-3}{3} & \cdots & \\ \vdots & \vdots & \vdots & & \binom{n-2k+3}{1} \\ \binom{n}{2k} & \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \binom{n-2k+3}{3} \end{vmatrix} \\
&= 1.
\end{aligned}$$

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Some Infinite Series of Fibonacci Numbers

By

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Abstract

In his paper [1], J.G. Goggins has shown a simple formula which relates π and Fibonacci numbers. In this note, we shall prove a generalized formula (4) with some integer parameter k . Then Goggins's formula can be regarded as the special case $k = 1$ of our new formula (4).

2000 Mathematics Subject Classification. Primary 11B39;
Secondary 40A05, 11A99

Introduction

In [1], J.G. Goggins has shown the following simple but very interesting formula

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}), \quad (1)$$

where F_n is the n th Fibonacci number. This formula is also given as the formula (f) in the text [5] chapter 3. Firstly, we shall rewrite this formula to the following two forms. Since $F_1 = 1$ and $\pi/4 = \tan^{-1}(1/F_1)$, (1) is equivalent to the following formula

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}). \quad (2)$$

From the facts $F_{-2k} = -F_{2k}$ and $F_{-2k-1} = F_{2k+1}$, (2) is also equivalent to the following formula

$$\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(1/F_{2n+1}). \quad (3)$$

The purpose of this short note is to generalize this formula to the following formula which holds for any integer parameter k ,

$$k\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}). \quad (4)$$

Remark. We note that, from the fact $F_2 = 1$, the above formula (3) is exactly the case $k = 1$ of this formula (4).

Let $\{G_n\}$ be generalized Fibonacci sequences which satisfy

$$G_{n+2} = G_{n+1} + G_n.$$

Using the induction on m , we can show the following addition theorem of G_ℓ .

Addition Theorem. (See for example [3]).

$$G_{m+\ell} = F_m G_{\ell+1} + F_{m-1} G_\ell, \quad \text{for any integer } m.$$

Substituting $G_{\ell+1} - G_{\ell-1}$ for G_ℓ , we have

$$\begin{aligned} G_{m+\ell} &= F_m G_{\ell+1} + F_{m-1} (G_{\ell+1} - G_{\ell-1}) = (F_m + F_{m-1}) G_{\ell+1} - F_{m-1} G_{\ell-1} \\ &= F_{m+1} G_{\ell+1} - F_{m-1} G_{\ell-1}. \end{aligned}$$

Thus we have obtained a modified version of this addition theorem.

Corollary 1.

$$G_{m+\ell} = F_{m+1} G_{\ell+1} - F_{m-1} G_{\ell-1}, \quad \text{for any integer } m.$$

Let us consider the special case when $G = F$ and $\ell = 2n$ is even and $m = 2k - 1$ is odd in Corollary 1. Then we have $F_{2n+2k-1} = F_{2k} F_{2n+1} - F_{2k-2} F_{2n-1}$. Hence we have:

Corollary 2.

$$F_{2k-2} F_{2n-1} + F_{2n+2k-1} = F_{2k} F_{2n+1}.$$

Let us consider the special case when $G = F$ and $\ell = 2n$ is even and $m = -2n - 2k + 2$ in Corollary 1. Then we have $F_{-2k+2} = F_{-2n-2k+3}F_{2n+1} - F_{-2n-2k+1}F_{2n-1}$, which is equivalent to $-F_{2k-2} = F_{2n+2k-3}F_{2n+1} - F_{2n+2k-1}F_{2n-1}$. Thus we have shown:

Corollary 3.

$$F_{2n+2k-1}F_{2n-1} - F_{2k-2} = F_{2n+2k-3}F_{2n+1}.$$

Using these corollaries, we can show the following proposition.

Proposition.

$$\tan^{-1}\left(\frac{F_{2k-2}}{F_{2n+2k-1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n-1}}\right) = \tan^{-1}\left(\frac{F_{2k}}{F_{2n+2k-3}}\right).$$

Proof. From Corollaries 2 and 3, we have

$$\begin{aligned} \frac{\frac{F_{2k-2}}{F_{2n+2k-1}} + \frac{1}{F_{2n-1}}}{1 - \frac{F_{2k-2}}{F_{2n+2k-1}F_{2n-1}}} &= \frac{F_{2k-2}F_{2n-1} + F_{2n+2k-1}}{F_{2n+2k-1}F_{2n-1} - F_{2k-2}} = \frac{F_{2k}F_{2n+1}}{F_{2n+2k-3}F_{2n+1}} \\ &= \frac{F_{2k}}{F_{2n+2k-3}}, \text{ which completes the proof.} \end{aligned}$$

This proposition and the fact

$$\lim_{n \rightarrow \pm\infty} \tan^{-1}(F_{2m}/F_{2n+1}) = 0$$

for any fixed m imply that

$$\sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k-2}}{F_{2n+1}}\right) + \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n-1}}\right) = \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k}}{F_{2n+1}}\right).$$

Put $A(k) = \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k}}{F_{2n+1}}\right)$. Then this relation can be written as

$$A(k-1) + A(1) = A(k). \quad (5)$$

We note that $A(1) = \pi$ from the formula (3). Then, from this relation (5) and

the induction on k , we can show the infinite series $A(k)$ is convergent for any integer k . Using the same relation (5), we can also verify that $A(k)$ satisfies the formula (4). Now we have completed the proof of the following theorem.

Theorem. *With the above notations, we have*

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}) = k\pi,$$

or equivalently

$$\sum_{n=0}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}) = \frac{k\pi}{2} \text{ for any integer } k.$$

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On Some Formulas for $\pi/2$

By

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Abstract

In his paper [1], J.G. Goggins has shown a formula which relates π and Fibonacci numbers. In our paper [2], we have proved a generalized version of this formula. In this note, we shall prove formulas which generalize Fibonacci number to certain binary recurrence sequences.

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Introduction

In [1], J.G. Goggins has shown the following simple but very interesting formula

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}), \quad (1)$$

where F_n is the n th Fibonacci number. We note this formula is also given as the formula (f) in the text [5] chapter 3. Since $F_1 = 1$, we see $\frac{\pi}{4} = \tan^{-1}(1/F_1)$. Thus (1) is equivalent to the following formula

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}). \quad (2)$$

The purpose of this short note is to generalize this formula on Fibonacci number to two formulas on binary recurrence sequences, that is, to the following

two formulas

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}), \quad (3)$$

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}). \quad (4)$$

Here $\{u_n\}$ is the Lucas sequences associated to the parameter $(t, -1)$ and $\{v_n\}$ is the companion Lucas sequences associated to the parameter $(t, -1)$, respectively.

First of all, let us recall the fundamental properties of u_n and v_n . Let t be a positive integer and $\{u_n\}$ and $\{v_n\}$ be the binary recurrence sequences defined by putting

$$\begin{cases} u_{n+2} = tu_{n+1} + u_n, \\ v_{n+2} = tv_{n+1} + v_n, \end{cases}$$

with initial terms $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = t$.

Put $\varepsilon = (t + \sqrt{t^2 + 4})/2$ and $\bar{\varepsilon} = (t - \sqrt{t^2 + 4})/2$. Then one knows the following Binet's formula

$$\begin{cases} u_n = (\varepsilon^n - \bar{\varepsilon}^n)/\sqrt{t^2 + 4}, \\ v_n = \varepsilon^n + \bar{\varepsilon}^n. \end{cases}$$

Put $\alpha_{2n} = \tan^{-1}(1/u_{2n})$ and $\alpha_{2n-1} = \tan^{-1}(t/u_{2n-1})$ for any positive index n . Then we can show the following proposition.

Proposition 1. *For any integer $n \geq 1$, $\alpha_{2n} = \alpha_{2n+1} + \alpha_{2n+2}$.*

Proof. We have

$$\begin{aligned} \tan(\alpha_{2n+1} + \alpha_{2n+2}) &= \frac{t/u_{2n+1} + 1/u_{2n+2}}{1 - t/(u_{2n+1}u_{2n+2})} = \frac{tu_{2n+2} + u_{2n+1}}{u_{2n+1}u_{2n+2} - t} \\ &= \frac{u_{2n+3}}{u_{2n+1}u_{2n+2} - t}. \end{aligned}$$

By virtue of the Binet's formula, we see

$$\begin{aligned} u_{2n+1}u_{2n+2} - t &= (\varepsilon^{2n+1} - \bar{\varepsilon}^{2n+1})(\varepsilon^{2n+2} - \bar{\varepsilon}^{2n+2})/(t^2 + 4) - t \\ &= (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} + \varepsilon + \bar{\varepsilon})/(t^2 + 4) - t = (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - t^3 - 3t)/(t^2 + 4). \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} u_{2n}u_{2n+3} &= (\varepsilon^{2n} - \bar{\varepsilon}^{2n})(\varepsilon^{2n+3} - \bar{\varepsilon}^{2n+3})/(t^2 + 4) \\ &= (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - \varepsilon^3 - \bar{\varepsilon}^3)/(t^2 + 4) = (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - t^3 - 3t)/(t^2 + 4). \end{aligned}$$

Thus we have shown

$$\tan(\alpha_{2n+1} + \alpha_{2n+2}) = \frac{1}{u_{2n}} = \tan(\alpha_{2n}),$$

which completes the proof.

From this proposition, we have $\alpha_{2n} - \alpha_{2n+2} = \alpha_{2n+1}$ for any $n \geq 1$. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \tan^{-1}(t/u_{2n+1}) &= \sum_{n=1}^{\infty} \alpha_{2n+1} = \sum_{n=1}^{\infty} (\alpha_{2n} - \alpha_{2n+2}) \\ &= (\alpha_2 - \alpha_4) + (\alpha_4 - \alpha_6) + \cdots + (\alpha_{2n} - \alpha_{2n+2}) + \cdots = \alpha_2. \end{aligned}$$

Since $\alpha_2 = \tan^{-1}(1/t) = \frac{\pi}{2} - \tan^{-1}(t/u_1)$, we have shown the formula (3).

Now we shall show the formula (4) similarly. Put $\beta_{2n} = \tan^{-1}(t/v_{2n})$ and $\beta_{2n-1} = \tan^{-1}(2/v_{2n-1})$ for any positive index n . Then we can show the following proposition.

Proposition 2. *For any integer $n \geq 1$, $2\beta_{2n} = \beta_{2n-1} - \beta_{2n+1}$.*

Proof. We have

$$\begin{aligned} \tan(\beta_{2n-1} - \beta_{2n+1}) &= \frac{2/v_{2n-1} - 2/v_{2n+1}}{1 + 4/(v_{2n-1}v_{2n+1})} = \frac{2(v_{2n+1} - v_{2n-1})}{v_{2n-1}v_{2n+1} + 4} \\ &= \frac{2tv_{2n}}{v_{2n-1}v_{2n+1} + 4}. \end{aligned}$$

By virtue of the Binet's formula, we see

$$\begin{aligned} v_{2n-1}v_{2n+1} + 4 &= (\varepsilon^{2n+1} + \bar{\varepsilon}^{2n+1})(\varepsilon^{2n-1} + \bar{\varepsilon}^{2n-1}) + 4 \\ &= (\varepsilon^{4n} + \bar{\varepsilon}^{4n}) - (\varepsilon^2 + \bar{\varepsilon}^2) + 4 = (\varepsilon^{4n} + \bar{\varepsilon}^{4n}) - (t^2 + 2) + 4 \\ &= (\varepsilon^{2n} + \bar{\varepsilon}^{2n})^2 - t^2 = v_{2n}^2 - t^2. \end{aligned}$$

On the other hand, we have

$$\tan(2\beta_{2n}) = \frac{t/v_{2n} + t/v_{2n}}{1 - (t/v_{2n})^2} = \frac{2tv_{2n}}{v_{2n}^2 - t^2}.$$

Thus we have shown

$$\tan(\beta_{2n-1} - \beta_{2n+1}) = \tan(2\beta_{2n}),$$

which completes the proof.

From this proposition, we have $\beta_{2n-1} - \beta_{2n+1} = 2\beta_{2n}$ for any $n \geq 1$. Then we have

$$\sum_{n=1}^{\infty} 2 \tan^{-1}(t/v_{2n}) = \sum_{n=1}^{\infty} 2\beta_{2n} = \sum_{n=1}^{\infty} (\beta_{2n-1} - \beta_{2n+1}) = \beta_1 = \tan^{-1}(2/t).$$

Since $v_{-2n} = v_{2n}$, one knows that $\tan^{-1}(t/v_{-2n}) = \tan^{-1}(t/v_{2n})$.

Hence we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}) &= 2 \left(\sum_{n=1}^{\infty} \tan^{-1}(t/v_{2n}) \right) + \tan^{-1}(t/v_0) \\ &= \tan^{-1}(2/t) + \tan^{-1}(t/2) = \frac{\pi}{2}, \end{aligned}$$

which completes the proof of (4).

Now we have completely proved two formulas of (4), which we shall state as the following theorem.

Theorem. *With the above notations, we have the following formulas,*

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}),$$

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}).$$

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Existence of Global and Bounded Solutions for Damped Sublinear Wave Equations

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Abstract

We study the initial-boundary value problem for the sublinear wave equations with a linear damping : $u'' - \Delta u - \omega \Delta u' + \delta u' = \gamma |u|^{p-2}u$ with the homogeneous Dirichlet boundary condition and $H_0^1(\Omega) \times L^2(\Omega)$ -data condition under $\omega \geq 0$ and $\delta > -\omega \lambda_1$. When $1 < p < 2$, we show that the (local) weak solutions are global and uniformly bounded in time $t \geq 0$.

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1 Introduction

We consider the initial-boundary value problem for the following semilinear wave equation :

$$u'' - \Delta u - \omega \Delta u' + \delta u' = f(u), \quad u = u(x, t), \quad \text{in } \Omega \times [0, \infty) \quad (1)$$

with homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega$$

and initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad u'(x, 0) = u_1(x),$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial^2/\partial x_j^2$ is Laplacian, ω and δ are constants such that $\omega \geq 0$

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and $\delta > -\omega\lambda_1$ with λ_1 being the first eigenvalue of the operator $-\Delta$ under the homogeneous Dirichlet boundary condition, that is,

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2},$$

and

$$f(u) = \gamma|u|^{p-2}u, \quad \gamma > 0, \quad p > 1. \quad (2)$$

In the superlinear case ($p > 2$), it is well known that the so-called potential well method is useful to the analysis of global existence for problem (1). (see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20], and also, [8], [9], [14], [18]), and moreover, the concavity method is applied to the analysis of finite time blow-up phenomena (see Tsutsumi [23], Levine [10], [11], and also, [1], [2], [3], [7], [13], [15], [16], [22]).

In order to explain some known results for $p > 2$, we define the total energy associated with (1) by

$$E(u, u') = \frac{1}{2}\|u'\|^2 + J(u)$$

where we put

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 - \frac{\gamma}{p}\|u\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u\|^2 + \frac{1}{p}I(u) \end{aligned}$$

with

$$I(u) = \|\nabla u\|^2 - \gamma\|u\|_p^p,$$

and we define the mountain pass level d (also known as the potential well depth) by

$$d = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \left(\sup_{\lambda \geq 0} J(\lambda u) \right)$$

(see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20]).

When the power p in (2) satisfies that $p > 2$ and $p \leq 2(N-1)/(N-2)$ if $N \geq 3$, many authors have already studied on global existence or finite time blow-up of (local) weak solutions in the class $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ for the problem (1) with the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying suitable conditions : (i) if $E(u_0, u_1) < d$ and $I(u_0) > 0$, then there exists a unique global solution $u(t)$ satisfying $\|u(t)\|_{H^1} + \|u'(t)\| \rightarrow 0$ as $t \rightarrow \infty$; (ii) if $E(u_0, u_1) < d$ and $I(u_0) < 0$, then the local solution $u(t)$ blows up at some finite time, that

is, there exists a finite time $T^* < \infty$ such that $\|u(t)\|_{H^1} \rightarrow \infty$ as $t \rightarrow T^*$; moreover (iii) when $\omega = 0$, if $E(u_0, u_1) \geq d$, $I(u_0) < 0$, $\|u_0\| \geq \sup\{\|\phi\| \mid \phi \in H_0^1(\Omega) \setminus \{0\} \text{ with } (1/2 - 1/p)\|\nabla\phi\|^2 \leq E(u_0, u_1)\}$, and $\int_{\Omega} u_0 u_1 dx \geq 0$, then the local solution $u(t)$ blows up at some finite time (see Gazzola-Squassina [5]). We note that if $E(u(t), u'(t)) \geq d$ for all $t \geq 0$, then $\lim_{t \rightarrow \infty} E(u(t), u'(t))$ exists, and when $\omega = 0$, $p > 2$, and $p < 2(N-1)/(N-2)$ if $N \geq 3$ or $p \leq 6$ if $N = 2$, the local solution $u(t)$ is global and bounded (see Esquivel-Avila [4]).

On the other hand, when the power p in (2) satisfies that $1 < p \leq 2$, we see that for the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the (local) weak solution $u(t)$ in the class $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ is global. In particular, from Remark 3.10 in Gazzola and Squassina [5], we know that the global solution $u(t)$ satisfies

$$\begin{aligned} \|u(t)\|_{H^1} + \|u'(t)\| &\leq C(1+t)^{p/(4-2p)} \quad \text{if } p < 2 \\ \|u(t)\|_{H^1} + \|u'(t)\| &\leq Ce^{\alpha t} \quad \text{if } p = 2 \end{aligned}$$

with some $\alpha > 0$, for $t \geq 0$, but we can not know boundedness of global solutions.

The purpose of this paper is to show boundedness of global solutions of (1) in the case $1 < p < 2$ (i.e. sublinear case).

Our main result is as follows.

Theorem 1.1 *Let $1 < p < 2$, and let $\omega \geq 0$ and $\delta > -\omega\lambda_1$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ satisfying*

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq C + CI_0 e^{-\tilde{k}t}$$

with some constants $C > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

On the other hand, in the case $p = 2$ we have the following.

Theorem 1.2 *Let $p = 2$, and let $\omega \geq 0$ and $\delta > -\omega\lambda_1$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ satisfying that $\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0 e^{\tilde{\alpha}t}$, and moreover, if $\gamma < \lambda_1$,*

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0 e^{-\tilde{k}t}$$

with some constants $C > 0$, $\tilde{\alpha} > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

We use only familiar functional spaces and omit the definitions. We denote $L^p(\Omega)$ -norm by $\|\cdot\|_p$ (we often write $\|\cdot\| = \|\cdot\|_2$ for simplicity). Positive constants will be denoted by C and will change from line to line.

2 Proofs

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem (e.g. see [6], [12], [17], [19]).

Proposition 2.1 *Let $p > 1$ and $p \leq 2(N-1)/(N-2)$ if $N \geq 3$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, there exists a unique (local) weak solution $u(t)$ in the class $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of problem (1), that is,*

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u'(t)w \, dx + \int_{\Omega} \nabla u(t) \nabla w \, dx + \omega \int_{\Omega} \nabla u'(t) \nabla w \, dx \\ + \delta \int_{\Omega} u'(t)w \, dx = \int_{\Omega} f(u(t))w \, dx \end{aligned}$$

a.e. in $(0, T)$ for every $w \in H_0^1(\Omega)$.

Moreover, if $\sup_{0 \leq t < T} (\|u(t)\|_{H^1} + \|u'(t)\|) < \infty$, then the solution $u(t)$ can be continued to $T + \varepsilon$ for some $\varepsilon > 0$.

Proof of Theorem 1.1. Multiplying (1) by u' and integrating it over Ω , we have

$$\frac{d}{dt} E_1(t) + \omega \|\nabla u'(t)\|^2 + \delta \|u'(t)\|^2 = \int_{\Omega} f(u(t))u'(t) \, dx, \quad (3)$$

where $E_1(t)$ is defined by

$$E_1(t) = E_1(u(t), u'(t)) = \frac{1}{2} (\|u'(t)\|^2 + \|\nabla u(t)\|^2).$$

And, multiplying (1) by u and integrating it over Ω , we have

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \left(\omega \|\nabla u(t)\|^2 + \delta \|u(t)\|^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right) \\ + \|\nabla u(t)\|^2 - \|u'(t)\|^2 = \int_{\Omega} f(u(t))u(t) \, dx. \end{aligned} \quad (4)$$

Then, taking (3) + $\varepsilon \times$ (4) for any small $\varepsilon > 0$, we have

$$\frac{d}{dt} F_1(t) + G_1(t) = \int_{\Omega} f(u(t)) (u'(t) + \varepsilon u(t)) \, dx, \quad (5)$$

where $F_1(t)$ and $G_1(t)$ are defined by

$$F_1(t) = E_1(t) + \frac{\varepsilon}{2} \left(\omega \|\nabla u(t)\|^2 + \delta \|u(t)\|^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right)$$

and

$$G_1(t) = (\omega \|\nabla u'(t)\|^2 + \delta \|u'(t)\|^2) + \varepsilon (\|\nabla u(t)\|^2 - \|u'(t)\|^2) .$$

Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that

$$F_1(t) \leq C(\|u'(t)\|^2 + \|\nabla u(t)\|^2) \quad (6)$$

and

$$G_1(t) \geq (\delta + \omega \lambda_1 - \varepsilon) \|u'(t)\|^2 + \varepsilon \|\nabla u(t)\|^2 . \quad (7)$$

Thus, if $\delta + \omega \lambda_1 > 0$, choosing small $\varepsilon > 0$, we have from (5)–(7) that

$$\frac{d}{dt} F_1(t) + 2k F_1(t) \leq \int_{\Omega} f(u(t)) (u'(t) + \varepsilon u(t)) \, dx \quad (8)$$

with some constant $k > 0$. Moreover, we observe from the Young inequality and the Poincaré inequality with $p < 2$ that

$$\begin{aligned} \int_{\Omega} f(u(t)) (u'(t) + \varepsilon u(t)) \, dx &\leq C \|u'(t)\|_p \|u(t)\|_p^{p-1} + C \|u(t)\|_p^p \\ &\leq C \|u'(t)\| \|\nabla u(t)\|^{p-1} + C \|\nabla u(t)\|^p \end{aligned} \quad (9)$$

and by $\delta + \omega \lambda_1 > 0$,

$$\begin{aligned} F_1(t) &\geq E_1(t) + \frac{\varepsilon}{2} ((\delta + \omega \lambda_1) \|u(t)\|^2 - 2 \|u(t)\| \|u'(t)\|) \\ &\geq \frac{1}{2} (1 - C\varepsilon) \|u'(t)\|^2 + \frac{1}{2} \|\nabla u(t)\|^2 . \end{aligned} \quad (10)$$

Thus, choosing small $\varepsilon > 0$, we have from (8)–(10) that

$$\frac{d}{dt} F_1(t) + 2k F_1(t) \leq C F_1(t)^{p/2} \leq k F_1(t) + C , ,$$

where we used the Young inequality together with the fact that $1/2 < p/2 < 1$ at the last inequality, and hence,

$$F_1(t) \leq \frac{C}{k} + F_1(0) e^{-kt} . \quad (11)$$

Therefore, we obtain from (10) and (11) that

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \leq C F_1(t) \leq C + C I_0^2 e^{-kt}$$

for $t \geq 0$. \square

Proof of Theorem 1.2. Since $p = 2$ in (2), we have from (3) and the Poincaré inequality that

$$\frac{d}{dt}E_1(t) \leq \gamma\|u(t)\|\|u'(t)\| \leq \alpha E_1(t)$$

with some constant $\alpha > 0$, and hence, we have

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \leq 2E_1(0)e^{\alpha t}$$

for $t \geq 0$.

Next, let $\gamma < \lambda_1$. Since $p = 2$ in (2), we have from (3) and (4) that

$$\frac{d}{dt}E(t) + \omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2 = 0 \quad (12)$$

and

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_{\Omega}u(t)u'(t)dx\right) \\ + \|\nabla u(t)\|^2 - \gamma\|u(t)\|^2 - \|u'(t)\|^2 = 0, \end{aligned} \quad (13)$$

respectively, where we write

$$E(t) = E_1(t) - \frac{\gamma}{2}\|u(t)\|^2 = \frac{1}{2}(\|u'(t)\|^2 + \|\nabla u(t)\|^2 - \gamma\|u(t)\|^2)$$

for simplicity. Then, taking (12) + $\varepsilon \times$ (13) for any small $\varepsilon > 0$, we have

$$\frac{d}{dt}F(t) + G(t) = 0 \quad (14)$$

where $F(t)$ and $G(t)$ are defined by

$$F(t) = E(t) + \frac{\varepsilon}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_{\Omega}u(t)u'(t)dx\right)$$

and

$$G(t) = (\omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2) + \varepsilon(\|\nabla u(t)\|^2 - \gamma\|u(t)\|^2 - \|u'(t)\|^2).$$

Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that

$$F(t) \leq C(\|u'(t)\|^2 + \|\nabla u(t)\|^2) \quad (15)$$

and

$$G(t) \geq (\delta + \omega\lambda_1 - \varepsilon)\|u'(t)\|^2 + \varepsilon(1 - \gamma/\lambda_1)\|\nabla u(t)\|^2. \quad (16)$$

Thus, if $\delta + \omega\lambda_1 > 0$, choosing small $\varepsilon > 0$, we have from (14)–(16) that

$$\frac{d}{dt}F(t) + kF(t) \leq 0 \quad (17)$$

with some constant $k > 0$. Moreover, we observe from the Young inequality and the Poincaré inequality that

$$\begin{aligned} F(t) &\geq \frac{1}{2} (\|u'(t)\|^2 + (1 - \gamma/\lambda_1)\|\nabla u(t)\|^2) \\ &\quad + \frac{\varepsilon}{2} ((\delta + \omega\lambda_1)\|u(t)\|^2 - 2\|u(t)\|\|u'(t)\|) \\ &\geq \frac{1}{2}(1 - C\varepsilon)\|u'(t)\|^2 + \frac{1}{2}(1 - \gamma/\lambda_1)\|\nabla u(t)\|^2. \end{aligned} \quad (18)$$

Thus, choosing small $\varepsilon > 0$, we have from (17) and (18) that

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \leq CF(t) \leq CF(0)e^{-kt}$$

for $t \geq 0$. \square

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A Numerical Method for Distinction between Blow-up and Global Solutions of the Nonlinear Heat Equation

By

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Abstract

The famous one-dimensional nonlinear heat equation is considered. To this equation a numerical method for distinction between blow-up and global solutions is proposed. Difficulty is in the treatment of the global solution which is defined in the infinite interval. The bounding transform is used to overcome this difficulty. Numerical experiments show the validity of our method.

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Introduction

Following the unique paper[7] there have been a lot of preceding researches on blow-up solutions for nonlinear heat equations. In the paper, the initial and boundary value problem governed by the famous one-dimensional nonlinear heat equation as follows:

Problem 1 For two parameters $\alpha \geq 0$, and $T > 0$, find $u(t, x)$ such that

$$\begin{aligned} u_t &= u_{xx} + u^2, & 0 < t < T, \quad 0 < x < 1, \\ u(t, 0) &= 0, & 0 \leq t < T, \\ u(t, 1) &= 0, & 0 \leq t < T, \\ u(0, x) &= \alpha \sin \pi x, & 0 < x < 1. \end{aligned}$$

Numerical methods have been proposed to such a problem with the blow-up solution [14, 3, 9, 4]. They adopt the adaptive control on the time increment, i.e.

the time increment varies depending on the solution. This technique is useful for the computation of the blow-up time. The blow-up time T_b in Problem 1 with $\alpha = 100$ was computed to be approximately 0.01098[9].

By the way, it is well-known that Problem 1 has global solutions for small initial data and blow-up solutions for large initial data[7, 5, 6, 12, 13]. Numerical results by FDM \oplus EE mentioned in §1 shows these situations(Fig.1).

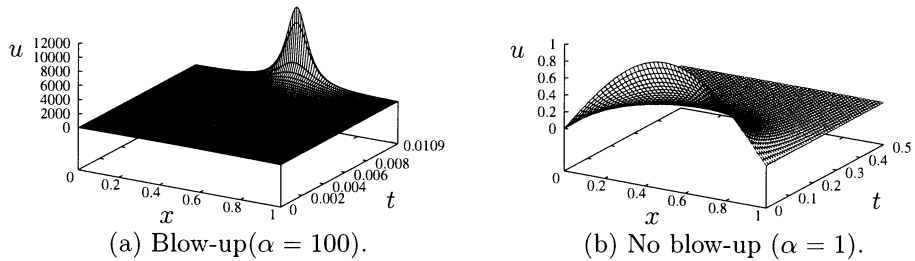


Fig. 1. Solution profiles.

For $\alpha = 100$ overflow easily occurs beyond $t = 0.0109$, so Fig.1(a) is recognized to show the profile of the blow-up solution. On the other hand, without any theoretical results it is vague that Fig.1(b) show the profile of the global decreasing solution because numerical computation is local.

In the paper a numerical method for distinction between blow-up and global solutions is proposed. Difficulty is global computation in time. To overcome this difficulty the bounding transform[10] is adopted. For precise numerical computation spectral collocation method is adopted. This method may offer new possibility of computation of the blow-up time.

1 Our numerical method

We consider the more complicated problem which is derived from Problem 1 by using the following transformations:

$$\tau = t/\beta, \quad \hat{u}(\tau, x) = \beta u(t, x)$$

where β is a positive constant. Then, Problem 2 is obtained.

Problem 2 For three parameters $\alpha \geq 0$, $\beta > 0$, and $T > 0$, find $\hat{u}(\tau, x)$ such that

$$\begin{aligned} \hat{u}_\tau &= \beta \hat{u}_{xx} + \hat{u}^2, & 0 < \tau < T/\beta, \quad 0 < x < 1, \\ \hat{u}(\tau, 0) &= 0, & 0 \leq \tau < T/\beta, \\ \hat{u}(\tau, 1) &= 0, & 0 \leq \tau < T/\beta, \\ \hat{u}(0, x) &= \beta \alpha \sin \pi x, & 0 < x < 1. \end{aligned}$$

If $\beta = 1$ Problem 2 is equivalent to Problem 1. For small β the blow-up time $\tau_b (= T_b/\beta)$ becomes large. In this case numerical distinction that the solution is the global one or the blow-up one becomes difficult. For example, $T_b = 0.11$ in Problem 1 corresponds to $\tau_b = 0.11 \times 10^8$ in Problem 2 with $\beta = 10^{-8}$. The following bounding transform on τ is introduced for the treatment of the global solution[10].

$$\tau = \frac{s}{1-s^2} \left(s(\tau) = \frac{2\tau}{1 + \sqrt{1 + 4\tau^2}} \right).$$

The interval $[0, \infty)$ on τ is mapped onto the interval $[0, 1)$ on s . From this transform Problem 2 becomes the following Problem 3.

Problem 3 For three parameters $\alpha \geq 0$, $\beta > 0$, and $T > 0$, find $\tilde{u}(\tau, x)$ such that

$$\begin{aligned} \tilde{u}_s &= \frac{1+s^2}{(1-s^2)^2} (\beta \tilde{u}_{xx} + \tilde{u}^2), & 0 < s < s(T/\beta), \quad 0 < x < 1, \\ \tilde{u}(s, 0) &= 0, & 0 \leq s < s(T/\beta), \\ \tilde{u}(s, 1) &= 0, & 0 \leq s < s(T/\beta), \\ \tilde{u}(0, x) &= \beta \alpha \sin \pi x, & 0 < x < 1. \end{aligned}$$

This problem is defined in the bounded domain and $s(T/\beta) = 1$ means $T = \infty$. Thus, the global solution in Problem 2 can be computed by solving Problem 3.

Our method for distinction between blow-up and global solutions is as follows.

Numerical computation is carried out by two types of discretization. One is FDM \oplus EE(second order finite difference method in space and first order explicit Euler method in time) and another is SCM(spectral collocation method) which is easily applicable to nonlinear problems[2]. In SCM Chebyshev-Gauss-Lobatto(CGL) collocation points are used in space, and Chebyshev-Gauss-Radau(CGR) collocation points or CGL points are used in time. Discretized equations by SCM are nonlinear, so Newton method is used for solving them. If

exponential convergence of numerical solutions by SCM is obtained, then (converged) numerical solutions are very accurate and reliable[2, 11]. In FDM \oplus EE the adaptive control on the time increment is not used for global computing. FDM \oplus EE is so simple that it is firstly applied for rough numerical computation.

Concrete procedure is as follows:

- 0) Set $s_l = 0$ and choose the time increment $\Delta s (> 0)$ for EE adequately.
- 1) For $s_l \leq s$ compute the solution profile by using FDM \oplus EE. If necessary Δs may be varied or multiple precision is adopted. (If overflow occurs at $s = s_e (< 1)$ then the solution is probably of the blow-up type. If numerical computation works well until $s = s_e = 1$ then the solution is probably of the global type.)
- 2) Referring the solution profile obtained for $s_l \leq s \leq s_e$ in the above step 1), choose the interval $[s_l, s_r] (s_r \leq s_e)$ where the profile seems to be smooth and carry out numerical computation by SCM in this interval. $s_r (\leq s_e)$ should be chosen for realizing exponential convergence. In this interval there is no blow-up solution. In the case where $s_r < 1$ and Newton method does not converge determination of the blow-up solution is done referring the solution profile.
- 3) If distinction between blow-up and global solutions can not be clear in the above step 2), set $s_l = s_r$ and go to the step 1) with the initial data at $s = s_r$ that is computed in the step 2).

The above procedure is not rigorous. In the practical situation trial and error is inevitable.

We should remark that our method is not perfect. For example, the grow-up solution in Problem 2 becomes the discontinuous solution at $s = 1$ in Problem 3. Numerical computation to such a solution is very difficult[16]. However, our method can realize global numerical computation and it may offer a new field of numerical analysis.

2 Numerical results

Numerical computation is basically carried out in double precision. However, some results are computed in multiple precision[8].

In SCM N_s, N_x denote approximation orders on the time and space variables, respectively. In FDM \oplus EE N_s, N_x denote division numbers on the time and space variables, respectively. $\Delta s = 2/N_s, \Delta x = 2/N_x$. Moreover, $N_x = 20$ in FDM \oplus EE because FDM \oplus EE is used only for rough computation of the

solution profile. The following Err_N is used for checking the convergence of solutions by SCM.

$$Err_N = \frac{\max_{0 \leq i, j \leq 50} |v_N(s_j, x_i) - v_{N+10}(s_j, x_i)|}{\max_{0 \leq i, j \leq 50} |v_{N+10}(s_j, x_i)|},$$

$$x_i = i/50, \quad s_j = (s_e - s_l)j/50 + s_l, \quad i, j = 0, \dots, 50$$

where $v_N(s_j, x_i)$ is the interpolant for the data $\tilde{u}(s_j, x_i)$ which is computed in $0 \leq x \leq 1$, $s_l \leq s \leq s_e$ by SCM with $N = N_x = N_s$. s_j is the CGL or CGR points. x_i is the CGL points. In SCM discretized equations are nonlinear, so Newton method is used. The convergence of Newton method is determined whether the absolute relative difference of numerical solutions is smaller than ε ($\varepsilon = 10^{-13}$ in double precision, $\varepsilon = 10^{-47}$ in 50 digits) or not in 20 iterations.

(i) In the case of $\beta = 1$.

Numerical results for $\alpha = 1$ by FDM \oplus EE are shown in Fig.2 and Table 1.

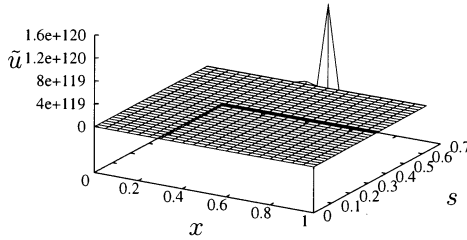


Fig. 2. Solution profile by FDM \oplus EE
($\beta = 1$, $\alpha = 1$, $N_x = 20$, $\Delta s = 5 \times 10^{-4}$).

Table 1. Overflow time
($\beta = 1$, $\alpha = 1$, $N_x = 20$).

Δs	time
5×10^{-4}	0.6235
10^{-4}	0.8269
10^{-5}	0.94328
10^{-6}	0.981787

Table 1 shows that the overflow time approaches to 1 as Δs becomes small. So, the solution profile in Fig.2 seems not to be one of the blow-up solution. Then, numerical computation by SCM with CGL points on x and CGR points on s is carried out for $0 \leq s < 1$ (Fig.3).

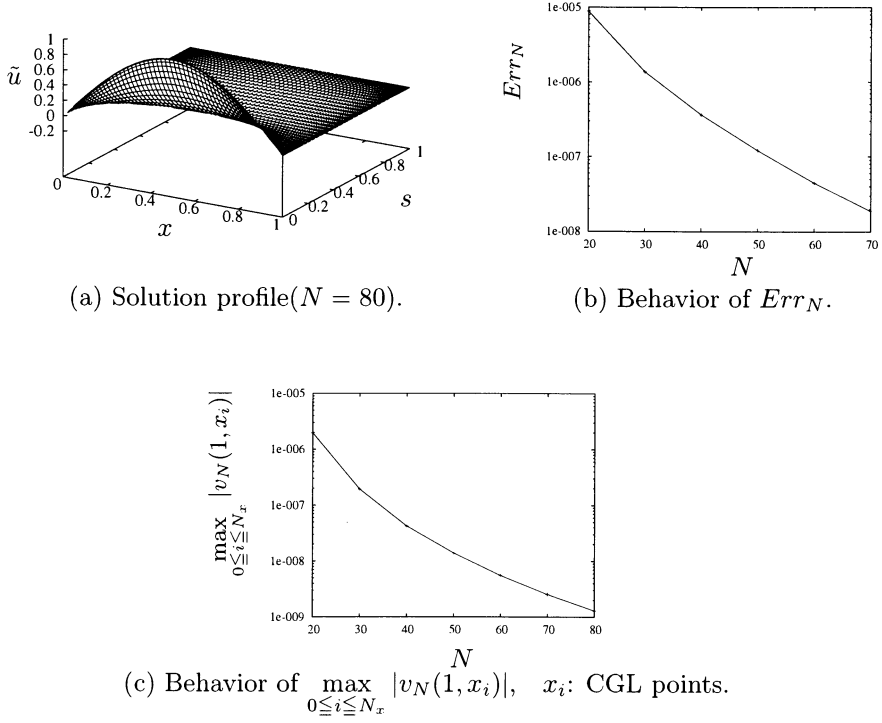


Fig. 3. Numerical results by SCM with $CGL \oplus CGR(\beta = 1, \alpha = 1)$.

Fig.3(b) shows exponential convergence of numerical solutions. So, the converged numerical solution is recognized to be reliable. Fig.3(c) shows that values of numerical solutions at $s = 1$ ($T = \infty$ in Problem 1 or 2) converge to 0, then it suggests the existence of the global decreasing solution for $\beta = 1, \alpha = 1$.

Numerical results for $\alpha = 100$ by $FDM \oplus EE$ are shown in Fig.4 and Table 2. From Table 2 the overflow time approaches to a constant $s_b < 1$ as Δs becomes small. This suggests the existence of the blow-up solution with the blow-up time s_b . Thus, numerical computation by SCM with CGL points is carried out for $0 \leq s \leq 0.0109$ (Fig.5). The interval on s is divided in numerical computation for realizing exponential convergence(Figs.5(a-2),(b-2)). Fig.5(c) is obtained by joining Figs.5(a) and (b).

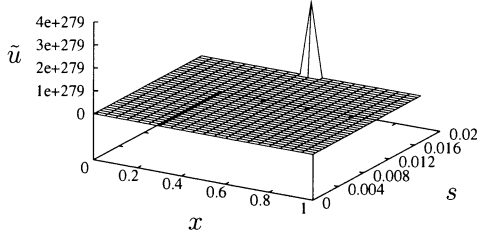
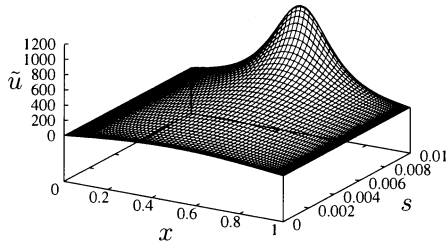


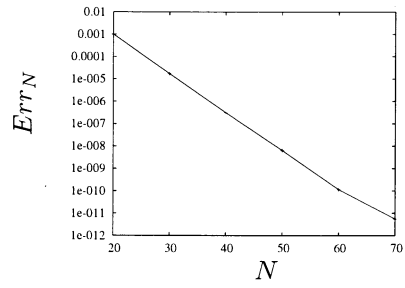
Fig. 4. Solution profile by FDM \oplus EE ($\beta = 1, \alpha = 100, N_x = 20, \Delta s = 5 \times 10^{-4}$).

Table 2. Overflow time ($\beta = 1, \alpha = 100, N_x = 20$).

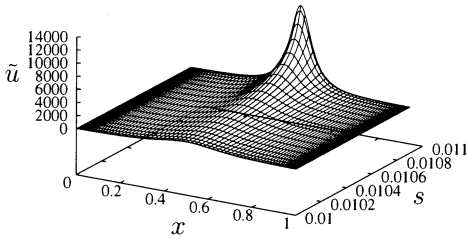
Δs	time
5×10^{-4}	0.0175
10^{-4}	0.0124
10^{-5}	0.01113
10^{-6}	0.010983
10^{-7}	0.0109666



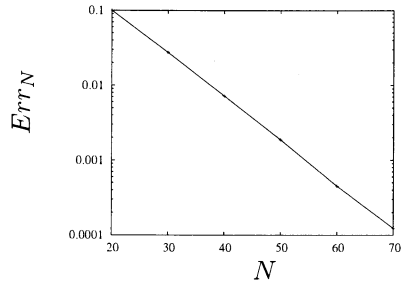
(a-1) Solution profile($N = 80$)
(a) $0 \leq s \leq 0.01$.



(a-2) Behavior of Err_N



(b-1) Solution profile($N = 80$)
(b) $0.01 \leq s \leq 0.0109$.



(b-2) Behavior of Err_N

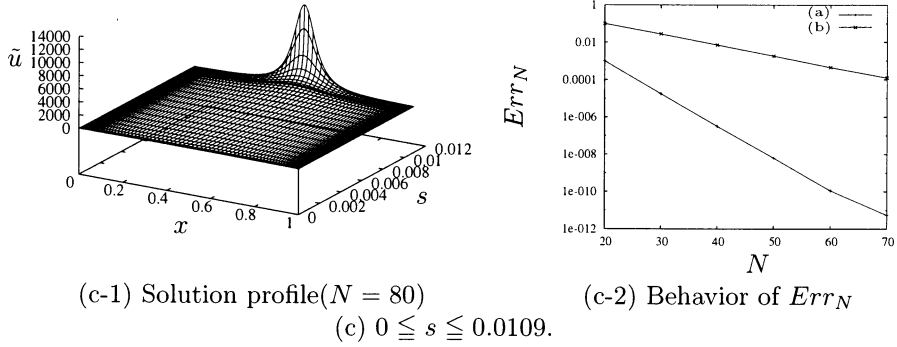
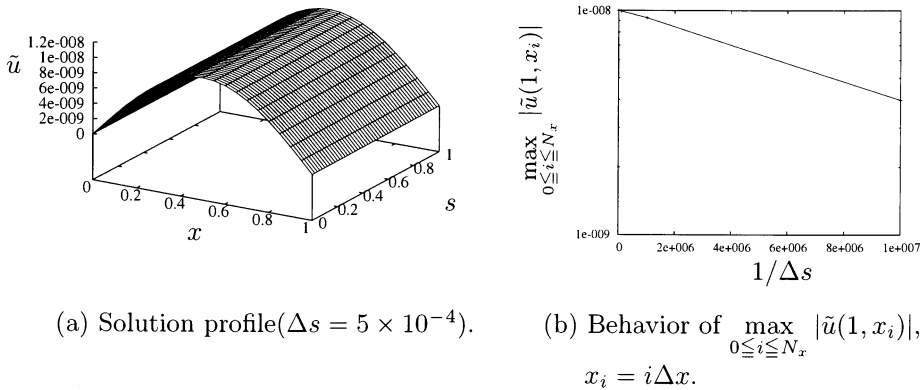
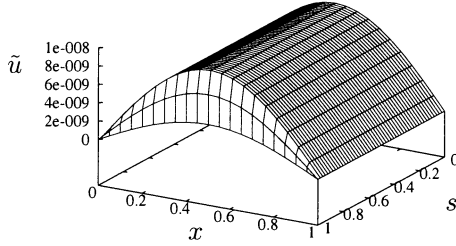


Fig. 5. Numerical results by SCM($\beta = 1$, $\alpha = 100$).

(ii) In the case of $\beta = 10^{-8}$.

Numerical results for $\alpha = 1$ by FDM \oplus EE are shown in Fig.6. Fig.6(a) with $\Delta s = 5 \times 10^{-4}$ suggests the global solution. From Fig.6(b) values of solutions at $s = 1(T = \infty)$ converge to 0 as Δs becomes small. Thus, the solution profile in Fig.6(a) is not precise. The solution profile with $\Delta s = 10^{-7}$ in Fig.6(c) may be precise and it suggests the existence of the global decreasing solution. In Fig.6(c) the view angle is different from that in Fig.6(a).

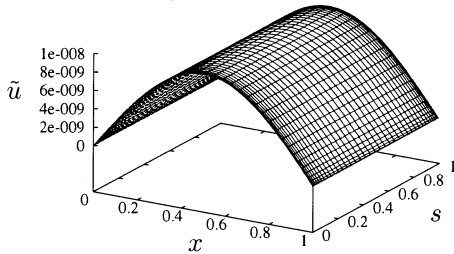




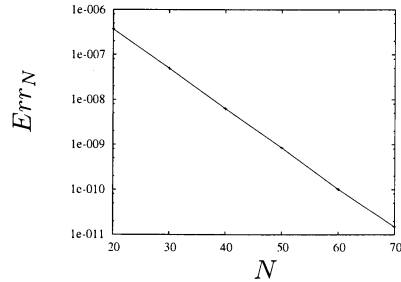
(c) Solution profile($\Delta s = 10^{-7}$).

Fig. 6. Solution profiles by FDM \oplus EE($\beta = 10^{-8}$, $\alpha = 1$, $0 \leq s \leq 1$).

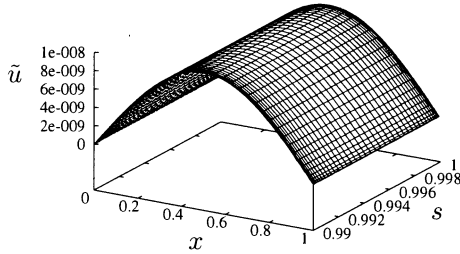
For more precise numerical computation SCM is applied in divided intervals for $0 \leq s < 1$ referring rough numerical results by FDM \oplus EE in Fig.6. Numerical results are shown in Fig.7. Solution profiles for $0 \leq s \leq 0.9999$ by SCM in Figs.7(a-1),(a-3) are reliable because Figs.7(a-2),(a-4) show exponential convergence in each interval. For patching data across the interval CGL points are used on s . Fig.7(b-1) shows the rough solution profile for $0.9999 \leq s \leq 1$ by FDM \oplus EE. From Fig.7(b-1) the solution is smooth for $0.9999 \leq s \leq 0.999999$. Then, SCM is applied in this interval and it gives the solution profile in Fig.7(b-2) which is reliable due to exponential convergence in this interval(Fig.7(b-3)). Rough numerical computation for $0.999999 \leq s \leq 1$ by FDM \oplus EE is shown in Fig.7(c-1). Multiple precision is used because double precision induces oscillation. Referring this solution profile SCM in multiple precision is applied in divided intervals for $0.999999 \leq s \leq 1$ (Figs.7(c-2)~(c-5)). Values of solutions at $s = 1$ in Fig.7(c-6) show the existence of the global decreasing solution for $\beta = 10^{-8}$, $\alpha = 1$. Fig.7(d) is obtained by joining Figs.7(a)~(c).



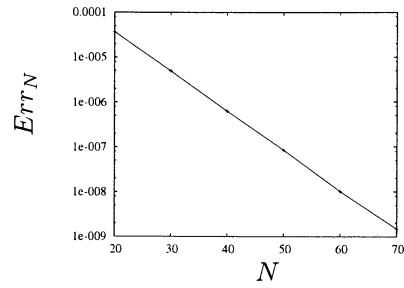
(a-1) Solution profile by SCM
($0 \leq s \leq 0.99$, $N = 80$).



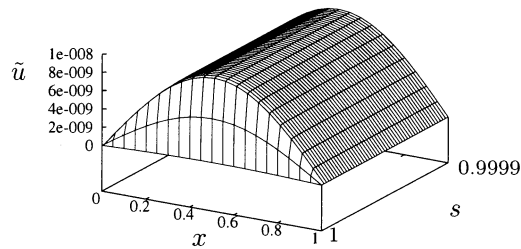
(a-2) Behavior of Err_N
($0 \leq s \leq 0.99$).



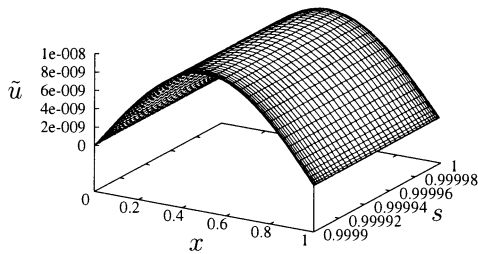
(a-3) Solution profile by SCM
($0.99 \leq s \leq 0.9999$, $N = 80$).



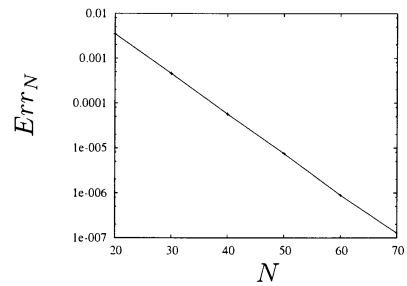
(a-4) Behavior of Err_N
($0.99 \leq s \leq 0.9999$).



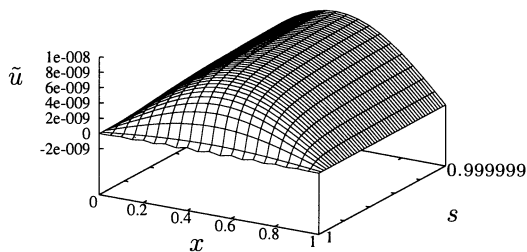
(b-1) Solution profile by FDM+EE($0.9999 \leq s \leq 1$, $\Delta s = 5 \times 10^{-8}$).



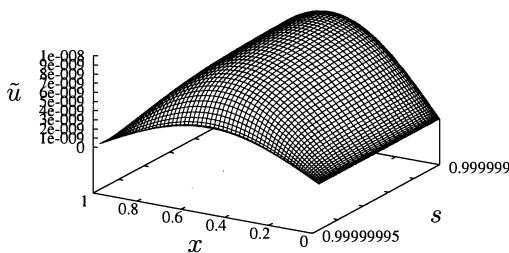
(b-2) Solution profile by SCM
($0.9999 \leq s \leq 0.999999$, $N = 80$).



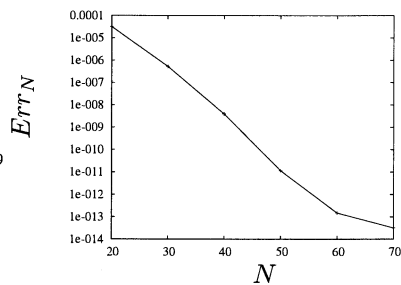
(b-3) Behavior of Err_N
($0.9999 \leq s \leq 0.999999$).



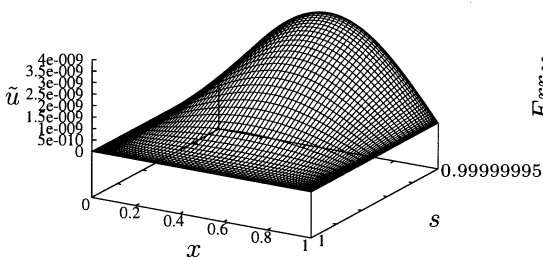
(c-1) Solution profile by FDM \oplus EE($0.999999 \leq s \leq 1$, $\Delta s = 5 \times 10^{-10}$, 200digits).



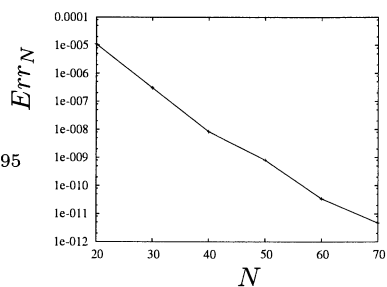
(c-2) Solution profile by SCM ($0.999999 \leq s \leq 0.99999995$, $N = 80$, 50digits).



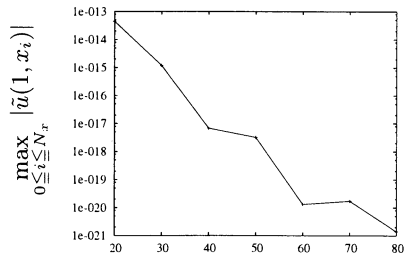
(c-3) Behavior of Err_N ($0.999999 \leq s \leq 0.99999995$).



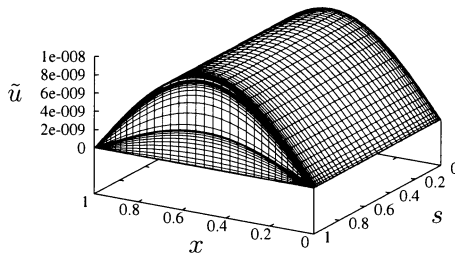
(c-4) Solution profile by SCM ($0.99999995 \leq s \leq 1$, $N = 80$, 50digits).



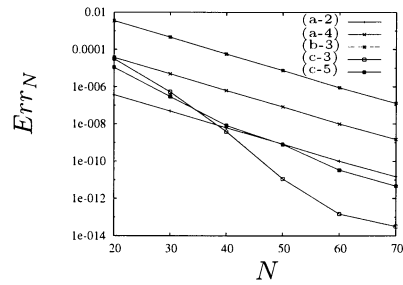
(c-5) Behavior of Err_N ($0.99999995 \leq s \leq 1$).



(c-6) Behavior of $\max_{0 \leq i \leq N_x} |\tilde{u}(1, x_i)|$, x_i : CGL points.



(d-1) Solution profile by SCM
($0 \leq s \leq 1$).



(d-2) Behavior of Err_N
($0 \leq s \leq 1$).

Fig. 7. Numerical results ($\beta = 10^{-8}$, $\alpha = 1$).

Numerical results for $\alpha = 100$ by FDM \oplus EE are shown in Fig.8. Fig.8(a) is obtained with $\Delta s (= 5 \times 10^{-4})$ which is not so small and it suggests that the solution is global. However, Fig.8(b) shows that the value of the solution at $s = 1 (T = \infty)$ grows as Δs becomes small. Fig.8(c) shows the solution profile with $\Delta s = 10^{-6}$ and it suggests the existence of the blow-up solution.

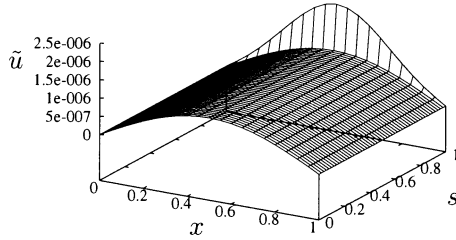
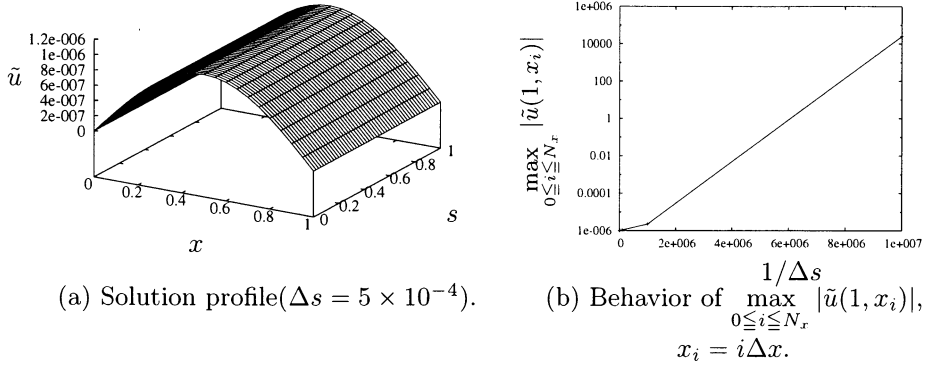
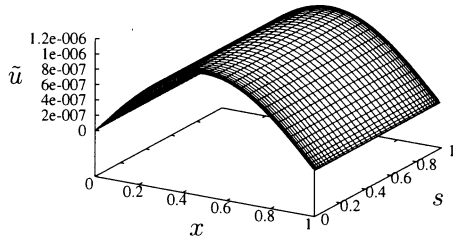
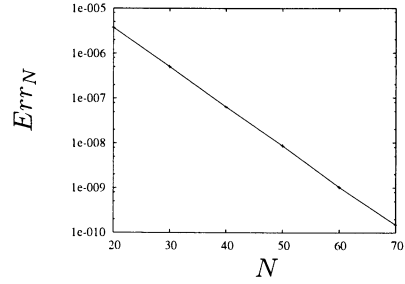


Fig. 8. Solution profiles by FDM \oplus EE($\beta = 10^{-8}$, $\alpha = 100$, $0 \leq s \leq 1$).

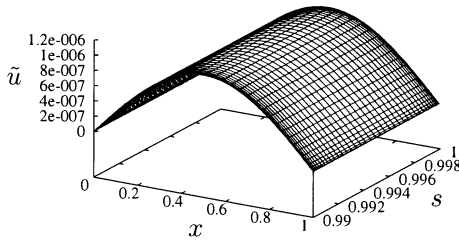
For more precise numerical computation SCM is applied in divided intervals for $0 \leq s < 1$ referring rough numerical results by FDM \oplus EE in Fig.8. Numerical results are shown in Fig.9. The procedure is same as in Fig.7. Solution profiles for $0 \leq s \leq 0.99999954$ by SCM in Figs.9(a-1, 3, 5, 7, 9) are reliable because Figs.9(a-2, 4, 6, 8, 10) show exponential convergence in each interval. Fig.9(b) is obtained by joining Fig.9(a). Here, $s = 0.99999954$ corresponds to $\tau = 0.10869563 \times 10^7$. This means that numerical computation about the blow-up time is satisfactory.



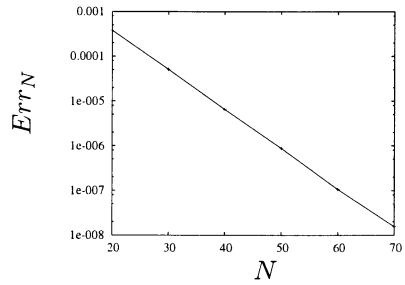
(a-1) Solution profile by SCM
($0 \leq s \leq 0.99$, $N = 80$).



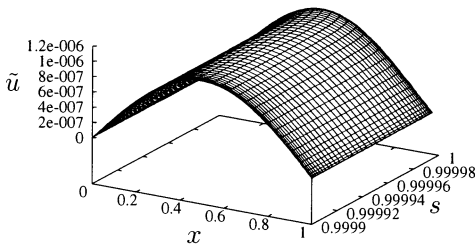
(a-2) Behavior of Err_N
($0 \leq s \leq 0.99$).



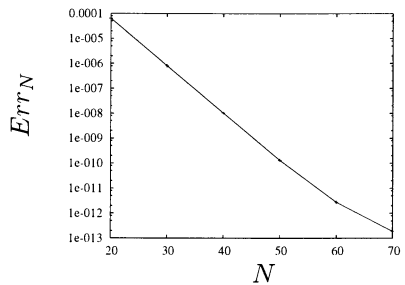
(a-3) Solution profile by SCM
($0.99 \leq s \leq 0.9999$, $N = 80$).



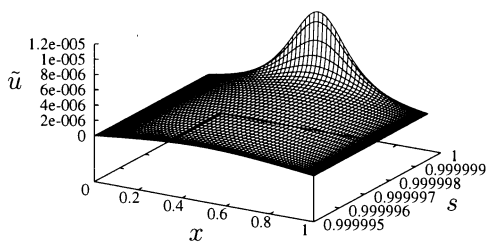
(a-4) Behavior of Err_N
($0.99 \leq s \leq 0.9999$).



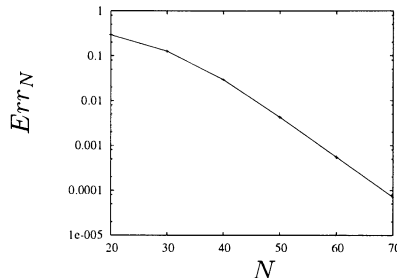
(a-5) Solution profile by SCM
($0.9999 \leq s \leq 0.999995$, $N = 80$).



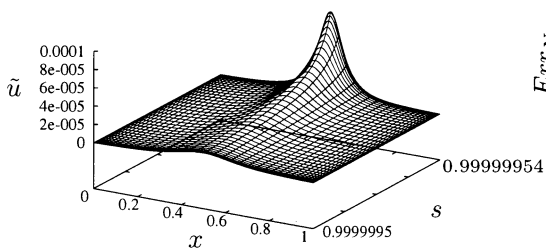
(a-6) Behavior of Err_N
($0.9999 \leq s \leq 0.999995$).



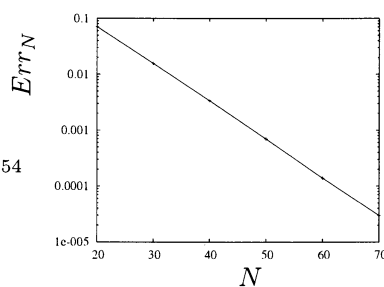
(a-7) Solution profile by SCM
 ($0.999995 \leq s \leq 0.9999995$, $N = 80$).



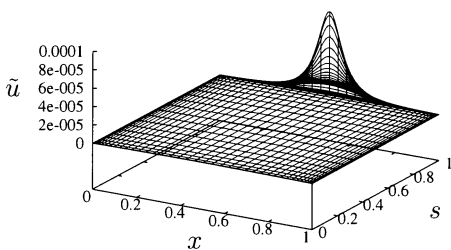
(a-8) Behavior of Err_N
 ($0.999995 \leq s \leq 0.9999995$).



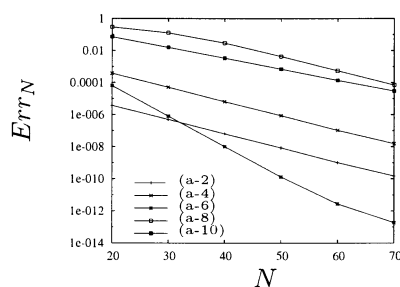
(a-9) Solution profile by SCM
 ($0.9999995 \leq s \leq 0.99999954$, $N = 80$).



(a-10) Behavior of Err_N
 ($0.9999995 \leq s \leq 0.99999954$).



(b-1) Solution profile by SCM
 ($0 \leq s \leq 0.99999954$).



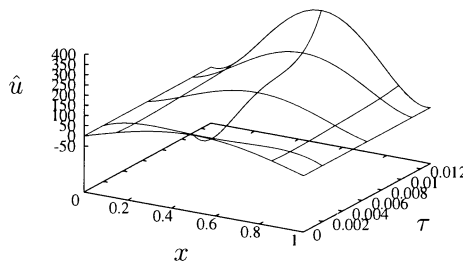
(b-2) Behavior of Err_N
 ($0 \leq s \leq 0.99999954$).

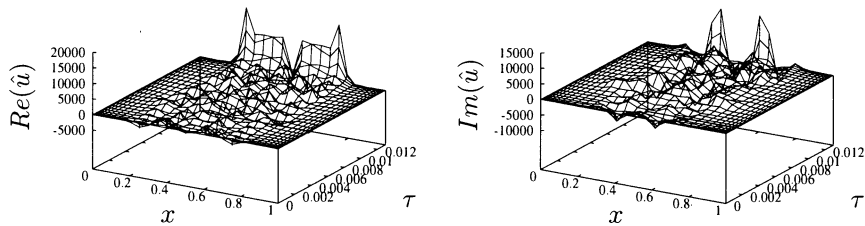
Fig. 9. Numerical results($\beta = 10^{-8}$, $\alpha = 100$).

(iii) Blow-up time and complex Newton method

Former numerical methods[9, 14] for computing the blow-up time used the adaptive control on the time increment depending on the solution. From the view point of the solution profile we may propose the different idea for computing the blow-up time as follows. The profile of the blow-up solution has singularity in the bounded domain. On the other hand, the numerical method SCM determines the solution profile and it is very sensitive to singularity even if singularity exists outside the domain[15]. These suggest that SCM can feel the blow-up time as singularity. Together with numerical continuation[11]. SCM may offer a new approach for computing the blow-up time.

The solutions of problems considered here do not exist beyond the blow-up time[1]. This suggests that SCM fails in the time-space domain including the blow-up time. Problem 2 with $\beta = 1$, $\alpha = 100$ has the blow-up time $\tau_b \approx 0.0109$ [9]. In the time-space domain as $0 \leq \tau \leq 0.013$ discretized equations by low order SCM($N_x = N_s = 4$) can be solved by Newton method. The solution profile is shown in Fig.10(a). On the other hand, discretized equations by higher order SCM cannot be solved by Newton method. If Newton method does not converge it gives apprehension rather than no information. Thus, application of complex Newton method is considered. It works well and it gives the solution profiles as Fig.10(b). From these numerical results and Fig.5(c) the blow-up time τ_b of Problem 2 with $\beta = 1$, $\alpha = 100$ is estimated as $0.010901 \cdots \leq \tau_b \leq 0.013$. It is interesting that estimation from above is numerically obtained.

(a) Real Newton method, $N_x = N_s = 4$.

(b) Complex Newton method, $N_x = N_s = 30$.Fig. 10. Solution profiles by SCM($\beta = 1$, $\alpha = 100$, $0 \leq \tau \leq 0.013$).

3 conclusion

In the paper a numerical method for distinction between blow-up and global solutions is proposed. It consists of finite difference method, explicit Euler method, spectral collocation method, bounding transform, Newton method and multiple precision arithmetic. Our method is applied to a famous one-dimensional nonlinear heat equation. Numerical results are satisfactory. Moreover, the blow-up time is estimated from above in some case. In the paper complex Newton method is also used. Its new applicability is our future work.

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