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Existence of Global and Bounded Solutions for Damped Sublinear Wave Equations

By

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Abstract

We study the initial–boundary value problem for the sublinear wave equations with a linear dampping : $u'' - \Delta u - \omega\Delta u' + \delta u' = \gamma|u|^{p-2}u$ with the homogeneous Dirichlet boundary condition and $H_0^1(\Omega) \times L^2(\Omega)$-data condition under $\omega \geq 0$ and $\delta > -\omega\lambda_1$. When $1 < p < 2$, we show that the (local) weak solutions are global and uniformly bounded in time $t \geq 0$.

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1 Introduction

We consider the initial-boundary value problem for the following semilinear wave equation :

$$u'' - \Delta u - \omega\Delta u' + \delta u' = f(u)\,, \quad u = u(x,t)\,, \quad \text{in } \Omega \times [0,\infty) \tag{1}$$

with homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega$$

and initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x)\,,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $\Delta = \nabla\cdot\nabla = \sum_{j=1}^N \partial^2/\partial x_j^2$ is Laplacian, ω and δ are constants such that $\omega \geq 0$

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and $\delta > -\omega\lambda_1$ with λ_1 being the first eigenvalue of the operator $-\Delta$ under the homogeneous Direchlet boundary condition, that is,

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{||\nabla u||^2}{||u||^2} ,$$

and

$$f(u) = \gamma |u|^{p-2} u \,, \quad \gamma > 0 \,, \qquad p > 1 \,. \tag{2}$$

In the supperlinear case ($p > 2$), it is well known that the so-called potential well method is useful to the analysis of global existence for problem (1). (see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20], and also, [8], [9], [14], [18]), and moreover, the concavity method is applied to the analysis of finite time blow-up phenomena (see Tsutsumi [23], Levine [10], [11], and also, [1], [2], [3], [7], [13], [15], [16], [22]).

In order to explain some known results for $p > 2$, we define the total energy associated with (1) by

$$E(u, u') = \frac{1}{2}||u'||^2 + J(u)$$

where we put

$$\begin{aligned} J(u) &= \frac{1}{2}||\nabla u||^2 - \frac{\gamma}{p}||u||_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)||\nabla u||^2 + \frac{1}{p} I(u) \end{aligned}$$

with

$$I(u) = ||\nabla u||^2 - \gamma ||u||_p^p ,$$

and we define the mountain pass level d (also known as the potential well depth) by

$$d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \left(\sup_{\lambda \geq 0} J(\lambda u) \right)$$

(see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20]).

When the power p in (2) satisfies that $p > 2$ and $p \leq 2(N-1)/(N-2)$ if $N \geq 3$, many authors have already studied on global existence or finite time blow-up of (local) weak solutions in the class $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$ for the problem (1) with the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying suitable conditions : (i) if $E(u_0, u_1) < d$ and $I(u_0) > 0$, then there exists a unique global solution $u(t)$ satisfying $||u(t)||_{H^1} + ||u'(t)|| \to 0$ as $t \to \infty$; (ii) if $E(u_0, u_1) < d$ and $I(u_0) < 0$, then the local solution $u(t)$ blows up at some finite time, that is, there exists a finite time $T^* < \infty$ such that $\|u(t)\|_{H^1} \to \infty$ as $t \to T^*$; moreover (iii) when $\omega = 0$, if $E(u_0, u_1) \geq d$, $I(u_0) < 0$, $\|u_0\| \geq \sup\{\|\phi\| \mid \phi \in H_0^1(\Omega)\backslash\{0\}$ with $(1/2 - 1/p)\|\nabla\phi\|^2 \leq E(u_0, u_1)\}$, and $\int_\Omega u_0 u_1\, dx \geq 0$, then the local solution $u(t)$ blows up at some finite time (see Gazzola-Squassina [5]). We note that if $E(u(t), u'(t)) \geq d$ for all $t \geq 0$, then $\lim_{t\to\infty} E(u(t), u'(t))$ exists, and when $\omega = 0$, $p > 2$, and $p < 2(N-1)/(N-2)$ if $N \geq 3$ or $p \leq 6$ if $N = 2$, the local solution $u(t)$ is global and bounded (see Esquivel-Avila [4]).

On the other hand, when the power p in (2) satisfies that $1 < p \leq 2$, we see that for the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the (local) weak solution $u(t)$ in the class $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$ is global. In particular, from Remark 3.10 in Gazzola and Squassina [5], we know that the global solution $u(t)$ satisfies

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq C(1+t)^{p/(4-2p)} \quad \text{if } p < 2$$
$$\|u(t)\|_{H^1} + \|u'(t)\| \leq Ce^{\alpha t} \quad \text{if } p = 2$$

with some $\alpha > 0$, for $t \geq 0$, but we can not know boundedness of global solutions.

The purpose of this paper is to show boundedness of global solutions of (1) in the case $1 < p < 2$ (i.e. sublinear case).

Our main result is as follows.

Theorem 1.1 *Let $1 < p < 2$, and let $\omega \geq 0$ and $\delta > -\omega\lambda_1$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ satisfying*

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq C + CI_0 e^{-\tilde{k}t}$$

with some constants $C > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

On the other hand, in the case $p = 2$ we have the following.

Theorem 1.2 *Let $p = 2$, and let $\omega \geq 0$ and $\delta > -\omega\lambda_1$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ satisfying that $\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0 e^{\tilde{\alpha}t}$, and moreover, if $\gamma < \lambda_1$,*

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0 e^{-\tilde{k}t}$$

with some constants $C > 0$, $\tilde{\alpha} > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

We use only familiar functional spaces and omit the definitions. We denote $L^p(\Omega)$-norm by $\|\cdot\|_p$ (we often write $\|\cdot\| = \|\cdot\|_2$ for simplicity). Positive constants will be denoted by C and will change from line to line.

2 Proofs

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem (e.g. see [6], [12], [17], [19]).

Proposition 2.1 *Let $p > 1$ and $p \leq 2(N-1)/(N-2)$ if $N \geq 3$. Suppose that the initial data (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$. Then, there exists a unique (local) weak solution $u(t)$ in the class $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$ of problem (1), that is,*

$$\frac{d}{dt}\int_\Omega u'(t) w\, dx + \int_\Omega \nabla u(t) \nabla w\, dx + \omega \int_\Omega \nabla u'(t) \nabla w\, dx + \delta \int_\Omega u'(t) w\, dx = \int_\Omega f(u(t)) w\, dx$$

a.e. in $(0,T)$ for every $w \in H_0^1(\Omega)$.

Moreover, if $\sup_{0 \leq t < T} (\|u(t)\|_{H^1} + \|u'(t)\|) < \infty$, then the solution $u(t)$ can be continued to $T + \varepsilon$ for some $\varepsilon > 0$.

Proof of Theorem 1.1. Multiplying (1) by u' and integrating it over Ω, we have

$$\frac{d}{dt}E_1(t) + \omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2 = \int_\Omega f(u(t)) u'(t)\, dx\,, \tag{3}$$

where $E_1(t)$ is defined by

$$E_1(t) = E_1(u(t), u'(t)) = \frac{1}{2}\left(\|u'(t)\|^2 + \|\nabla u(t)\|^2\right).$$

And, multiplying (1) by u and integrating it over Ω, we have

$$\frac{d}{dt}\frac{1}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_\Omega u(t)u'(t)\, dx\right) + \|\nabla u(t)\|^2 - \|u'(t)\|^2 = \int_\Omega f(u(t)) u(t)\, dx\,. \tag{4}$$

Then, taking (3) $+\, \varepsilon \times$ (4) for any small $\varepsilon > 0$, we have

$$\frac{d}{dt}F_1(t) + G_1(t) = \int_\Omega f(u(t))\left(u'(t) + \varepsilon u(t)\right)\, dx\,, \tag{5}$$

where $F_1(t)$ and $G_1(t)$ are defined by

$$F_1(t) = E_1(t) + \frac{\varepsilon}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_\Omega u(t)u'(t)\, dx\right)$$

and

$$G_1(t) = \left(\omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2\right) + \varepsilon\left(\|\nabla u(t)\|^2 - \|u'(t)\|^2\right) .$$

Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that

$$F_1(t) \le C(\|u'(t)\|^2 + \|\nabla u(t)\|^2) \tag{6}$$

and

$$G_1(t) \ge (\delta + \omega\lambda_1 - \varepsilon)\|u'(t)\|^2 + \varepsilon\|\nabla u(t)\|^2 . \tag{7}$$

Thus, if $\delta + \omega\lambda_1 > 0$, choosing small $\varepsilon > 0$, we have from (5)–(7) that

$$\frac{d}{dt}F_1(t) + 2kF_1(t) \le \int_\Omega f(u(t))\,(u'(t) + \varepsilon u(t))\;dx \tag{8}$$

with some constant $k > 0$. Moreover, we observe from the Young inequality and the Poincaré inequality with $p < 2$ that

$$\begin{aligned}\int_\Omega f(u(t))\,(u'(t) + \varepsilon u(t))\;dx &\le C\|u'(t)\|_p\|u(t)\|_p^{p-1} + C\|u(t)\|_p^p \\ &\le C\|u'(t)\|\|\nabla u(t)\|^{p-1} + C\|\nabla u(t)\|^p \end{aligned} \tag{9}$$

and by $\delta + \omega\lambda_1 > 0$,

$$\begin{aligned} F_1(t) &\ge E_1(t) + \frac{\varepsilon}{2}\left((\delta + \omega\lambda_1)\|u(t)\|^2 - 2\|u(t)\|\|u'(t)\|\right) \\ &\ge \frac{1}{2}(1 - C\varepsilon)\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 . \end{aligned} \tag{10}$$

Thus, choosing small $\varepsilon > 0$, we have from (8)–(10) that

$$\frac{d}{dt}F_1(t) + 2kF_1(t) \le CF_1(t)^{p/2} \le kF_1(t) + C, ,$$

where we used the Young inequality together with the fact that $1/2 < p/2 < 1$ at the last inequality, and hence,

$$F_1(t) \le \frac{C}{k} + F_1(0)e^{-kt} . \tag{11}$$

Therefore, we obtain from (10) and (11) that

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \le CF_1(t) \le C + CI_0^2 e^{-kt}$$

for $t \ge 0$. □

Proof of Theorem 1.2. Since $p = 2$ in (2), we have from (3) and the Poincaré inequality that

$$\frac{d}{dt}E_1(t) \le \gamma\|u(t)\|\|u'(t)\| \le \alpha E_1(t)$$

with some constant $\alpha > 0$, and hence, we have

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \le 2E_1(0)e^{\alpha t}$$

for $t \ge 0$.

Next, let $\gamma < \lambda_1$. Since $p = 2$ in (2), we have from (3) and (4) that

$$\frac{d}{dt}E(t) + \omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2 = 0 \tag{12}$$

and

$$\frac{d}{dt}\frac{1}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_\Omega u(t)u'(t)\,dx\right) + \|\nabla u(t)\|^2 - \gamma\|u(t)\|^2 - \|u'(t)\|^2 = 0\,, \tag{13}$$

respectively, where we write

$$E(t) = E_1(t) - \frac{\gamma}{2}\|u(t)\|^2 = \frac{1}{2}\left(\|u'(t)\|^2 + \|\nabla u(t)\|^2 - \gamma\|u(t)\|^2\right)$$

for simplicity. Then, taking (12) $+\,\varepsilon \times$ (13) for any small $\varepsilon > 0$, we have

$$\frac{d}{dt}F(t) + G(t) = 0 \tag{14}$$

where $F(t)$ and $G(t)$ are defined by

$$F(t) = E(t) + \frac{\varepsilon}{2}\left(\omega\|\nabla u(t)\|^2 + \delta\|u(t)\|^2 + 2\int_\Omega u(t)u'(t)\,dx\right)$$

and

$$G(t) = \left(\omega\|\nabla u'(t)\|^2 + \delta\|u'(t)\|^2\right) + \varepsilon\left(\|\nabla u(t)\|^2 - \gamma\|u(t)\|^2 - \|u'(t)\|^2\right)\,.$$

Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that

$$F(t) \le C(\|u'(t)\|^2 + \|\nabla u(t)\|^2) \tag{15}$$

and

$$G(t) \ge (\delta + \omega\lambda_1 - \varepsilon)\|u'(t)\|^2 + \varepsilon(1 - \gamma/\lambda_1)\|\nabla u(t)\|^2\,. \tag{16}$$

Thus, if $\delta + \omega\lambda_1 > 0$, choosing small $\varepsilon > 0$, we have from (14)–(16) that

$$\frac{d}{dt}F(t) + kF(t) \le 0 \tag{17}$$

with some constant $k > 0$. Moreover, we observe from the Young inequality and the Poincaré inequality that

$$\begin{aligned} F(t) &\ge \frac{1}{2}\left(\|u'(t)\|^2 + (1-\gamma/\lambda_1)\|\nabla u(t)\|^2\right) \\ &\quad + \frac{\varepsilon}{2}\left((\delta+\omega\lambda_1)\|u(t)\|^2 - 2\|u(t)\|\|u'(t)\|\right) \\ &\ge \frac{1}{2}(1-C\varepsilon)\|u'(t)\|^2 + \frac{1}{2}(1-\gamma/\lambda_1)\|\nabla u(t)\|^2 . \end{aligned} \tag{18}$$

Thus, choosing small $\varepsilon > 0$, we have from (17) and (18) that

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \le CF(t) \le CF(0)e^{-kt}$$

for $t \ge 0$. □

References

[1] S. Alinhac, Blowup for Nonlinear Hyperbolic Equations, Progress in Nonlinear Differential Equations and Their Applications, **17** 1995.

[2] J.M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math. Oxford Ser. (2), 28 (1977), 473–486.

[3] Y. Ebihara, S. Kawashima, and H.A. Levine, On solutions to $u_{tt} - |x|^\alpha \Delta u = f(u)$ $(\alpha > 0)$, Funkcial. Ekvac., **38** (1995), 539–544.

[4] J.A. Esquivel-Avila, The dynamics of a nonlinear wave equation, J. Math. Anal. Appl., **279** (2003), 135–150.

[5] F. Gazzola and M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006), 185–207.

[6] V. Georgiev and G. Todorova, Global solution for three-dimensional weak viscosity, C. R. Acad. Bulgare Sci., **47** (1994), 17–20.

[7] R.T. Glassey, Blow-up theorems for nonlinear wave equations, Math. Z., **132** (1973), 183–203.

[8] R. Ikehata and T. Suzuki, Stable and unstable sets for evolution equations of parabolic and hyperbolic type., Hiroshima Math. J., **26** (1996), 475–491.

[9] H. Ishii, Asymptotic stability and blowing up of solutions of some nonlinear equations, J. Differential Equations, **26** (1977), 291–319.

[10] H.A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$, Trans. Amer. Math. Soc., **192** (1974), 1–21.

[11] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J. Math. Anal., **5** (1974), 138–146.

[12] H.A. Levine and J. Serrin, Global nonexistence theorems for quasilinear evolution equations with dissipation, Arch. Rational Mech. Anal, **137** (1997), 341–361.

[13] H.A. Levine and G. Todorova, Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy, Proc. Amer. Math. Soc., **129** (2001), 793–805.

[14] M. Nakao and K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, Math. Z., **214** (1993), 325–342.

[15] M. Ohta, Remarks on blowup of solutions for nonlinear evolution equations of second order, Adv. Math. Sci. Appl, **8** (1998), 901–910.

[16] K. Ono, Blowup phenomena for nonlinear dissipative wave equations, J. Math. Tokushima Univ., **30** (1996), 19–43.

[17] K. Ono, Global existence and decay properties of solutions for some mildly degenerate nonlinear dissipative Kirchhoff strings, Funkcial. Ekvac., **40** (1997), 255–270.

[18] K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, J. Math. Anal. Appl., **216** (1997), 321–342.

[19] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation, Math. Methods Appl. Sci., **20** (1997), 151–177.

[20] L.E. Payne and D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math., **22** (1975), 273–303.

[21] D.H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Rational Mech. Anal., **30** (1968), 148–172.

[22] P. Souplet, Nonexistence of global solutions to some differential inequalities of the second order and applications, Portugal. Math., **52** (1995), 289–299.

[23] M. Tsutsumi, On solutions of semilinear differential equations in a Hilbert space, Math. Japon., **17** (1972), 173–193.