J. Math. Univ. Tokushima
Vol. 42 (2008), 13–17

On Some Formulas for $\pi/2$

By

Shin-ichi KATAYAMA

Department of Mathematical and Natural Sciences,
Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minamijosanjima-cho 1-1, Tokushima 770-8502, JAPAN
e-mail address : katayama@ias.tokushima-u.ac.jp
(Received September 30, 2008)

Abstract

In his paper [1], J.G. Goggins has shown a formula which relates π and Fibonacci numbers. In our paper [2], we have proved a generalized version of this formula. In this note, we shall prove formulas which generalize Fibonacci number to certain binary recurrence sequences.

2000 Mathematics Subject Classification. Primary 11B39;
Secondary 40A05, 11A99

Introduction

In [1], J.G. Goggins has shown the following simple but very interesting formula

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}), \tag{1}$$

where F_n is the nth Fibonacci number. We note this formula is also given as the formula (f) in the text [5] chapter 3. Since $F_1 = 1$, we see $\frac{\pi}{4} = \tan^{-1}(1/F_1)$. Thus (1) is equivalent to the following formula

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}). \tag{2}$$

The purpose of this short note is to generalize this formula on Fibonacci number to two formulas on binary recurrence sequences, that is, to the following two formulas

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}), \tag{3}$$

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}). \tag{4}$$

Here $\{u_n\}$ is the Lucas sequences associated to the parameter $(t, -1)$ and $\{v_n\}$ is the companion Lucas sequences associated to the parameter $(t, -1)$, respectively.

First of all, let us recall the fundamental properties of u_n and v_n. Let t be a positive integer and $\{u_n\}$ and $\{v_n\}$ be the binary recurrence sequences defined by putting

$$\begin{cases} u_{n+2} = tu_{n+1} + u_n, \\ v_{n+2} = tv_{n+1} + v_n, \end{cases}$$

with initial terms $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = t$.

Put $\varepsilon = (t + \sqrt{t^2+4})/2$ and $\bar{\varepsilon} = (t - \sqrt{t^2+4})/2$. Then one knows the following Binet's formula

$$\begin{cases} u_n = (\varepsilon^n - \bar{\varepsilon}^n)/\sqrt{t^2+4}, \\ v_n = \varepsilon^n + \bar{\varepsilon}^n. \end{cases}$$

Put $\alpha_{2n} = \tan^{-1}(1/u_{2n})$ and $\alpha_{2n-1} = \tan^{-1}(t/u_{2n-1})$ for any positive index n. Then we can show the following proposition.

Proposition 1. *For any integer $n \geq 1$, $\alpha_{2n} = \alpha_{2n+1} + \alpha_{2n+2}$.*

Proof. We have

$$\tan(\alpha_{2n+1} + \alpha_{2n+2}) = \frac{t/u_{2n+1} + 1/u_{2n+2}}{1 - t/(u_{2n+1}u_{2n+2})} = \frac{tu_{2n+2} + u_{2n+1}}{u_{2n+1}u_{2n+2} - t}$$
$$= \frac{u_{2n+3}}{u_{2n+1}u_{2n+2} - t}.$$

By virtue of the Binet's formula, we see

$$u_{2n+1}u_{2n+2} - t = (\varepsilon^{2n+1} - \bar{\varepsilon}^{2n+1})(\varepsilon^{2n+2} - \bar{\varepsilon}^{2n+2})/(t^2+4) - t$$
$$= (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} + \varepsilon + \bar{\varepsilon})/(t^2+4) - t = (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - t^3 - 3t)/(t^2+4).$$

On the other hand, we also have

$$u_{2n}u_{2n+3} = (\varepsilon^{2n} - \bar{\varepsilon}^{2n})(\varepsilon^{2n+3} - \bar{\varepsilon}^{2n+3})/(t^2+4)$$
$$= (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - \varepsilon^3 - \bar{\varepsilon}^3)/(t^2+4) = (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - t^3 - 3t)/(t^2+4).$$

Thus we have shown

$$\tan(\alpha_{2n+1} + \alpha_{2n+2}) = \frac{1}{u_{2n}} = \tan(\alpha_{2n}),$$

which completes the proof.

From this proposition, we have $\alpha_{2n} - \alpha_{2n+2} = \alpha_{2n+1}$ for any $n \geq 1$. Then we have

$$\sum_{n=1}^{\infty} \tan^{-1}(t/u_{2n+1}) = \sum_{n=1}^{\infty} \alpha_{2n+1} = \sum_{n=1}^{\infty} (\alpha_{2n} - \alpha_{2n+2})$$
$$= (\alpha_2 - \alpha_4) + (\alpha_4 - \alpha_6) + \cdots + (\alpha_{2n} - \alpha_{2n+2}) + \cdots = \alpha_2.$$

Since $\alpha_2 = \tan^{-1}(1/t) = \frac{\pi}{2} - \tan^{-1}(t/u_1)$, we have shown the formula (3).

Now we shall show the formula (4) similarly. Put $\beta_{2n} = \tan^{-1}(t/v_{2n})$ and $\beta_{2n-1} = \tan^{-1}(2/v_{2n-1})$ for any positive index n. Then we can show the following proposition.

Proposition 2. *For any integer $n \geq 1$, $2\beta_{2n} = \beta_{2n-1} - \beta_{2n+1}$.*

Proof. We have

$$\tan(\beta_{2n-1} - \beta_{2n+1}) = \frac{2/v_{2n-1} - 2/v_{2n+1}}{1 + 4/(v_{2n-1}v_{2n+1})} = \frac{2(v_{2n+1} - v_{2n-1})}{v_{2n-1}v_{2n+1} + 4}$$
$$= \frac{2tv_{2n}}{v_{2n-1}v_{2n+1} + 4}.$$

By virtue of the Binet's formula, we see

$$v_{2n-1}v_{2n+1} + 4 = (\varepsilon^{2n+1} + \bar{\varepsilon}^{2n+1})(\varepsilon^{2n-1} + \bar{\varepsilon}^{2n-1}) + 4$$
$$= (\varepsilon^{4n} + \bar{\varepsilon}^{4n}) - (\varepsilon^2 + \bar{\varepsilon}^2) + 4 = (\varepsilon^{4n} + \bar{\varepsilon}^{4n}) - (t^2+2) + 4$$
$$= (\varepsilon^{2n} + \bar{\varepsilon}^{2n})^2 - t^2 = v_{2n}^2 - t^2.$$

On the other hand, we have

$$\tan(2\beta_{2n}) = \frac{t/v_{2n} + t/v_{2n}}{1-(t/v_{2n})^2} = \frac{2tv_{2n}}{{v_{2n}}^2 - t^2}.$$

Thus we have shown

$$\tan(\beta_{2n-1} - \beta_{2n+1}) = \tan(2\beta_{2n}),$$

which completes the proof.

From this proposition, we have $\beta_{2n-1} - \beta_{2n+1} = 2\beta_{2n}$ for any $n \geq 1$.
Then we have

$$\sum_{n=1}^{\infty} 2\tan^{-1}(t/v_{2n}) = \sum_{n=1}^{\infty} 2\beta_{2n} = \sum_{n=1}^{\infty}(\beta_{2n-1} - \beta_{2n+1}) = \beta_1 = \tan^{-1}(2/t).$$

Since $v_{-2n} = v_{2n}$, one knows that $\tan^{-1}(t/v_{-2n}) = \tan^{-1}(t/v_{2n})$.
Hence we have

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}) = 2\left(\sum_{n=1}^{\infty}\tan^{-1}(t/v_{2n})\right) + \tan^{-1}(t/v_0)$$

$$= \tan^{-1}(2/t) + \tan^{-1}(t/2) = \frac{\pi}{2},$$

which completes the proof of (4).

Now we have completely proved two formulas of (4), which we shall state as the following theorem.

Theorem. *With the above notations, we have the following formulas,*

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}),$$

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}).$$

References

[1] J. G. Goggins, *Formula for* $\pi/4$, Mathematical Gazette, **57** (1973), 13.

[2] S.-I. Katayama, *Some infinite series of Fibonacci numbers*, Journal of Mathematics, the University of Tokushima, **42** (2008), 9–12.

[3] S. Nakamura, The Micro Cosmos of Fibonacci Numbers (in Japanese), Nihonhyoronsha, Tokyo, 2002.

[4] P. Ribenboim, The New Book of Prime Number Records, Springer-Verlag, New York, 1995.

[5] S. Vajida, Fibonacci & Lucas Numbers, and The Golden Section, John Wiley and Sons, New York, 1989.