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Some Infinite Series of Fibonacci Numbers

By

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Abstract

In his paper [1], J.G. Goggins has shown a simple formula which relates π and Fibonacci numbers. In this note, we shall prove a generalized formula (4) with some integer parameter k. Then Goggins's formula can be regarded as the special case $k = 1$ of our new formula (4).

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Introduction

In [1], J.G. Goggins has shown the following simple but very interesting formula

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}), \tag{1}$$

where F_n is the nth Fibonacci number. This formula is also given as the formula (f) in the text [5] chapter 3. Firstly, we shall rewrite this formula to the following two forms. Since $F_1 = 1$ and $\pi/4 = \tan^{-1}(1/F_1)$, (1) is equivalent to the following formula

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}). \tag{2}$$

From the facts $F_{-2k} = -F_{2k}$ and $F_{-2k-1} = F_{2k+1}$, (2) is also equivalent to the following formula

$$\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(1/F_{2n+1}). \tag{3}$$

The purpose of this short note is to generalize this formula to the following formula which holds for any integer parameter k,

$$k\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}). \tag{4}$$

Remark. We note that, from the fact $F_2 = 1$, the above formula (3) is exactly the case $k = 1$ of this formula (4).

Let $\{G_n\}$ be generalized Fibonacci sequences which satisfy

$$G_{n+2} = G_{n+1} + G_n.$$

Using the induction on m, we can show the following addition theorem of G_ℓ.

Addition Theorem. (*See for example* [3]).

$G_{m+\ell} = F_m G_{\ell+1} + F_{m-1} G_\ell$, *for any integer* m.

Substituting $G_{\ell+1} - G_{\ell-1}$ for G_ℓ, we have

$$\begin{aligned} G_{m+\ell} &= F_m G_{\ell+1} + F_{m-1}(G_{\ell+1} - G_{\ell-1}) = (F_m + F_{m-1})G_{\ell+1} - F_{m-1}G_{\ell-1} \\ &= F_{m+1}G_{\ell+1} - F_{m-1}G_{\ell-1}. \end{aligned}$$

Thus we have obtained a modified version of this addition theorem.

Corollary 1.

$G_{m+\ell} = F_{m+1}G_{\ell+1} - F_{m-1}G_{\ell-1}$, for any integer m.

Let us consider the special case when $G = F$ and $\ell = 2n$ is even and $m = 2k-1$ is odd in Corollary 1. Then we have $F_{2n+2k-1} = F_{2k}F_{2n+1} - F_{2k-2}F_{2n-1}$. Hence we have:

Corollary 2.

$F_{2k-2}F_{2n-1} + F_{2n+2k-1} = F_{2k}F_{2n+1}$.

Let us consider the special case when $G = F$ and $\ell = 2n$ is even and $m = -2n - 2k + 2$ in Corollary 1. Then we have $F_{-2k+2} = F_{-2n-2k+3}F_{2n+1} - F_{-2n-2k+1}F_{2n-1}$, which is equivalent to $-F_{2k-2} = F_{2n+2k-3}F_{2n+1} - F_{2n+2k-1}F_{2n-1}$. Thus we have shown:

Corollary 3.

$$F_{2n+2k-1}F_{2n-1} - F_{2k-2} = F_{2n+2k-3}F_{2n+1}.$$

Using these corollaries, we can show the following proposition.

Proposition.

$$\tan^{-1}\left(\frac{F_{2k-2}}{F_{2n+2k-1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n-1}}\right) = \tan^{-1}\left(\frac{F_{2k}}{F_{2n+2k-3}}\right).$$

Proof. From Corollaries 2 and 3, we have

$$\frac{\dfrac{F_{2k-2}}{F_{2n+2k-1}} + \dfrac{1}{F_{2n-1}}}{1 - \dfrac{F_{2k-2}}{F_{2n+2k-1}F_{2n-1}}} = \frac{F_{2k-2}F_{2n-1} + F_{2n+2k-1}}{F_{2n+2k-1}F_{2n-1} - F_{2k-2}} = \frac{F_{2k}F_{2n+1}}{F_{2n+2k-3}F_{2n+1}}$$

$= \dfrac{F_{2k}}{F_{2n+2k-3}}$, which completes the proof.

This proposition and the fact

$$\lim_{n\to\pm\infty} \tan^{-1}(F_{2m}/F_{2n+1}) = 0$$

for any fixed m imply that

$$\sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k-2}}{F_{2n+1}}\right) + \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n-1}}\right) = \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k}}{F_{2n+1}}\right).$$

Put $A(k) = \displaystyle\sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{F_{2k}}{F_{2n+1}}\right)$. Then this relation can be written as

$$A(k-1) + A(1) = A(k). \tag{5}$$

We note that $A(1) = \pi$ from the formula (3). Then, from this relation (5) and the induction on k, we can show the infinite series $A(k)$ is convergent for any integer k. Using the same relation (5), we can also verify that $A(k)$ satisfies the formula (4). Now we have completed the proof of the following theorem.

Theorem. *With the above notations, we have*

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}) = k\pi,$$

or equivalently

$$\sum_{n=0}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}) = \frac{k\pi}{2} \text{ for any integer } k.$$

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