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Pandiagonal Constant Sum Matrices

By

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Abstract

In the present paper, we study square matrices in which the sum of elements in any row, in any column , in any extended diagonal add up to a constant. We call such a matrix a pandiagonal constant sum matrix. We will show that the number of independents elements in a pandiagonal constant sum matrix of order n is $n^2 - 4n + 3$ if n is odd or $n^2 - 4n + 4$ if n is even.

2000 Mathematics Subject Classification. 05B20

Introduction

Let Σ be a a set of n different elements. A latin square of order n is a square matrix with n entries of elements in Σ, none of them occurring twice within any row or column of the matrix. A *matrix of the same number* n is defined to be a square matrix with n^2 entries of n different elements, each appeared exactly n times. A latin square of order n is a matrix of the same number n. A magic square of order n is an arrangement of n^2 consecutive integers in a square, such that the sums of each row each column and each of the main diagonal are the same. If also the sum of each extended diagonal is the same, the magic square is called pandiagonal. Two latin squares $A = (a_{ij}$ and $B = (b_{ij})$ of order n are said to be orthogonal if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) . We can define that two matrices of the same number n are orthogonal similarly. Let $A = (a_{ij}),\ \ a_{ij} \in R$@be a square matrix of order n. It is called a *constant sum matrix* if the sums of each row and each column are the same. If moreover the sum of each main diagonal is the same , it is called a*diagonal constant sum matrix* and if the sum of each extended diagonal is the same, it is called a *pandiagonal constant sum matrix.* In the present paper, we take $0, 1, \cdots, n-1$ as n consecutive integers and put

$\Sigma = \{0, 1, \cdots, n-1\}$. A pandiagonal latin square on Σ is a pandaiagonal constant sum matrix of the same number n.

Let A and B are orthogonal matrices of the same number n. Put $C = nA + B$. Then it is known [3] that if A and B are diagonal(resp. pandiagonal) constant sum matrices, C is a magic (resp. pandiagonal magic) square.

1. Pandiagonal constant sum matrices

Let $A = (a_{ij}),\ a_{ij} \in R$ be a pandiagonal constant sum matrix of order n. In the present paper, subscripts have the range $0, 1, \cdot, n-1 \ \ (mod\ \ n)$. Then we have the following equations.

$$\sum_{i=0}^{n-1} a_{ij} = C,\ 0 \le j \le n-1, \quad (1)$$

$$\sum_{j=0}^{n-1} a_{ij} = C,\ 1 \le i \le n-1, \quad (2)$$

$$\sum_{i=0}^{n-1} a_{in+j-i} = C,\ 1 \le j \le n-1, \quad (3)$$

$$\sum_{i=0}^{n-1} a_{ii+j} = C,\ 1 \le j \le n-1, \quad (4)$$

where C is a constant. Notice that from (1) and (2), (3),(4) it follows

$$\sum_{j=0}^{n} a_{0j} = C,$$

$$\sum_{i=0}^{n} a_{in-i} = C,$$

$$\sum_{i=0}^{n} a_{ii} = C.$$

When n is even, that is, $n = 2m$, there is a redundant equation in (3) and (4). In fact, if we set $2j = 2k + 2i \quad (mod\ \ 2m)$, we have

$$\sum_{j=0}^{m-1} \sum_{i=0}^{2m-1} a_{i2m+1+2j-i} = \sum_{j=0}^{m-1} \sum_{i=0}^{2m-1} a_{i1+2j-i} = \sum_{k=0}^{m-1} \sum_{i=0}^{2m-1} a_{i1+2k+i} = mC.$$

Hence, we can consider the equation

$$\sum_{i=0}^{2m-1} a_{i1+i} = C$$

is redundant. Now, when $n = 2m$, we set

$$\begin{aligned}
\sum_{i=0}^{2m-1} a_{ij} &= C, \ 0 \leq j \leq 2m-1, & (5)\\
\sum_{j=0}^{2m-1} a_{ij} &= C, \ 1 \leq i \leq 2m-1, & (6)\\
\sum_{i=0}^{2m-1} a_{i2m+j-i} &= C, \ 1 \leq j \leq 2m-1, & (7)\\
\sum_{i=0}^{2m-1} a_{ii+j} &= C, \ 2 \leq j \leq 2m-1. & (8)
\end{aligned}$$

Theorem 1 When n is an odd number, the equations (1),(2),(3) and (4) are independent, and when $n = 2m$, the equations (5),(6),(7) and (8) are independent.

Proof. Set

$$x_i = a_{0i}, \quad y_i = a_{1i}, \quad z_i = a_{2i}, \quad 0 \leq i \leq n-1.$$

Then we have

$$\begin{aligned}
A_j : x_j + y_j + k_j \sum_{i=3}^{n-1} a_{ij} &= C, \ 0 \leq j \leq n-1,\\
B_j : x_j + y_{j-1} + z_{j-2} + \sum_{i=0}^{n-1} a_{ij-i} &= C, \ 1 \leq j \leq n-1,\\
D_1 : \sum_{j=0}^{n-1} y_j &= C,\\
D_2 : \sum_{j=0}^{n-1} z_j &= C,\\
D_j : \sum_{i=0}^{n-1} a_{ji} &= C, \ 3 \leq j \leq n-1.
\end{aligned}$$

(1) Now suppose that n is odd, that is $n = 2m+1$. Then it holds

$$C_j : x_j + y_{j+1} + z_{j+2} + \sum_{i=3}^{n-1} a_{ii+j} = C, \ \ 1 \le j \le n-1.$$

Now we represent simply the above equations as

$$\begin{aligned} A_j &= (x_j, y_j, z_j, *), \ \ 0 \le j \le n-1, \\ B_j &= (x_j, y_{j-1}, z_{j-2}, *), \ \ 1 \le j \le n-1, \\ C_j &= (x_j, y_{j+1}, z_{j+2}, *), \ \ 1 \le j \le n-1 \\ D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\ D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\ D_j &= (0,0,0,*), \ \ 3 \le j \le n-1. \end{aligned}$$

Put

$$\begin{aligned} B_j(1) &= B_j - A_j = (0, y_{j-1} - y_j, z_{j-2} - z_j, *), \ \ 1 \le j \le n-1 && (9) \\ B_{n-1}(2) &= B_{n-1}(1) = (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), && (10) \\ B_j(2) &= B_j(1) + B_{j+1}(2) \\ &= (0, y_{j-1} - y_{n-1}, z_{j-2} + z_{j-1} - z_{n-2} - z_{n-1}, *), && (11) \end{aligned}$$

$1 \le j \le n-2.$

Especially, we have

$$B_1(2) = (0, y_0 - y_{n-1}, z_0 - z_{n-2}.*)$$

Next, we get

$$\begin{aligned} C_j(1) &= C_j - A_j = (0, -y_j + y_{j+1}, -z_j + z_{j+2}, *), \ \ 1 \le j \le n-1, \\ C_j(2) &= C_j(1) + B_{j+1}(1) = (0,0, z_j - z_{j+1} - z_{j+2} + z_{j+3}, *), \ \ 1 \le j \le n-2, \\ C_{n-1}(2) &= C_{n-1}(1) - B_1(2) = (0,0, z_{n-2} - z_{n-1} - z_0 + z_1, *). \end{aligned}$$

Now, set

$$\begin{aligned} C_j(3) &= C_j(2) + C_{j+1}(2) = (0,0, z_{j-1} - 2z_{j+1} + z_{j+3}, *), \ \ 1 \le j \le n-2, \\ C_{n-1}(3) &= C_{n-1}(2) = (0,0, z_{n-2} - z_{n-1} - z_0 + z_1, *). \end{aligned}$$

Put

$$C_{2m-1}(4) = C_{2m-1}(3), \quad C_{2m-3}(4) = C_{2m-3}(3) + 2C_{2m-1}(4),$$

$$C_{2(m-k)+1}(4) = C_{2(m-k)+1}(3)+2C_{2(m-k+1)+1}(4)-C_{2(m-k+2)+1}(4), \quad 3 \leq k \leq m.$$

Then we have

$$C_{2(m-k)+1}(4) = (0, 0, z_{2(m-k)} - (k+1)z_{2m} + kz_1, *), \quad 1 \leq k \leq m.$$

Next, we put

$$C_{2m}(4) = C_{2m}(3), \quad C_{2m-2}(4) = C_{2m-2}(3) + 2C_{2m}(4),$$

$$C_{2(m-k)}(4) = C_{2(m-k)}(3) + 2C_{2(m-k+1)}(4) - C_{2(m-k+2)}(4), \quad 2 \leq k \leq m-1.$$

Then, we get

$$C_{2(m-k)}(4) = (0, 0, z_{2(m-k)-1} - (k+1)(z_{2m} - z_1) - z_0, *), \quad 0 \leq k \leq m-1.$$

Set

$$\begin{aligned} C_2(5) &= \frac{1}{2m+1}(C_2(4) + C_1(4)) = (0, 0, z_1 - z_{2m}, *), \\ C_{2(m-k)+1}(5) &= C_{2(m-k)+1}(4) - kC_2(5), \quad 1 \leq k \leq m, \\ C_{2(m-k)}(5) &= C_{2(m-k)}(4) - (k+1)C_2(5) + C_1(5), \quad 0 \leq k \leq m-2. \end{aligned}$$

Thus we obtain

$$C_j(5) = (0, 0, z_{j-1} - z_{2m}, *), \quad 1 \leq j \leq n-1 = 2m.$$

Now the equations

$$\begin{aligned} A_j &= (x_j, y_j, z_j, *), \;\; 0 \leq j \leq n-1, \\ B_j(2) &= (, y_{j-1} - y_{n-1}, z_{j-2} + -z_{j-1} - z_{n-2} - z_{n-1}, *), \;\; 1 \leq j \leq n-2, \\ B_{n-1}(2) &= (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), \\ D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\ C_j(5) &= (0, 0, z_{j-1} - z_{2m}, *), \;\; 1 \leq j \leq n-1, \\ D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\ D_j &= (0, 0, 0, *), \;\; 3 \leq j \leq n-1 \end{aligned}$$

are equivalent to the equations given at first. It is evident that the rank of the coefficient matrix of the equations is $4n-3$. Hence, these equations are independent.

(2) Suppose that n is even, that is, $n = 2m$. By using the similar notations, we consider the following $4n-4$ equations

$$
\begin{aligned}
A_j &= (x_j, y_j, z_j, *), \ \ 0 \le j \le n-1, \\
B_j &= (x_j, y_{j-1}, z_{j-2}, *), \ \ 1 \le j \le n-1, \\
C_j &= (x_{j+1}, y_{j+2}, z_{j+3}, *), \ \ 1 \le j \le n-2 \\
.D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\
D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\
D_j &= (0,0,0,*), \ \ 3 \le j \le n-1.
\end{aligned}
$$

We define $B_j(1), B_j(2), \ 1 \le j \le n-1$ as similarly as in (9),(10),(11).
We put

$$
\begin{aligned}
C_j(1) &= C_j - A_{j+1} = (0, -y_{j+1} + y_{j+2}, -z_{j+1} + z_{j+3}, *), \ \ 1 \le j \le n-2, \\
C_j(2) &= C_j(1) + B_{j+2}(1) = (0, 0, z_j - z_{j+1} - z_{j+2} + z_{j+3}, *), \ \ 1 \le j \le n-3, \\
C_{n-2}(2) &= C_{n-2}(1) - B_1(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}
$$

Now, set

$$
\begin{aligned}
C_j(3) &= C_j(2) + C_{j+1}(2) = (0, 0, z_j - 2z_{j+2} + z_{j+4}, *), \ \ 1 \le j \le n-2, \\
C_{n-2}(3) &= C_{n-2}(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}
$$

Put

$$C_{2m-2}(4) = C_{2m-2}(3), \quad C_{2m-4}(4) = C_{2m-4}(3) + 2C_{2m-2}(4)$$

$$C_{2(m-k)}(4) = C_{2(m-k)}(3) + 2C_{2(m-k+1)}(4) - C_{2(m-k+2)}, \quad 3 \le j \le m-1.$$

Then we have

$$C_{2(m-k)}(4) = (0, 0, z_{2(m-k)} - k(z_{2m-1} - z_1) - z_0, *), \quad 1 \le k \le m-1.$$

Next, we put

$$C_{2m-3}(4) = C_{2m-3}(3), \quad C_{2m-5}(4) = C_{2m-5}(3) + 2C_{2m-3}(4),$$

$$C_{2(m-k)-1}(4) = C_{2(m-k)-1}(3) + 2C_{2(m-k+1)-1}(4) - C_{2(m-k+2)-1}, \quad 3 \le k \le m-1.$$

Then we get

$$C_{2(m-k)-1}(4) = (0, 0, z_{2(m-k)-1} - (k+1)z_{2m-1} + kz_1, *) \quad 1 \le k \le m-1.$$

Set

$$C_1(5) = \frac{1}{m} C_1(4) = (0, 0, z_1 - z_{2m-1}, *),$$

$$C_{2(m-k)}(5) = C_{2(m-k)}(4) - kC_1(5) = (0, 0, z_{2(m-k)} - z_0, *), \quad 1 \le k \le m-1,$$

$$C_{2(m-k)-1}(5) = C_{2(m-k)-1}(4) - kC_1(5) = (0, 0, z_{2(m-k)-1} - z_{2m-1}, *), \quad 1 \le k \le m-1.$$

Now the equations

$$\begin{aligned}
A_j &= (x_j, y_j, z_j, *), \ \ 0 \le j \le n-1, \\
B_j(2) &= (, y_{j-1} - y_{n-1}, z_{j-2} + -z_{j-1} - z_{n-2} - z_{n-1}, *), \ \ 1 \le j \le n-2, \\
B_{n-1}(2) &= (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), \\
D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\
C_1(5) &= (0, 0, z_1 - z_{2m-1}, *), \\
C_{2(m-k)}(5) &= (0, 0, z_{2(m-k)} - z_0, *), \quad 1 \le k \le m-1, \\
C_{2(m-k)-1}(5) &= (0, 0, z_{2(m-k)-1} - z_{2m-1}, *), \quad 1 \le k \le m-1, \\
D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\
D_j &= (0, 0, 0, *), \ \ 3 \le j \le n-1
\end{aligned}$$

are equivalent to the equations given at first. It is evident that the rank of the coefficient matrix of the equations is $4n - 4$. Hence, these equations are independent.

2. Pandiagonal zero sum matrices

Let $A = (a_{ij})$, $a_{ij} \in R$ be a pandiagonal constant sum matrix of order n with constant C. In this section, we have from here on , subtracted S/n from every elements in the matrix so that the sum of the elements in any row, column or diagonal will be zero, and we call such modified a *zero-sum matrix.* The results in this section mainly owe to W. R. Andress [1]. We introduce operators R, C such that

$$Ra_{i,j} = a_{i+1,j} \quad Ca_{i,j} = a_{i,j+1}.$$

Set

$$L_n(R) = \sum_{i=0}^{n-1} R^i, \quad D_n(R,C) = \sum_{i=0}^{n-1} R^{n-1-i}C^i.$$

Then we have

$$\begin{aligned} \text{columm} : L_n(R)a_{i,j} &= 0, \\ \text{row} : L_n(C)a_{i,j} &= 0, \\ \text{diagonal} : L_n(RC)a_{i,j} &= 0, \\ \text{diagonal} : D_n(R,C)a_{i,j} &= 0. \end{aligned}$$

Lemma Let $Q_{i,j}$ be elements of a square matrix of order n. If any one of the three conditions (1) $(C-1)Q_{i,j} = 0, \sum_{j=0}^{n-1} Q_{i,j} = 0$, (2) $(R-1)Q_{i,j} = 0, \sum_{i=0}^{n-1} Q_{i,j} = 0$, (3) $(R-C)Q_{i,j} = 0, \sum_{i=0}^{n-1} Q_{i,i} = 0$ holds, it follows that $Q_{i,j} = 0$.

Proof. Assume that the condition (1) holds. Then $Q_{i,j} = 0$ is independent of column. Hence using the second equation of (1), we get $Q_{i,j} = 0$. The other results follow similarly.

From $(L_n(RC) - L_n(R))a_{i,j} = R(C-1)\sum_{i=1}^{n-1} R^{i-1}L_{i-1}(C)a_{i,j} = 0$, using Lemma, we get $\sum_{i=1}^{n-1} R^{i-1}L_{i-1}(C)a_{i,j} = 0$. Since this is true for all values of i, j, it is convenient to suppress the operand $a_{i,j}$ so that

$$\sum_{i=0}^{n-2} R^i L_i(C) = 0. \tag{12}$$

This is a triangle-invariant. We may interchange the operations R, C in the above formula so that

$$\sum_{i=0}^{n-2} C^i L_i(R) = 0. \tag{13}$$

The triangle-invariant (12) remains invariant if we replace R by $1/R$ and multiple R^{n-1} as this merely represents a reflection in a horizontal line, and give

$$\sum_{i=0}^{n-2} R^i L_{n-2-i}(C) = 0. \tag{14}$$

Put

$$S_n = L_n(R)L_n(C).$$

Then, this presents a square of order n. Subtracting (12)-(13), and justifying the removal of the factor $R - C$ by Lemma 1, we obtain the invariant

$$\sum_{i=0}^{\lfloor (n-3)/2 \rfloor} (RC)^i S_{n-2-2i} = 0. \tag{15}$$

Subtracting (14) from (15), adding $C^{-1}D_n$ and multiplying $(RC)^{-1}$, we get the invariant

$$\sum_{i=0}^{\lfloor (n-5)/2 \rfloor} (RC)^i S_{n-3-2i} + R^{n-2}C^{n-2} = 0. \tag{16}$$

Subtracting $\sum_{i=0}^{n-3} R^i L_n(R)$ from (15) and adding $(C^{n-3} + C^{n-2})L_n(R)$ gives the invariant

$$\sum_{i=0}^{\lfloor (n-5)/2 \rfloor} (RC)^i S_{n-4-2i} + (RC)^{n-3} S_2 = 0. \tag{17}$$

From now on, we assume that n is odd, that is, $n = 2m + 1$.
The triangle-invariant (12) remains invariant if we replace C by $1/C$ as this merely represents a reflection in a vertical line, and give

$$\sum_{i=0}^{n-2} R^i L_i(C^{-1}) = 0. \tag{18}$$

Subtracting this from (12) and removing $R(C - C^{\ 1})$, we get

$$\sum_{i=0}^{m-1} R^i \sum_{j=0}^{i} \lfloor (i{+}2{-}j)/2 \rfloor (C^j{+}C^{-j}) + \sum_{i=0}^{m-2} R^i \sum_{j=0}^{m-2-i} \lfloor (m{-}i{-}j)/2 \rfloor (C^j{+}C^{-j}) = 0. \tag{19}$$

Theorem 2 Let $A = (a_{i,j})$ be a pandiagonal constant sum matrix of order n. For an odd n, if n^2-4n+3 elements $a_{i,j},\quad 0 \le i \le n-4,\quad 0 \le j \le n-2$ are given, the other elements decided uniquely. For an even n, if n^2-4n+4 elements $a_{i,j},\quad 0 \le i \le n-4,\quad 0 \le j \le n-2$ and any any one of $a_{n-3,j},\quad 0 \le j \le n-1$ are given, the other elements decided uniquely.

Proof. Using (12), we can get $a_{i,n-1},\quad 0 \le i \le n-4$. Assume that n is an odd number. Using (19), we obtain $a_{n-3,j},\quad 0 \le j \le n-1$ and then $a_{n-2,j}, a_{n-1,j},\quad 0 \le j \le n-1$. Next, Let n be an even number. Using (16), we can determine $a_{n-2,j},\quad 0 \le j \le n-1$. Then using (17), from any one of $a_{n-3,j},\quad 0 \le j \le n-1$, we obtain $a_{n-3,j},\quad 0 \le j \le n-1$. Now, it is easy to get $a_{n-1,j},\quad 0 \le j \le n-1$.

3. Orthogonal matrices of the same number

A *matrix of the same number* n is a square matrix with n^2 entries of n different elements, each appeared exactly n times. Two matrices of the same number n $A = (a_{ij}$ and $B = (b_{ij})$ are defined to be orthogonal if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) . It is well known that the largest value of r for which there exist r mutually orthogonal Latin squares of order n is less than n. Now, we have

Theorem 3. Denote by $N(n)$ the largest value of r for which there exist r mutually orthogonal matrices of order n. it holds

$$N(n) \leq n + 1.$$

Proof. Suppose $A_1, \cdots, A_t$ are mutually orthogonal matrices of order n on the symbols $\{0, 1, \cdots, n-1\}$. Take an n-square matrix $S = (s_{i,j})$ whose n^2 positions are labelled $0, 1, \cdots, n^2 - 1$ as follows: $S_{i,j} = ni + j, \quad 0 \leq i \leq j, 0 \leq j \leq n-1$. Then consider the collection of subsets $B_{r,m}$ defined by

$$B_{r,m} = \{x : x \text{ is the label in} S \text{of aposition in which } A_r \text{ has entry } m\},$$

where $1 \leq r \leq t, \ \ 0 \leq m \leq n-1$. There are tn subsets of size n. It follows from the orthogonality of the A_r that no pair of elements can occur in more than one block. Suppose for example x and y both occur in B_{r_1,m_1} and B_{r_2,m_2}. Then A_{r_1} has the same entry m_1 in x and y, while A_{r_2} has entry m_2 in these positions. Hence the pair (m_1, m_2) occurs twice, contradicting to the orthogonality of A_1 and A_2. Note the number of pairs of elements in the subsets is

$$tn\binom{n}{2} = \frac{1}{2}tn^2(n-1).$$

This number must be more than $\binom{n^2}{2}$. Hence

$$\frac{1}{2}tn^2(n-1) \leq \frac{1}{2}n^2(n^2-1)$$

gives $t \leq n + 1$.

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Signed Graphs and Hushimi Trees

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Abstract

The operation of local switching is introduced by Cameron, Seidel and Tsaranov. It acts on the set of all signed graphs on n vertices. In this paper, mainly, we study how local switching acts on trees. We show that two trees on the same vertices are isomorphic if and only if one is transformed to the other by a sequence of local switchings. There is a correspondence between signed graphs and a root lattice. Any signed graph corresponding to the lattice A_n is transformed by a sequence of local switchings to the tree which is regarded as the Dynkin diagram of A_n.

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Introduction

Following [?], we state basic facts about signed graphs. A graph $G = (V, E)$ consists of an n-set V(the vertices) and a set E of unordered pairs from V(the edges). A *signed graph* (G, f) is a graph G with a signing $f : E \to \{1, -1\}$ of the edges. We set $E^+ = f^{-1}(+1)$ and $E^- = f^{-1}(-1)$. For any subset $U \subseteq V$ of vertices, let f_U denote the signing obtained from f by reversing the sign of each edge which has one vertex in U. This defines on the set of signings an equivalence relation, called *switching*. The equivalence classes $\{f_U : U \subseteq V\}$ are the *signed switching classes* of the graph $G = (V, E)$. The *adjacency matrix* $A = (A_{ij})$ is defined by $A_{ij} = f(\{i, j\})$ for $\{i, j\} \in E$; else $A_{ij} = 0$ otherwise. The matrix $2I + A$ is called the *intersection matrix* , and interpreted as the Gram matrix of the inner product of n base vectors $a_1, \cdots, a_n$ in a (possibly indefinite) n-dimensional inner product space $R^{p,q}$. These vectors are roots(which have length $\sqrt{2}$) at angles $\pi/2, \pi/3$, or $2\pi/3$. Their integral linear combinations form a root lattice (an even integral lattice spanned by vectors of norm 2), which we denote by $L(G, f)$. The reflection w_i in the hyperplane orthogonal to the root a_i is given by

$$w_i(x) = x - \frac{2(a_i, x)}{(a_i, a_i)} a_i = x - (x, a_i)a_i.$$

The Weyl group $W(\Gamma, f)$ of $L(G, f)$ is generated by the reflections w_i, $(i = 1, \cdots n)$.

Let $i \in V$ be a vertex of G, and $V(i)$ be the set of neighbours of i. The *local graph* of (G, f) at i has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of G for which $f(i,j)f(j,k)f(k,i) = -1$. A *rim* of (G, f) at i is any union of connected components of local graph at i. Let J be any rim at i, and let $K = V(i) \backslash J$. *Local switching* of (G, f) with respect to (i, J) is the following operation:(i) delete all edges of G between J and K; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j, k\}$ with sign chosen so that $f(i,j)f(j,k)f(k,i) = -1$; (iii) change the signs of all edges from i to J; (iv) leave all other edges and signs unaltered. Let Ω_n be the set of switching classes of signed graphs of order n. Local switching , applied to any vertex and any rim at the vertex, gives a relation on Ω_n which is symmetric but not transitive. The equivalence classes of its transitive closure are called the *clusters* of order n. If two signed graphs (G_1, f_1) and (G_2, f_2) are in the same cluster, we say that (G_1, f_1) and (G_2, f_2) are *equivalent by local switching.* They are equivalent by local switching if and only if (G_1, f_1) is transformed to (G_2, f_2) by a sequence of switchings and local switchings.

Let L be a root lattice , B the set of ordered root bases, and B^* the subset of B consisting of bases which arise from signed graphs. Then, $(a_1, \cdots, a_n) \in \mathsf{B}^*$ if and only if $(a_i, a_j) \in \{0, +1, -1\}$ for all $i \neq j$. Many natural operations on B do not preserve B^*. Consider the map

$$\sigma_{ij} : (a_1, \cdots, a_n) \mapsto (a_1, \cdots, w_i(a_j), \cdots, a_n).$$

For any i, the map σ_{ii} just changes the sign of the vector a_i. Hence, they generate the equivalence relation induced by switching and preserve B^*. If i and j are non-adjacent, then σ_{ij} is the identity. So assume that i and j are adjacent. By switching, we may ensure that $(a_i, e_k) \geq 0$ for all k. Then $(a_i, a_j) = 1$ and $(w_i(a_j), a_k) = (a_j, a_k) - (a_i, a_k)$. Hence, if $(a_i, a_k) = 1$ and $(a_j, a_k) = -1$, B^* is not preserved by σ_{ij}. However the product of the commuting maps σ_{ij} and σ_{ik} preserve B^*. Let J be any set of neighbours of i and let $(a_1, \cdots, a_n)$ be a root base in B^*. Then $\prod_{j \in J} \sigma_{ij}$ maps $(a_1, \cdots, a_n)$ to a base in B^* if and only if J is a rim at i. This is the reason why the notion of local switching is defined as above.

We investigate how local switching acts on trees. For this purpose, we need to treat with Hushimi trees. In section 2, we discuss Hushimi trees which are related to the lattice A_n. We show in section 3 that these Hushimi trees are equivalent by local switching to trees with only two leaves. In section 4, we prove that two trees are equivalent by local switching if and only if one is obtained by rearrangement of vertices of the other. We deal with signed cycles in section 5. A signed cycle with odd parity is equivalent to a tree which may be regarded as the Dynkin diagram $[D_n]$ of the lattice D_n. Any signed graph corresponding to the lattice D_n is also equivalent to the tree $[D_n]$.

1. The lattice A_n and signed Hushimi trees

A connected graph $G = (V, E)$ is called *Hushimi tree* if each block of G is a complete graph. A complete graph is a Hushimi tree of one block. Let a be a cut-vertex of a Hushimi tree G. If G is divided into m connected components when the cut-vertex a is deleted, in the present paper, we say that *the Hushimi degree(simply H-degree) of the cut-vertex a is m* . If a vertex a of G is not a cut-vertex, its H-degree is defined to be 1. A connected subgraph of a Hushimi tree G is called a *sub-Hushimi tree* if it consists of some blocks of G. A block of Hushimi tree is said to be *pendant* if it has only one cut-vertex. It is evident that any Hushimi tree has at least two pendant blocks.

Definition. In this paper, a Hushimi tree is said to be *simple* if the H-degree of any its cut-vertex is 2. A Hushimi tree is said to be *semi-simple* if its each block has at most two cut-vertices whose H-degree are greater than 2. A signed Hushimi tree is called a Hushimi tree *with positive sign* (or simply a *positive* Hushimi tree) if we can switch all signs of edges into $+1$. A tree with only two leaves is said to be *a line-tree.*

A tree is always considered as a Hushimi tree with positive sign. The lattice A_n is spanned by vectors $e_i - e_j$, $1 \le i \ne j \le n+1$, where $\{e_1, \cdots, e_{n+1}\}$ is the orthonormal base of the euclidean $(n+1)$-space R^{n+1}. There is the one-to -one correspondence between ordered root bases of A_n and connected signed graphs associated with A_n. A line-tree with n vertices may be considered as the Dynkin diagram $[A_n]$ of the lattice A_n.

Theorem 1. Any connected signed graph is a signed graph associated with A_n if and only if it is a positive simple Hushimi tree.

Proof. Let G be a signed graph corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$ of the lattice A_n. If we replace a_i by $-a_i$, then the sign of G is switched with respect to $\{a_i\}$. Hence there is no problem whether we take a_i or $-a_i$. There are no induced cycles in G whose length are more than 3. In fact, if $a_{i_1}, a_{i_2}, \cdots, a_{i_m}, (m > 3)$ make an induced cycle, then we can assume that $a_{i_1} = e_{j_1} - e_{j_2}, a_{i_2} = e_{j_2} - e_{j_3}, a_{i_3} = e_{j_3} - e_{j_4}, \cdots, a_{i_m} = e_{j_m} - e_{j_1}$. But this implies that $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$ are not linearly independent. If $a_{i_1}, a_{i_2}, a_{i_3}$ make an induced cycle, then we can assume that $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}$, $a_{i_3} = e_j - e_{j_3}$. We have induced cycles of this type only in G. Now take a block B of G consisting of vertices $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$. Two vertices a_{i_1} and a_{i_2} must be on an induced cycle in B. We may assume that $a_{i_1}, a_{i_2}, a_{i_3}$ make an induced cycle. Then we can put $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}, a_{i_3} = e_j - e_{j_3}$. Two vertices a_{i_1} and a_{i_4} are also on an induced cycle in B, which may consist of $a_{i_1}, a_{i_4}, a_{i_5}$. Then we can put $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$ or $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$, where $j_4 \neq j$. Assume that $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$. Two vertices a_{i_2} and a_{i_4} are also on an induced cycle in B. Then we have $j_4 = j_2$, a contradiction. Hence, we get $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$. By this way, we get $a_{i_k} = e_j - e_{j_k}$, $1 \leq k \leq m$. Hence any block of G is a complete graph whose edges have sign $+ 1$. Suppose that the vertex $a_{i_1} = e_j - e_{j_1}$ of a block B is a cut-vertex. If two vertex a_k, a_ℓ which are not in B are adjacent with a_{i_1}, then we can put $a_k = e_{j_1} - e_{k_1}$, $a_\ell = e_{j_1} - e_{\ell_1}$. Hence a_{i_1}, a_k, a_ℓ are contained in another block of G. Hence we show that $G - a_{i_1}$ has two connected components. Thus G is a Hushimi tree with positive sign and the H-degree of any cut-vertex of G is 2.

Conversely, let G be a positive Hushimi tree whose any cut-vertex has the H-degree 2. Assume that G has m blocks. If $m = 1$, it is evident that G is a connected signed graph associated with A_n. Now suppose that the result is true for positive Hushimi trees with m blocks whose any cut-vertex has the H-degree 2. Let G be a positive Hushimi tree with $m + 1$ blocks. Let B be a pendant block of G and $a_1 = e_{i_1} - e_{i_2}$ be its cut-vertex. Let G' be the positive Hushimi tree which is made from G by deleting $B \setminus \{a_1\}$. Then G' is a connected signed graph associated with A_n and corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$, where A_n is spanned by vectors $e_i - e_j$, $1 \leq i \neq j \leq n + 1$. Now, all the ℓ vertices of B are adjacent with a vertex $a_1 = e_{i_1} - e_{i_2}$. We can assume that e_{i_2} is not used in any other a_j. Then, we can consider that the block B consists of $e_{i_2} - e_{n+2}, e_{i_2} - e_{n+3}, \cdots, e_{i_2} - e_{n+\ell+1}$ and a_1, where $\{e_1, \cdots, e_{n+1}, e_{n+2}, \cdots, e_{n+\ell+1}\}$ is the orthonormal base of the euclidean $(n + \ell + 1)$-space $R^{n+\ell+1}$. Hence we regard G as a connected signed graph associated with $A_{n+\ell}$.

3. Line-trees

Theorem 2 A complete graph with positive sign is equivalent to a line-tree by local switching.

We will prove a little stronger result as follows.

Lemma 3. Let G be a complete graph with positive sign. Take any two vertices a and b of G. Then it can be transformed to a line-tree, by a sequence of local switchings, without adopting local switchings at a and b. Conversely, any line-tree is transformed to a complete graph with positive sign, by a sequence of local switchings, without adopting local switchings at its two leaves.

Proof. Let G consist of vertices $a_1, a_2, \cdots, a_k$. We may assume that $a = a_1$ and $b = a_n$. Set $J = \{a_1\}$ and $K = \{a_3, a_4, \cdots, a_k\}$. By local switching with respect to (a_2, J), we obtain a positive Hushimi tree with two blocks $\{a_1, a_2\}$ and $\{a_2, \cdots, a_k\}$. Next, set $J = \{a_2\}$ and $K = \{a_4, \cdots, a_k\}$. By local switching with respect to (a_3, J), we obtain a positive Hushimi tree with three blocks $\{a_1, a_2\}$, $\{a_2, a_3\}$ and $\{a_3, \cdots, a_k\}$. By this way, we can get a line-tree , by a sequence of local switchings, without adopting local switchings at a_1 and a_n. The converse is obtained by the reverse sequence of local switchings.

We show

Theorem 4. Let G be a positive simple Hushimi tree. Then G is equivalent to a line-tree by local switching. Conversely, a line-tree is transformed to a positive simple Hushimi tree, by any sequence of local switchings.

Firstly, we prepare two lemmas for proving the above theorem.

Lemma 5. Let G be a positive Hushimi tree consisting of two blocks, B_1 and B_2. Then, it can be transformed to a positive complete graph, by local switching.

Proof. We can set $B_1 = \{a_1, a_2, \cdots, a_m\}$ and $B_2 = \{a_1, b_1, \cdots, b_k\}$. Then the vertex a_1 is the cut-vertex. Put $J = \{a_2, a_3, \cdots, a_m\}$ and $K = \{b_1, b_2, \cdots, b_k\}$. By local switching with respect to (a_1, J), G is transformed to a complete graph.

Lemma 6. Let G be a positive Hushimi tree. If a is a vertex of G with H-degree 1(resp. 2), then, by local switching, from G, we get a positive Hushimi tree, in which the H-degree of a is 2(resp. 1) and the H-degrees of all the other vertices are not altered.

Proof. Take any vertex a of G. If the H-degree of the vertex a is 2, then there are two blocks B_1 and B_2 which contain a. By local switching at the vertex a, we join B_1 and B_2 and get a positive Hushimi tree where the H-degree of the vertex a is 1 and the H-degrees of all the other vertices are not altered. If the H-degree of a is 1, then there is a block B which contains a. By local switching at a, B is divided into two blocks B_1 and B_2 which contain a. The vertex a has H-degree 2 as a vertex of the new positive Hushimi tree. The H-degrees of all the other vertices are not altered in this case either.

Proof of Theorem 4. If G has only one block, we get the result by Theorem 2. Suppose the result is true for any positive Hushimi tree with m blocks which satisfies the assumption. Now, assume that G has $m+1$ blocks. Take a pendant block B_1 of G with cut-vertex b. Let B_2 be the other block with cut-vertex b. Put $i = b, J = B_1 \backslash b, K = B_2 \backslash b$. By local switching with respect to (b, J), we obtain a positive Hushimi tree with m blocks, which can be transformed to a positive complete graph, by a sequence of local switchings.

It follows from lemma 6 that a positive Hushimi tree whose any cut-vertex has H-degree 2 is transformed to a positive Hushimi tree whose any cut-vertex has H-degree 2, by any local switching. As a line-tree is a positive Hushimai tree whose any cut-vertex has H-degree 2, we get easily that a line-tree is transformed to a positive Hushimi tree whose any cut-vertex has H-degree 2, by any sequence of local switchings.

3. Trees

We show the following results in this section.

Theorem 7. Let G be a positive semi-simple Hushimi tree. Then, G is equivalent to a tree by local switching. Conversely, if a tree is transformed to a positive Hushimi tree G by a sequence of local switchings, then, G is a positive semi-simple Hushimi tree.

Let T be a tree with vertices $\{a_1, \cdots, a_n\}$. Let $\alpha = (a_{i_1}, \cdots, a_{i_n})$ be a permutation of $\{a_1, \cdots, a_n\}$. For each $j, 1 \leq j \leq n$, by replacing a_j with a_{i_j} we get a new tree T' from T. We call T' a *permutation* of T. It is evident that T' is isomorphic to T.

Theorem 8. A tree T_1 is equivalent to a tree T_2 by local switching if and only if T_2 is a permutation of T_1.

From lemma 6, the following is evident.

Lemma 9. Let G be a positive Hushimi tree. Then, it can be transformed to a positive Hushimi tree which has no cut-vertex with H-degree 2, by a sequence of local switchings.

Lemma 10. Let G be a positive Hushimi tree which consists of k-blocks $B_1, \cdots, B_k$,$(k \geq 3)$ and has a unique cut-vertex v contained in all blocks. Hence, the H-degree of v is k. Take any vertex a in one block and b in another block which are not the cut-vertex v. By any sequence of local switchings, we can not construct a complete block containing both a and b. Hence, any positive Hushimi tree which is equivalent to G by local switching and has no cut-vertices with H-degree 2 is isomorphic to G.

Proof. Firstly, let $k = 3$. We may assume $a \in B_1 \backslash \{v\}$ and $b \in B_2 \backslash \{v\}$. Take any vertex $c \in B_3 \backslash \{v\}$.

Case 1. To construct a block containing a and b, from G, we get a signed graph G_1 by local switching with respect to (v, $J = B_1 \backslash \{v\}$). Then, there are the edges ac, ab, but there is no edge bc. To get a complete block containing a and b, we need to join b to c or delete the edge ac (or ab) by local switching. Firstly, we want to join b to c. Take any vertex $a' \in B_1$. By local switching with respect to $(a', J = B_2 \cup B_1 \backslash \{a'\})$, we get a singed graph where there is the edge bc but there is no edge ac if $a' \neq a$ or there is no edge vc if $a' = a$. In fact, in this case, each vertex in $B_2 \backslash \{v\}$ is jointed to each vertex in $B_3 \backslash \{v\}$ but all the edges between $B_3 \backslash \{v\}$ and $B_1 \backslash \{a'\}$ are deleted. Hence this signed graph is similar to G_1. Thus, we can not get a complete block containing a and b. Take any vertex $a' \in B_1 \backslash \{a\}$. Next, we delete the edge ac by local switching with respect to $(a', J = B_1 \cup B_2 \backslash \{a'\})$. Then, we get the edge bc. In the signed graph obtained, any two vertices in $B_1 \cup B_2$ are jointed and each vertex in $B_2 \backslash \{v\}$ is jointed to each vertex in $B_3 \backslash \{v\}$, but all the edges between $B_1 \backslash \{a'\}$ and $B_3 \backslash \{v\}$ are deleted. This signed graph is also similar to G_1.

Case 2. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, , a \in B_{11}, b \in B_2, c \in B_3$. By local switching with respect to $(v, J = B_{11} \backslash \{v\}))$, we obtained a signed graph G_2. By the same argument as in Case 1, we can show that there is no complete block containing a, b.

Case 3. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, B_2 = B_{21} \cup B_{22}, B_{21} \cap B_{22} = \emptyset, B_3 = B_{31} \cup B_{32}, B_{31} \cap B_{32} = \emptyset, a \in B_{11}, b \in B_{21}, c \in B_{31}$. By local switching with respect to $(v, J = B_{11} \cup B_{22} \cup B_{32} \backslash \{v\})$, we obtained a signed graph G_3. Then, G_3 has the edges ab, ac, but has no edge bc. By similar discussion about B_{11}, B_{21}, B_{31} as in Case 1, we can not get the three edges ab, ac, bc at the same time. In G_3, each vertex in $B_{21} \backslash \{v\}$ is jointed to each vertex in $G_{32} \backslash \{v\}$ and each vertex in $B_{31} \backslash \{v\}$ is jointed to each vertex in $G_{22} \backslash \{v\}$. Even if we ignore these facts, we can not construct a complete block containing B_{11}, B_{21}, B_{31} by the same reason as in Case 1.

Assume $k = 4$.

Case 4. Let $a \in B_1, b \in B_2, c \in B_3$. Set $B'_3 = B_3 \cup B_4$. By local switching with respect to $(v, J = B_1 \backslash \{v\}))$, we get a signed graph G_4. Then, G_4 has the edges ab, ac , but has no edge bc. Even if B'_3 was a complete block, we could not construct a complete block containing B_1, B_2, B'_3.

Case 5. Let $a \in B_1, b \in B_2, c \in B_3, d \in B_4$. By local switching with respect to $(v, J = B_1 \cup B_4 \backslash \{v\}))$, we get a signed graph G_5. Then. G_5 has the edges ab, ac, db, dc, but has no edges bc, ad. We show as similarly as in Case 1 that we can not construct a complete block containing a, b by deleting the edge bc. By local switching at some vertex, for example d, we will try to join b and c. By local switching with respect to $(d, J = B_3 \cup B_4 \backslash \{v, d\})$, we get a signed graph. Then, each vertex in $B_2 \backslash \{v\}$ is jointed to each vertex in $B_3 \backslash \{v\}$. But, all the edges jointing v and vertices in $B_3 \backslash \{v\}$ are deleted, and if $B_4 \backslash \{v, d\})$ is not empty, all the edges between $B_2 \backslash \{v\}$ and $B_4 \backslash \{v, d\})$ are deleted. The block containing B_1, B_2, B_3 must contain B_4. But, B_4, B_2, B_3 can not make a complete block as we can show by the same argument for B_1, B_2, B_3 in Case 1.

Case 6. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, B_2 = B_{21} \cup B_{22}$, $B_{21} \cap B_{22} = \emptyset$, $B_3 = B_{31} \cup B_{32}, B_{31} \cap B_{32} = \emptyset, B_4 = B_{41} \cup B_{42}, B_{41} \cap B_{42} = \emptyset, a \in B_{11}, b \in B_{21}, c \in B_{31}, d \in B_{41}$. By local switching with respect to $(v, J = B_{11} \cup B_{41} \cup B_{22} \cup B_{32} \backslash \{v\}))$, we obtained a signed graph G_6. Then. G_6 has the edges ab, ac, db, dc, but has no edges bc, ad. By the same argument as in the case 5, even if we ignore $B_{12}, B_{22}, B_{32}, B_{42}$, we can not construct a complete block containing a, b, c, d.

Assume $k \geq 5$.

Case 7. Let $a \in B_1, b \in B_2, c \in B_3$,. Set $B'_3 = B_3 \cup B_4 \cup \cdots \cup B_k$. By local switching with respect to $(v, J = B_1 \backslash \{v\}))$, we get a signed graph G_7. Then, G_7 has the edges ab, ac , but has no edge bc. Even if B'_3 was a complete block, we could not construct a complete block containing B_1, B_2, B'_3.

Case 8. Let $a \in B_1, b \in B_2, c \in B_3, d \in B_\ell$, $(\ell \leq k)$. Set $B'_4 = B_\ell \cup B_{\ell+1} \cup \cdots \cup B_k$ and $B'_3 = B_3 \cup \cdots \cup B_{\ell-1}$. By local switching with respect to $(v, J = B_1 \cup B'_4 \backslash \{v\})$, we get a signed graph G_8. Then. G_8 has the edges ab, ac, db, dc, but has no edges bc, ad. Even if B'_3 and B'_4 were complete blocks, we could not construct a complete block containing a, b, c, d by the same argument in Case 5.

When we apply some local switching, it rather prevents from making a complete block to divide given blocks B_i's . Hence, in any cases, we can not construct a complete block containing vertices a, b.

Proof of Theorem 7. By lemma 9, we may assume that G has no cut-vertices with H-degree 2. Select an arbitrary vertex in each pendant block which is not a cut-vertex. We will show that G can be transformed to a tree, by a sequence of local switchings, without adopting local switchings at the selected vertices. Assume that G has m blocks. If $m = 1$, the result follows from Lemma 3. Now suppose that the result is true for Hushimi trees with m blocks which satisfy the assumption. Let G have $m + 1$ blocks. Take any pendant block B_1 with a cut-vertex b. Let $B_2, \cdots, B_k$ be all the other blocks of G which contain the vertex b. We get k sub-Hushimi trees $G_i (i = 1, \cdots k)$ of G, where each G_i contains B_i. Select b and an arbitrary vertex in each pendant block of G_i which is not a cut-vertex. Then, each G_i can be transformed to a tree, by a sequence of local switchings, without adopting local switchings at the selected vertices. Hence, we show the result for the Hushimi tree G.

Now, take a tree T. Then, it is a positive Hushimi tree and its each block has at most two cut-vertices whose H-degree are greater than 2. By lemma 6, by some sequence of local switchings at vertices with H-degree 2, we obtain from T the positive Hushimi tree G_1 which has no cut-vertices with H-degree 2. Take a cut-vertex v of G_1 whose H-degree is k,$(k \geq 3)$. Let G_2 be a signed graph obtained from G_1 by local switching at v. It is evident that G_2 is not a Hushimi tree. It follows from lemma 10 that by any sequence of local switchings, from G_2, we can not get a positive Hushimi tree which has no cut-vertices with H-degree 2 and is not isomorphic to G_1. Thus, we obtain the desired result.

We need the following lemma to prove theorem 8.

Lemma 11. Assume that a tree T has a vertex {v} with degree k and just k leaves. Let a_1 be one of the leaves and $a_1 a_2 \cdots a_\ell v$ be the path between a_1 and v. Take any vertex a_i, $1 \leq i \leq \ell$. Then, by a sequence of local switchings, from T, we get a new tree T' where v and a_i are interchanged and all the other vertices are not altered.

Proof. By a sequence of local switchings, from T, we get a positive Hushimi tree G_1 with k blocks. This Hushimi tree has the unique cut-vertex v with H-degree k. Let B_1 be a complete block with vertices $a_1, a_2, \cdots, a_\ell, v$. By local switching with respect to $(v, J = B_1 \backslash \{v\})$, from G_1 we get a signed graph G_2. By local switching with respect to $(a_i, J = B_1 \backslash \{a_i\})$, we get a positive Hushimi tree G_3 with k blocks. This G_3 has the unique cut-vertex a_i. By a sequence of local switchings, from G_3 we get the desired tree T'.

Lemma 12. Let T_1 and T_2 be line-trees of order n. Then, T_1 is equivalent to T_2 by local switching if and only if T_2 is a permutation of T_1.

Proof. Let T_1 be a line-tree $a_1 a_2 \cdots a_n$ and T_2 be its permutation $a_{i_1} a_{i_2} \cdots a_{i_n}$. Then, T_1 and T_2 are equivalent by local switching to the complete graph with vertices $\{a_1, a_2, \cdots, a_n\}$. Hence, they are equivalent by local switching. Conversely, if a line-tree T_1 is equivalent to a line-tree T_2 by local switching, it is evident that T_2 is a permutation of T_1.

Proof of Theorem 8. Let T_2 be a permutation of T_1. Then using lemmas 11 and 12, we can construct a sequence of local switchings by which T_1 is transformed to T_2. On the other hand, when a tree is transformed to another tree by local switchings, by taking account of lemma 9, we can use local switchings such that are treated in lemmas 11 and 12. Hence we only interchange vertices of the tree.

4. The lattice D_n and signed cycles

A $k-$cycle $C^k = (V, E)$, where $V = \{a_1, a_2, \cdots, a_k\}$, $E = \{a_1a_2, a_2a_3, \cdots, a_{k-1}a_k, a_ka_1\}$, will be denoted simply $C^k = a_1a_2\cdots a_ka_1$. For signed cycles, there are two switching classes, which are distinguished by the parity or the balance, where the parity of a signed cycle is the parity of the number of its edges which carry a positive sign and the balance is the product of the signs on its edges [?].

The lattice D_n is spanned by vectors $\pm e_i \pm e_j$, $(1 \leq i \neq j \leq n)$, where $\{e_1, \cdots, e_n\}$ is the orthonormal base of the euclidean n-space R^n. There is the one-to -one correspondence between ordered root bases of D_n and connected signed graphs associated with D_n.

Theorem 13. Let C^k be a $k-$cycle. Then, it is equivalent to a tree by local switching if and only if its parity is odd.

Proof. Let the parity be odd. If the parity of k is odd, then by switching, we may assume that signs of all edges are positive. Put $C^k = a_1a_2\cdots a_ka_1$. By a sequence of local switchings with respect to $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, $\cdots$, $(a_{k-1}, J = \{a_k\})$, we get a signed graph G, which is the graph obtained from the positive complete graph on vertices $\{a_1, a_2, \cdots, a_k\}$ by deleting the edge a_1a_k. By a sequence of local switchings with respect to $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\})$, $\cdots$, $(a_{k-1}, J = \{a_{k-2}\})$, from the graph G, we get a tree with edge set $E = \{a_2a_3, a_3a_4, \cdots, a_{k-2}a_{k-1}, a_{k-1}a_k, a_{k-1}a_1\}$, which may be regarded as the Dynkin diagram of D_k.

When the parity of k is even, we get a tree as similarly as above.

Now assume that the parity of C^k is even. For a cycle $a_1a_2a_3a_1$, we may assume that only the edge a_1a_2 has negative sign. Then we can not transform it to a tree by local switching. Next, every edge of a cycle $a_1a_2a_3a_4a_1$ has positive sign. We must transform it by local switching, for example, with respect to $(a_2, \{a_1\})$. Then, we have a signed graph with $E^+ = \{a_1a_2, a_1a_3, a_2a_3, a_4a_1\}$, $E^- = \{a_3a_4\}$. This graph can not be transformed to a tree by local switching. Now suppose that any $k-1$-cycle with even parity can not be transformed to a tree by a sequence of local switching. Take a k-cycle $a_1a_2\cdots c_kc_1$ with even parity. We must do some local switching, for example, with respect to $(a_1, J = \{a_k\})$. We get a signed graph and its induced cycle $a_2a_3a_4\cdots a_ka_2$ with even parity. Any local switching of the signed graph at some $a_j, 2 \leq j \leq k$, has the same effect on the induced cycle $a_2a_3a_4\cdots a_ka_2$ as local switching at a_j of the cycle $a_2a_3a_4\cdots a_ka_2$. As the cycle $a_2a_3a_4\cdots a_ka_2$ can not be transformed to a tree, the induced cycle $a_2a_3a_4\cdots a_ka_2$ and hence the k-cycle $a_1a_2\cdots c_kc_1$ can not be transformed to a tree by a sequence of local switchings.

We denote by $[D_k]$ the tree which is isomorphic to the Dynkin diagram of D_k, and by $K_k - e$ the graph obtained from the positive complete graph on k vertices by deleting one edge. In the above proof, we proved already

Theorem 14. Let C^k be a $k-$cycle with odd parity. Then, C_k, $[D_k]$ and $K_k - e$ are equivalent by local switching.

Theorem 15. Any signed graph associated to the lattice D_n is equivalent to the tree $[D_n]$ by local switching.

Proof. Let G be a signed graph corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$ of the lattice D_n. If we replace a_i by $-a_i$, then the sign of G is switched with respect to $\{a_i\}$. Hence there is no problem whether we take a_i or $-a_i$. If $a_i = e_j - e_\ell$ (resp. $a_i = e_j + e_\ell$) is contained in the ordered base, $e_j + e_\ell$ (resp. $e_j - e_\ell$) is not contained in it except one pair which we denote by $a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k$, $(1 \leq k \leq n$). It leaves the switching class of G invariant to replace $a_i = e_j - e_\ell$ (resp. $a_i = e_j + e_\ell$) by $e_j + e_\ell$ (resp. $e_j - e_\ell$). Hence, we always take $a_i = e_j - e_\ell, (j < \ell)$, if either of $e_j - e_\ell$ or $e_j + e_\ell$ is contained in the ordered base.

If G is a graph corresponding to the base $\{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \cdots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n\}$, G is just the tree $[D_n]$.

Assume that G is a graph corresponding to the base $\{a_1 = e_1 - e_2,\ a_2 = e_2 - e_3,\ \cdots, a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k, a_{k+1} = e_k - e_{k+1}, \cdots,\ a_n = e_{n-1} - e_n\}$, $(2 < k < n)$. Then it is a signed graph with edge sets $E^+ = \{a_1a_2, a_2a_3, a_3a_4, \cdots, a_{k-2}a_{k-1}, a_{k-2}a_k, a_{k-1}a_{k+1}, a_{k+1}a_{k+2}, \cdots, a_{n-1}a_n\}$ and $E^- = \{a_ka_{k+1}\}$. By a sequence of local switchings, from G, we get a signed graph G_1 with three blocks B_1, B_2 and B_3, where B_1 and B_3 are the positive complete graphs on vertices $\{a_1, \cdots, a_{k-2}\}$ and vertices $\{a_{k+1}, \cdots, a_n\}$, and B_2 is a 4-cycle $a_{k-2}a_{k-1}a_{k+1}a_ka_{k-2}$ with odd parity. By local switchings with respect to $(a_{k-2}, J = \{a_{k-1}, a_k\})$, $(a_{k+1}, J = \{a_{k-1}, a_k\})$ and $(a_k, J = B_1)$, from G_1, we get a signed graph which is isomorphic to $K_n - e$.

In general, let G be a signed graph corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$, where we may assume that $a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k$ is the particular pair. By a similar argument as in the proof of theorem 1, we can show that G consists of ℓ blocks $B_1, B_2, \cdots, B_\ell$ such that $B_1, B_2, \cdots, B_{\ell-1}$ are complete blocks and B_ℓ is given by $\{a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k, a_{i_1} = e_{k-1} - e_{j_1}, \cdots, a_{i_s} = e_{k-1} - e_{j_s}, a_{u_1} = e_k - e_{v_1}, \cdots, a_{u_t} = e_k - e_{v_t}\}$, where all $e_{k-1}, e_k, e_{j_1}, \cdots, e_{j_s}, e_{v_1}, \cdots, e_{v_t}$ are different. For any cut-vertex a of G, we can show as similarly as in the proof of theorem 1 that $G - a$ has two connected components. By a sequence of local switchings at all cut-vertices, from G, we get a signed graph G_1. If it is necessary, by rearrangement of vertices, G_1 can be expressed as follows. The subgraphs of G_1 on vertices $\{a_1, \cdots, a_{k-2}\}$ and on vertices $\{a_{k+1},\ \cdots, a_n\}$

are complete. Moreover G_1 has the edges $\{a_1a_{k-1},\ a_2a_{k-1}, \cdots, a_{k-2}a_{k-1}, a_1a_k, a_2a_k, \cdots, a_{k-2}a_k, a_{k+1}a_{k-1}, a_{k+2}a_{k-1}, \cdots, a_na_{k-1}\}$ with sign $+1$ and the edges $\{a_{k+1}a_k, a_{k+2}a_k, \cdots, a_na_k\}$ with sign - 1. By local switching with respect to $(a_{k-1}, J = \{a_1, \cdots, a_{k-2}\})$, from G_1, we get $K_n - \{a_{k-1}a_k\}$.

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On Finite Simple Groups of Cube Order

Dedicated to Professor Toru Ishihara on his 65th birthday

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Abstract

In [17], M. Newman, D. Shanks and H. C. Williams have shown that the order of a symplectic group $S_p(2n, \mathbf{F}_q)$ is square if and only if $n = 2$ and $q = p$. Here p is a prime called a NSW prime. In this paper, we shall show that there is no symplectic group of cube order.

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Introduction and Preliminaries

In their paper [17], M. Newman, D. Shanks and H. C. Williams have shown that a symplectic group $S_p(2n, \mathbf{F}_q)$ has a square order if and only if $n = 2$ and $q = p$, where p is a NSW prime. The main result given in [17] is the following.

Proposition 1. *The order of a symplectic group $S_p(2n, q)$ is square if and only if $(n, q) = (2, S_{2m+1})$, where S_{2m+1} is a NSW prime .*

Now we shall recall the definition of NSW numbers in P. Ribenboim's book [18]. We define a sequence $\{S_{2m+1}\}$ by putting

$$S_{2m+1} = \frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}.$$

We call a prime NSW number S_{2m+1} to be a NSW prime. For example, $S_3 = 7, S_5 = 41$ and $S_7 = 239$ are the first three NSW primes. In [9], we have verified the conjecture announced in [17] is true. Namely, we have shown that the order of any finite simple group G is not square when $G \neq S_p(4, q)$. Thus it is a natural problem to ask the existence of finite simple groups of higher powers. In this paper, we shall consider the existence of finite simple group of cube order. For the sake of simplicity, we restrict ourselves to the special case $G = S_p(2n, q)$. We shall show the following main theorem.

Theorem. *There is no symplectic group $G = S_p(2n, q)$ of cube order.*

Firstly we shall prepare the preliminary lemmas which we will use in later.

Lemma 1 (Bertrand's postulate). *If n is an integer > 2, there exists an odd prime p such that*

$$n/2 < p \leq n.$$

Lemma 2 (Breusch [3]). *For $n \geq 7$, there exists a prime p of the form $6k + 1$ such that*

$$n/2 < p \leq n.$$

Lemma 3 (Shorey, Bugeaud and et al [1], [19]). *For any $n \geq 3$, the diophantine equation*

$$\frac{x^{2n} - 1}{x^2 - 1} = y^3$$

has no integer solution in integers $x > 1, y > 1$.

Lemma 4 (Ljunggren [11]). *If $n \equiv 1, 2, 4$ mod 6 and ≥ 4, then the diophantine equation*

$$\frac{x^n - 1}{x - 1} = y^3$$

has no integer solution in integers $|x| > 1, y > 1$.

We note that $\left\{ \frac{x^n - 1}{x - 1} \right\}$ is the Lucas sequence associated to the pair $(x + 1, x)$ and satisfies the following elementary relation on the greatest common divisor.

Lemma 5 (Ribeiboim [18]).

$$\left(\frac{x^m - 1}{x - 1}, \frac{x^n - 1}{x - 1} \right) = \frac{x^{(m,n)} - 1}{x - 1} .$$

Lemma 6 (Delaunay [4], [5]). *The diophantine equation*

$$x^3 + dy^3 = 1 \quad (d > 1)$$

has at most one integer solution with $xy \neq 0$. Moreover the solution (x, y) corresponds to the binomial fundamental unit $x + y\sqrt[3]{d}$ in the ring $\mathbf{Z}[\sqrt[3]{d}]$.

1. Proof of the main result

We know the order of the symplectic group is

$$|S_p(2n, q)| = \frac{q^{n^2}}{d} \prod_{i=1}^{n} (q^{2i} - 1),$$

where $d = (2, q - 1)$. Hence we can write

$$|S_p(2n, q)| = \frac{q^{n^2}}{d} (q^2 - 1)^n \prod_{i=1}^{n} \left(\frac{q^{2i} - 1}{q^2 - 1} \right).$$

We shall treat the case $3|n$ and $3 \not| \, n$ separately. In the following, we shall consider the easier case $3|n$.

Case 1) $3|n$.

We can write $n = 3m$. Then we have

$$|S_p(2n, q)| = |S_p(6m, q)| = (q^{3m}(q^2 - 1))^{3m} \frac{1}{d} \prod_{i=1}^{3m} \left(\frac{q^{2i} - 1}{q^2 - 1} \right).$$

Then we see $|S_p(6m, q)|$ is cube if and only if $\frac{1}{d} \prod_{i=1}^{3m} \left(\frac{q^{2i} - 1}{q^2 - 1} \right)$ is cube. From Lemma 1, we can take an odd prime p which satisfies $3m/2 < p \leq 3m$ for any positive integer m. Take the factor $\frac{q^{2p} - 1}{q^2 - 1}$ of $|S_p(6m, q)|$. Then we see

$$\frac{1}{d} \prod_{i=1}^{3m} \left(\frac{q^{2i} - 1}{q^2 - 1} \right) = \left(\frac{q^{2p} - 1}{q^2 - 1} \right) \cdot \frac{1}{d} \prod_{i=1(\neq p)}^{3m} \left(\frac{q^{2i} - 1}{q^2 - 1} \right).$$

Here we note that $\frac{q^{2p} - 1}{q^2 - 1} = q^{2(p-1)} + \cdots + q^2 + 1$ is always odd. Hence we see $\left(\frac{q^{2p} - 1}{q^2 - 1}, d \right) = 1.$

Moreover, from Lemma 5, we have

$$\left(\frac{q^{2p}-1}{q^2-1}, \frac{q^{2i}-1}{q^2-1}\right) = 1,$$

for any $1 \leq i\ (\neq p) \leq 3m$. Thus we see if $|S_p(6m,q)|$ is cube then $\frac{q^{2p}-1}{q^2-1}$ must be cube. From Lemma 3, we know there is no integer solution with $q, y > 1$ for $\frac{q^{2p}-1}{q^2-1} = y^3$. Hence we have shown that $|S_p(6m,q)|$ is never a cube for any positive integer m.

Case 2) $3 \nmid n$.

In the next, we shall treat the case $3 \nmid n$. In the case $n \geq 7$, we can take a prime p of the form $6k+1$ which satisfies $n/2 < p \leq n$ from Lemma 2. Take the factor $\frac{q^{2p}-1}{q^2-1}$ of $|S_p(2n,q)|$. Then we have

$$\begin{aligned}
\left(\frac{q^{2p}-1}{q^2-1}, \frac{q^{2i}-1}{q^2-1}\right) &= 1 \text{ for any } 1 \leq i\ (\neq p) \leq n,\\
\left(\frac{q^{2p}-1}{q^2-1}, d\right) &= 1,\\
\left(\frac{q^{2p}-1}{q^2-1}, q^2-1\right) &= 1 \text{ or } p.
\end{aligned}$$

Thus if $|S_p(2n,q)|$ is cube, then the factor $\frac{q^{2p}-1}{q^2-1}$ must satisfy $\frac{q^{2p}-1}{q^2-1} = y^3$ or py^3 or p^2y^3 for some positive integer y. We note here that

$$\frac{q^{2p}-1}{q^2-1} = \left(\frac{q^p-1}{q-1}\right)\left(\frac{q^p+1}{q+1}\right)$$

with $\left(\frac{q^p-1}{q-1}, \frac{q^p+1}{q+1}\right) = 1$. Thus we can conclude that the assumption $|S_p(2n,q)|$ is cube implies

$$\frac{q^p-1}{q-1} = y^3 \text{ or } \frac{q^p+1}{q+1} = \frac{(-q)^p-1}{(-q)-1} = y^3 \text{ for some positive integer } y,$$

which contradicts Lemma 4. Thus we have shown $|S_p(2n,q)|$ is never a cube for $n \geq 7$.

Finally, we shall verify $|S_p(2n,q)|$ is not cube for remaining cases $n = 1, 2, 4$ and 5.

In the case $n = 1$, we have

$$|S_p(2,q)| = q(q+1)\left(\frac{q-1}{d}\right) \quad \text{with } d = (2, q-1).$$

Here we see $(q, q+1) = 1$, $\left(q, \frac{q-1}{d}\right) = 1$, and $\left(\frac{q-1}{d}, q+1\right) = 1$ or 2. Therefore, if $|S_p(2,q)|$ is cube, then we must have $q = x^3$ for some integer $x > 1$. Also we must have $q + 1 = y^3$ or $2y^3$ or $4y^3$ for some integer $y > 1$.
If $q + 1 = y^3$, then it contradicts the classical fact $x^3 + y^3 \neq z^3$ for $xyz \neq 0$. If $q + 1 = 2y^3$, then from Lemma 6 the solution (x, y) corresponds to the fundamental unit $x + y\sqrt[3]{2}$ of $\mathbf{Z}[\sqrt[3]{2}]$. Since the fundamental unit ε of $\mathbf{Z}[\sqrt[3]{2}]$ with $0 < \varepsilon < 1$ is $\varepsilon = -1 + \sqrt[3]{2}$, we must have $x = y = 1$, which contradicts the condition $q = x^3 > 1$.
If $q + 1 = 4y^3$, then in the same way as above the solution (x, y) corresponds to the fundamental unit $x + y\sqrt[3]{4}$ of $\mathbf{Z}[\sqrt[3]{4}]$. Since the fundamental unit η of $\mathbf{Z}[\sqrt[3]{4}]$ with $0 < \eta < 1$ is $\eta = \varepsilon^2 = 1 + \sqrt[3]{4} - \sqrt[3]{16}$, we know there is no solution which satisfies $x^3 + 1 = 4y^3$. Thus we can conclude $|S_p(2,q)|$ is never a cube for any q.

In the case $n = 2$, we have

$$|S_p(4,q)| = q^4\left(\frac{q^2-1}{d}\right)^2 \cdot d \cdot (q^2+1) \quad \text{with } d = (2, q-1).$$

Here we see $\left(q, \frac{q^2-1}{d}\right) = 1$, $(q, q^2+1) = 1$, $(q, d) = 1$, $(q^2+1, d) = 1$ or 2, and $\left(q^2+1, \frac{q^2-1}{d}\right) = 1$ or 2. Therefore, if $|S_p(4,q)|$ is cube, then we must have $q = x^3$ for some integer $x > 1$. Also we must have $q^2 + 1 = y^3$ or $2y^3$ or $4y^3$ for some integer $y > 1$.
If $q^2 + 1 = (x^2)^3 + 1 = y^3$, then it contradicts the classical fact $x^3 + y^3 \neq z^3$ for $xyz \neq 0$. If $q^2 + 1 = (x^2)^3 + 1 = y^3$ or $q^2 + 1 = (x^2)^3 + 1 = 4y^3$, then in the same way as in the case $n = 1$, we can see there are no solutions when $x, y > 1$ from Lemma 6. Thus we can conclude $|S_p(4,q)|$ is never a cube for any q.

In the case $n = 4$, we have

$$|S_p(8,q)| = \frac{1}{d}q^{16}(q^2-1)^2(q^4-1)^2(q^4+q^2+1)(q^4+1) \quad \text{with } d = (2, q-1).$$

It is easy to see if $|S_p(8,q)|$ is cube, then $q = x^3$ with some integer $x > 1$. Moreover we see $(q^4+1, d) = 1$ or 2, $(q^4+1, q) = 1$, $(q^4+1, q^2-1) = 1$ or 2, $(q^4+1, q^4-1) = 1$ or 2, and $(q^4+1, q^4+q^2+1) = 1$. Therefore, if $|S_p(8,q)|$ is cube, then we must have $q^4 + 1 = (x^4)^3 + 1 = y^3$ or $2y^3$ or $4y^3$ for some integer

$y > 1$. In the same way as in the case $n = 1$, we can see there are no solutions for $x, y > 1$ from Lemma 6. Thus we can conclude $|S_p(8,q)|$ is never a cube for any q.

Finally we shall consider the case $n = 5$. Then we have

$$|S_p(10,q)| = \frac{1}{d}q^{25}(q^2-1)^3(q^4-1)^2(q^4+q^2+1)(q^4+1)\left(\frac{q^{10}-1}{q^2-1}\right),$$

with $d = (2, q-1)$. It is easy to see if $|S_p(10,q)|$ is cube, then $q = x^3$ with some integer $x > 1$. Moreover we see $(q^4+1, d) = 1$ or 2, $(q^4+1, q) = 1$, $(q^4+1, q^4-1) = 1$ or 2, $(q^4+1, q^4+q^2+1) = 1$ or 2, and $\left(q^4+1, \frac{q^{10}-1}{q^2-1}\right) = 1$. Therefore, if $|S_p(10,q)|$ is cube, then we must have $q^4+1 = (x^4)^3+1 = y^3$ or $2y^3$ or $4y^3$ for some integer $y > 1$. In the same way as in the above cases, we can see there are no solutions for $x, y > 1$ from Lemma 6. Thus we can conclude $|S_p(10,q)|$ is never a cube for any q, which completes the proof of our main theorem.

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Examples of the Iwasawa Invariants and the Higher K-groups Associated to Quadratic Fields

Dedicated to Professor Toru Ishihara on his 65th birthday

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Abstract

We compute the Iwasawa invariants of $\mathbf{Q}(\sqrt{f}, \zeta_p)$ in the range $|f| < 200$ and $5 \le p < 200000$ (resp. $|f| < 10$ and $5 \le p < 1000000$). These computational results give us concrete information on the higher K-groups of the ring of integers of $\mathbf{Q}(\sqrt{f})$.

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Introduction

Let F be a number field and $\mathcal{O}_F$ the ring of integers of F. Put $K = F(\zeta_p)$ and denote by K_∞ the cyclotomic $\mathbf{Z}_p$-extension of K. Let L_∞ be the maximal unramified abelian p-extension of K_∞ and L'_∞ the maximal unramified abelian p-extension of K_∞ in which every prime divisor lying above p splits completely. Put $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $X'_\infty = \mathrm{Gal}(L'_\infty/K_\infty)$.

It is known that there are relations between Iwasawa modules X'_∞ and Quillen's K-groups $K_n(\mathcal{O}_F)$. The main purpose of this paper is to give concrete information on the Iwasawa invariants of X_∞ and the higher K-groups $K_n(\mathcal{O}_F)$ for quadratic fields F by using these relations.

Following [9, 10], we compute Iwasawa invariants and found some exceptional pairs. Using these pairs, we give exceptional examples of $K_n(\mathcal{O}_F)$. For example, we find that for $5 \le p < 1000000$, p divides the order of $K_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})})$ if and only if $p = 7$ or 157229 under the Quillen-Lichtenbaum conjecture.

1 Iwasawa invariants of $\mathbf{Q}(\sqrt{f_\chi}, \zeta_p)$

Let χ be a quadratic Dirichlet character and p an odd prime number. Assume that $\chi \neq \omega^{\frac{p-1}{2}}$, where $\omega = \omega_p$ is the Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$ such that $\omega(a) \equiv a \bmod p$. Put $F = F_\chi = \mathbf{Q}(\sqrt{f_\chi})$ and $K_n = \mathbf{Q}(\sqrt{f_\chi}, \zeta_{p^{n+1}})$. Let A_n (resp. A'_n) the p-part of the ideal class group (resp. p-ideal class group) of K_n.

Put $G_\infty = \mathrm{Gal}(K_\infty/F)$, $\Delta = \mathrm{Gal}(K_\infty/F_\infty)$ and $\Gamma = \mathrm{Gal}(K_\infty/K)$. Further put $\Delta' = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$ and $e'_\psi = \frac{1}{\sharp\Delta'}\sum_{\delta\in\Delta'}\psi(\delta)\delta^{-1}$ for a Dirichlet character ψ of Δ'. For a $\mathbf{Z}_p[\Delta']$-module A, we denote $e'_\psi A$ by A^ψ. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ (resp. $\lambda'_p(\psi)$, $\mu'_p(\psi)$ and $\nu'_p(\psi)$) be the Iwasawa invariants associated to X^ψ_∞ (resp. X'^ψ_∞), i.e.,

$$\sharp A^\psi_n = p^{\lambda_p(\psi)n+\mu_p(\psi)p^n+\nu_p(\psi)} \quad (\text{resp. } \sharp A'^\psi_n = p^{\lambda'_p(\psi)n+\mu'_p(\psi)p^n+\nu'_p(\psi)})$$

for sufficiently large n. By Ferrero-Washington's theorem, we have $\mu_p(\psi) = \mu'_p(\psi) = 0$ for all p and ψ.

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p-adic L-function is defined as follows. Let $L_p(s,\psi)$ be the p-adic L-function constructed by [6]. Let f_0 be the least common multiple of f_ψ and p. By [3, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying

$$G_\psi((1+f_0)^{1-s}-1) = L_p(s,\psi)$$

for all $s \in \mathbf{Z}_p$ if $\psi \neq \chi^0$. By [2], it was proved that p does not divide $G_\psi(T)$. Therefore, by the p-adic Weierstrass preparation theorem, we can uniquely write

$$G_\psi(T) = g_\psi(T)u_\psi(T),$$

where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Put $\tilde{\lambda}_p(\psi) = \deg g_\psi(T)$.

For a pair (p,ψ), we assume the following condition

$$\text{(C)} \qquad \psi(p) \neq 1 \text{ and } \psi^{-1}\omega(p) \neq 1.$$

If $\psi(p) \neq 1$, we have $\lambda_p(\psi) = \lambda'_p(\psi)$ and $\nu_p(\psi) = \nu'_p(\psi)$.

We extend the tables of [9, 10] to all primes below 200000.

Proposition 1 *For* $|f| < 200$ *and* $100000 < p < 200000$*, all exceptional pairs* $(p, \chi\omega^k)$ *are given in the following table. The meaning of the symbols are as follows:* $[\nu] : \nu(\chi\omega^k) > 0$, $[a_0] : v_p(a_0) > 1$, $[b_0] : v_p(b_0) > 1$, $[\mathrm{lmd}] : \tilde{\lambda}(\chi\omega^k) > 1$, *where* $a_0 = L_p(1,\chi\omega^k)$ *and* $b_0 = L_p(0,\chi\omega^k)$.

Exceptional pairs $(p, \chi\omega^k)$ for $100000 < p < 200000$

f	p	k	f	p	k	f	p	k
	$[\nu]$			$[a_0]$			[lmd]	
8	157229	140434	57	119627	53592	133	185189	119132
28	109829	45474	77	175843	43682	168	104971	21988
56	100937	93200	141	120823	39250	-187	166823	150305
-79	153059	68171	157	150401	101272			
-91	107449	81489		$[b_0]$				
-104	184157	53783	-19	165667	11685			
-119	112241	37701	53	167593	99386			
-120	126691	28093	-71	177473	58993			
149	109211	11960	137	124493	41762			
-183	104803	58845	-152	104399	90165			

Proposition 2 *For* $|f| < 10$, *i.e.,* $f = -3, 5, -4, -7, 8$ *or* -8 *and* $200000 < p < 1000000$, *there is only one exceptional pair* $(399181, \chi_{-4}\omega^{1683})$, *which satisfies* $\tilde{\lambda}(\chi_{-4}\omega^{1683}) > 1$.

In Figures 1-2, we compare the actual number of exceptional pairs with the expected number E in the range $200 < p < 200000$.

FIGURE 1. Exceptional pairs (quadratic, 1<|f|<200, 200<p<200000)

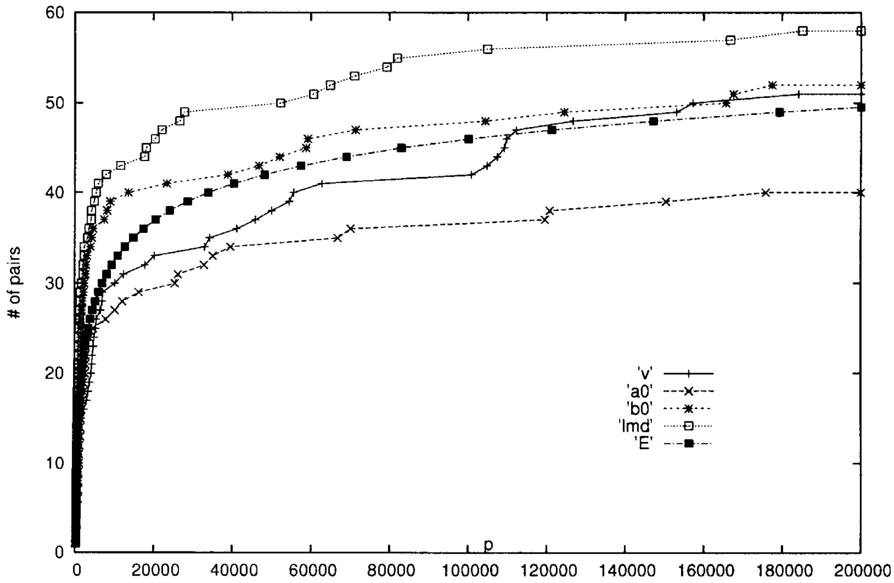
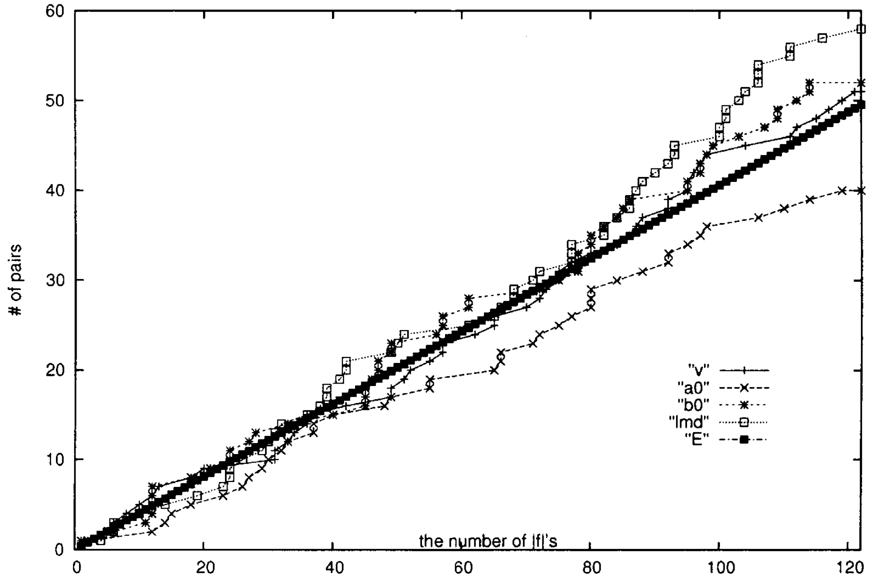


FIGURE 2. Exceptional pairs (Quadratic, 1<|f|<200, 200<p<200000)



From our data, the actual numbers still seem to be near to the expected numbers.

2 Higher K-groups of $\mathcal{O}_F$

We recall some results on Quillen's K-groups.

Theorem 1 (*Quillen*) *For all* $n \geq 0$, $K_n(\mathcal{O}_F)$ *is a finitely generated* $\mathbf{Z}$*-module.*

Theorem 2 (*Borel*) *For* $m \geq 1$,

$$\mathrm{rank}_{\mathbf{Z}}(K_{2m-1}(\mathcal{O}_F)) = \begin{cases} r_1(F) + r_2(F) & \text{if} \quad m \text{ is odd,} \\ r_2(F) & \text{if} \quad m \text{ is even,} \end{cases}$$

where $r_1(F)$ *is the number of real embeddings of* F, *and* $r_2(F)$ *is the number of pairs of complex embeddings of* F. *Further,*

$$K_{2m-2}(\mathcal{O}_F) \text{ is finite.}$$

Conjecture 1 (*The Quillen-Lichtenbaum conjecture*) *The natural map* (*via p-adic Chern characters*)

$$K_{2m-i}(\mathcal{O}_F) \otimes \mathbf{Z}_p \to H^i_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), \mathbf{Z}_p(m))$$

is an isomorphism for all $m \geq 2$, $i = 1, 2$ *and any odd prime number* p, *where* $A(m)$ *is the* m*-th Tate twist of a Galois module* A.

The surjectivity of p-adic Chern characters was proved by [1, 4, 7, 8]. We simply denote $H^i_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), A)$ by $H^i(\mathcal{O}_F, A)$.

Theorem 3 ([5, §3, §4]) *For $m \neq 0$, we have*

$$H^1(\mathcal{O}_F, \mathbf{Z}_p(m))_{tors} \simeq H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(m)).$$

For $m \neq 1$, we have an exact sequence

$$\begin{aligned} 0 \to X'_\infty(m-1)_{G_\infty} &\to H^2(\mathcal{O}_F, \mathbf{Z}_p(m)) \\ &\to \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m)) \to H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\vee \to 0, \end{aligned}$$

where $A^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(A, \mathbf{Q}_p/\mathbf{Z}_p)$.

From now on, we use the same notation as in the previous sections. For an even character $\chi\omega^{1-m}$, we write the Iwasawa polynomial $g_{\chi\omega^{1-m}}(T)$ for the p-adic L-function $L_p(s, \chi\omega^{1-m})$ in the form

$$g_{\chi\omega^{1-m}}(T) = \prod_{i=1}^{\tilde{\lambda}(\chi\omega^{1-m})} (T - \alpha_{\chi\omega^{1-m},i}), \quad \alpha_{\chi\omega^{1-m},i} \in \overline{\mathbf{Q}}_p.$$

We put

$$x(p, \chi, m-1) = \min\{\nu_p(\chi\omega^{1-m}),\ v_p(\prod_{i=1}^{\tilde{\lambda}} (1 - (1+f_0)^{m-1}(\alpha_{\chi\omega^{1-m},i} + 1)))\}.$$

For an odd character $\chi\omega^{1-m}$, we put $\alpha^*_{\chi\omega^m,i} = \frac{f_0 - \alpha_{\chi\omega^m,i}}{1+\alpha_{\chi\omega^m,i}}$,

$$g^*_{\chi\omega^m}(T) = \prod_{i=1}^{\tilde{\lambda}(\chi\omega^m)} (T - \alpha^*_{\chi\omega^m,i})$$

and

$$x^*(p, \chi, m-1) = v_p(\prod_{i=1}^{\tilde{\lambda}(\chi\omega^m)} (1 - (1+f_0)^{m-1}(\alpha^*_{\chi\omega^m,i} + 1))).$$

Further, for an integer m, we define the following sets of prime numbers

$$\begin{aligned} S_1(\chi, m-1) &= \{p : p'|(m-1), (p-1) \nmid (m-1), \chi\omega^{p'}(p) = 1, \chi\omega^{p'} \neq \chi^0\}, \\ S_2(\chi, m-1) &= \{p : (p-1)|(m-1),\ \chi(p) = 1\}, \end{aligned}$$

where $p' = \frac{p-1}{2}$. We put

$$y(p, \chi, m-1) = \begin{cases} v_p(m-1) + 1 & \text{if } p \in S_1(\chi, m-1) \cup S_2(\chi, m-1), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3 *Let χ be a quadratic Dirichlet character, p an odd prime number and $F = F_\chi$. For an odd character $\chi\omega^{1-m}$, if $(p, \chi\omega^m)$ satisfies* (C), *then*

$$\sharp X'_\infty(m-1)^\chi_{G_\infty} = p^{x^*(p,\chi,m-1)}.$$

For an even character $\chi\omega^{1-m}$, assume that $X'^{\chi\omega^{1-m}}_\infty$ is finite. If $(p, \chi\omega^{1-m})$ satisfies (C) *and if $g_{\chi\omega^{1-m}}(T)$ is an Eisenstein polynomial or of degree one, then*

$$\sharp X'_\infty(m-1)^\chi_{G_\infty} = p^{x(p,\chi,m-1)}.$$

Further, for an integer m, we have

$$\frac{\sharp \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m))^\chi}{\sharp H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\chi} = p^{y(p,\chi,m-1)}.$$

Proof. In [9, Proposition 4.1], we prove the above theorem when χ is even. In the same way, using an isomorphism

$$X'_\infty(m-1)^\chi_\Delta \simeq X'^{\chi\omega^{1-m}}_\infty \otimes \mathbf{Z}_p(m-1),$$

we can show the above equations. □

For a positive integer m and a prime number p, we denote by $K_{2m-2}(\mathcal{O}_F)(p)$ the p-Sylow subgroup of $K_{2m-2}(\mathcal{O}_F)$. Here we put

$$K'_{2m-2}(\mathcal{O}_F) = \bigoplus_{5 \le p < 200000} K_{2m-2}(\mathcal{O}_F)(p),$$

$$X'(\chi, m-1) = \prod_{5 \le p < 200000} \sharp X'_\infty(m-1)^\chi_{G_\infty} \text{ and}$$

$$Y'_i(\chi, m-1) = \prod_{p \in S_i(\chi,m),\, 5 \le p < 200000} \frac{\sharp \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m))^\chi}{\sharp H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\chi}.$$

Then, Theorem 3 and the surjectivity of p-adic Chern characters, we have

$$\sharp K'_{2m-2}(\mathcal{O}_F)^\chi \text{ is divided by } X'(\chi, m-1) \cdot Y'_1(\chi, m-1) \cdot Y'_2(\chi, m-1).$$

For an odd character $\chi\omega^{1-m}$, we can compute $v_p(X'(\chi, m-1))$ from the zeros of the Iwasawa polynomial by Proposition 3. In fact, we can easily obtain a lot of examples of (χ, m) with $X'(\chi, m-1) > 1$.

On the other hand, for an even character $\chi\omega^{1-m}$, it is more difficult to obtain examples of (χ, m) with $X'(\chi, m-1) > 1$. Since Vandiver's conjecture is true for all $p < 12000000$, $X'_\infty(m-1)^{\chi^0}_{G_\infty}$ is trivial for any odd integer m. Further we have $\sharp H^2(\mathbf{Q}_p, \mathbf{Z}_p(m)) = \sharp H^0(\mathbf{Q}_p, \mathbf{Q}_p/\mathbf{Z}_p(1-m)) = \sharp H^0(\mathbf{Z}, \mathbf{Q}_p/\mathbf{Z}_p(1-m))$. By Proposition 3 and our computational result [9, 10], we obtain such examples in the following tables.

Factors of $\sharp \mathbf{K}'_{\mathbf{2m-2}}(\mathcal{O}_{\mathbf{Q}(\sqrt{\mathbf{f}_\chi})})$ with $\mathbf{X}' = \mathbf{X}'(\chi, \mathbf{m}-\mathbf{1}) > \mathbf{1}$

$-200 < f_\chi < 0$ and $5 \leq p < 200000$

$2m-2$	f_χ	X'	$2m-2$	f_χ	X'
122	-4	379	46	-11	79
22	-11	173	5470	-15	4909
38	-19	37	58	-19	41
594	-19	2251	1714	-20	20261
34	-23	193	30	-31	131
1090	-31	821	26	-40	97
198	-51	557	5918	-51	6553
78178	-55	41189	46	-67	433
26	-71	17	14	-79	17
55534	-79	45943	169774	-79	153059
654	-84	10133	47958	-88	33049
30	-91	37	7550	-91	7069
51918	-91	107449	26	-103	17
102	-103	67	35102	-104	17837
260746	-104	184157	1034	-116	4363
149078	-119	112241	3846	-120	4177
197194	-120	126691	26	-127	67
1450	-131	853	14	-136	11
96258	-136	54547	4434	-139	4451
18	-148	23	490	-152	863
1398	-152	3019	6478	-155	12377
46	-163	79	1102	-167	797
91914	-183	104803	30	-187	79

We have $Y'_1(\chi, m-1) = Y'_2(\chi, m-1) = 1$ for all the above cases.

$1 < f_\chi < 200$ and $5 < p < 200000$

$2m-2$	f_χ	X'	Y_2'	$2m-2$	f_χ	X'	Y_2'
68372	8	34301	1	33588	8	157229	7
316	12	701	1	96	21	199	$5{\cdot}17$
128708	28	109829	47	44	33	53	1
20	37	43	11	936	53	1033	$7{\cdot}13^2{\cdot}37$
15472	56	100937	5	92652	56	55621	$43{\cdot}6619{\cdot}15443$
8	69	19	5	1220	85	3697	1
88	88	71	1	5124	101	5333	$43{\cdot}367$
8	104	19	5	20	113	43	11
3540	113	3373	$7{\cdot}11{\cdot}31$	140	124	197	11
380	124	239	11	76	129	67	1
9260	140	4751	1	1208	141	5431	5
20	149	43	1	108	149	71	$7{\cdot}19$
92	149	229	47	194500	149	109211	251
90936	156	50051	$5{\cdot}7{\cdot}19$	688	157	401	173
156	161	101	1	28	168	37	1
124	172	73	1	4	173	7	1
20	173	43	1	116	173	101	1
20	177	17	11	36	181	71	1
10724	181	6991	1	944	185	827	1
2904	188	1621	$23{\cdot}67{\cdot}727$	11380	193	62791	1
296	197	521	1				

We have $Y_1' = Y_1'(\chi, m-1) = 1$ for all the above cases.

Examples

There exist submodules A_i of K-groups such that

$$K_{122}(\mathcal{O}_{\mathbf{Q}(\sqrt{-4})}) \supseteq K'_{122}(\mathcal{O}_{\mathbf{Q}(\sqrt{-4})}) \supseteq A_1 \simeq \mathbf{Z}/(379\mathbf{Z}),$$

$$K_{68372}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq K'_{68372}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq A_2 \simeq \mathbf{Z}/(34301\mathbf{Z}), \text{ and}$$

$$K_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq K'_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq A_3 \simeq \mathbf{Z}/(7 \cdot 157229\mathbf{Z}).$$

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L^1 Estimate for the Dissipative Wave Equation in a Two Dimensional Exterior Domain

Dedicated to Professor Toru Ishihara on his 65th birthday

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Abstract

We consider the initial–boundary value problem in a two dimensional exterior domain for the dissipative wave equation $(\partial_t^2 + \partial_t - \Delta)u = 0$ with the homogeneous Dirichlet boundary condition. Using the so-called cut-off technique together with the local energy estimate and L^1 and L^2 estimates in the whole space, we derive the L^p estimates with $1 \leq p \leq \infty$ for the solution.

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1 Introduction and Results

Let Ω be an exterior domain in 2-dimensional Euclidean space $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and its complement $\Omega^c = \mathbb{R}^2 \setminus \Omega$ will be contained in the ball $B_{r_0} = \{x \in \mathbb{R}^2 \mid |x| < r_0\}$ with some $r_0 > 0$. We never impose any geometric condition on the domain Ω.

We investigate L^p estimates with $p \geq 1$ of the solution to the initial-boundary value problem for the dissipative wave equation :

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)u = 0\,, \quad u = u(x,t)\,, \quad \text{in } \Omega \times (0,\infty) \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \quad \text{and} \quad u\big|_{\partial\Omega} = 0\,, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$ and $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial_{x_j}^2$ is the Laplacian.

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In previous paper [18], for 2-dimensional case, we have already studied the following decay estimates of the solution of (1.1) :

$$\|u(t)\|_{L^p(\Omega)} \leq Cd_1(1+t)^{-(1-1/p)+\delta}$$

for $1 \leq p < \infty$ and

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leq d_1(1+t)^{-1+\delta}$$

for $t \geq 0$ with any small $\delta > 0$, where d_1 is the quantity given by

$$d_1 = \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{W^{1,1}(\Omega)} + \|u_1\|_{L^1(\Omega)}, \tag{1.2}$$

under the initial data $u_0 \in H_0^1(\Omega) \cap W^{1,1}(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^1(\Omega)$.

The purpose of this paper is an improvement of these estimates.

Our main result is as follows.

Theorem 1.1 *Let Ω be an exterior domain in $\mathbb{R}^2$. Suppose that the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap W^{1,1}(\Omega)$ and $u_1 \in H_0^1(\Omega) \cap L^1(\Omega)$. Then, the solution $u(t)$ of* (1.1) *satisfies that*

$$\|u(t)\|_{L^p(\Omega)} \leq Cd_2(1+t)^{-(1-1/p)}\log(2+t) \tag{1.3}$$

for $1 \leq p \leq \infty$ and

$$\|\partial_t^2 u(t)\|_{L^2(\Omega)} + \|\partial_t \nabla u(t)\|_{L^2(\Omega)} \leq Cd_2(1+t)^{-2}\log(2+t), \tag{1.4}$$

$$\|\partial_t u(t)\|_{H^1(\Omega)} + \|\nabla u(t)\|_{H^1(\Omega)} \leq Cd_2(1+t)^{-1}\log(2+t), \tag{1.5}$$

$$\|u(t)\|_{H^2(\Omega)} \leq Cd_2(1+t)^{-1/2}\log(2+t) \tag{1.6}$$

for $t \geq 0$, where d_2 is the quantity given by

$$d_2 = \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|u_0\|_{W^{1,1}(\Omega)} + \|u_1\|_{L^1(\Omega)}. \tag{1.7}$$

Theorem 1.1 follows from Theorems 3.1, 4.1, and 4.2, immediately.

We note that under the initial data belonging to some weighted energy space, the L^2 estimate $\|u(t)\|_{L^2(\Omega)} \leq C(1+t)^{-1/2}$ has been given by Ikehata and Matsuyama [8] (also, see Saeki and Ikehata [23] for the energy estimate, Ikehata [7], Nakao [12], [13] and the references cited therein).

On the other hand, for N-dimensional cases $\Omega \subset \mathbb{R}^N$ for $N \geq 3$, in previous paper [22], we have given the L^p estimates of the solutions

$$\|u(t)\|_{L^p(\Omega)} \leq C(1+t)^{-(N/2)(1-1/p)}$$

for $1 \leq p \leq 2$, and the L^2 estimates of the derivatives (see [18] for $N \leq 3$).

This paper is organized as follows. In Section 2, we prepare some Propositions for the proof of Theorem 1.1. In Section 3, we derive the L^1 estimate and the L^2 estimate of the solution. In Section 4, we give the energy and second energy estimates for (1.1).

We use only familiar functional spaces and omit the definitions. Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

In this Section, for the proof of Theorem 1.1, we will state some known results for the solution of (1.1).

First we state the result on the local energy decay estimate for (1.1) in 2-dimensional case, which was proved by W. Kawashita (W. Dan) in [2]. (Also, see Dan and Shibata [3], Shibata and Tsutsumi [24], Ono [22].)

Lemma 2.1 *Let Ω be an exterior domain in $\mathbb{R}^2$ and let $r > r_0$. Suppose that that initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ and*

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega_r ,$$

where $\Omega_r = \Omega \cap B_r$. Then, the solution $u(t)$ of (1.1) *satisfies that*

$$\|u(t)\|_{H^1(\Omega_r)} + \|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C(1 + t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})$$

for $t \geq 0$.

Next, we state the estimates of the solution and its derivatives to the Cauchy problem in the whole space $\mathbb{R}^2$:

$$\left\{ \begin{array}{ll} (\partial_t^2 + \partial_t - \Delta)v = 0\,, \quad v = v(x,t)\,, \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1)\,. \end{array} \right. \tag{2.1}$$

The following L^2 estimates are well-known (see Matsumura [10], and also Kawashima, Nakao and Ono [9]).

Lemma 2.2 *Let $m \geq 0$ be a non-negative integer. Suppose that the initial data $v_0 \in H^m(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $v_1 \in H^{m-1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then, the solution $v(t)$ of* (2.1) *satisfies that for $0 \leq k + b \leq m$,*

$$\begin{aligned} \|\partial_t^k \nabla^b v(t)\|_{L^2(\mathbb{R}^2)} &\leq C(1+t)^{-k-b/2-1/2} \\ &\quad \times (\|v_0\|_{H^m(\mathbb{R}^2)} + \|v_1\|_{H^{m-1}(\mathbb{R}^2)} + \|v_0\|_{L^1(\mathbb{R}^2)} + \|v_1\|_{L^1(\mathbb{R}^2)}) \end{aligned}$$

for $t \geq 0$.

By using the representation formula of the solution $v(t)$ of (2.1) (see Courant and Hilbert [1]), we have derived the L^1 estimate in previous papers [16], [17], [19]. (Also, see Nishihara [14], [15], Ono [20], [21]. Cf. Hosono and Ogawa [6], Milani and Han [11].)

Lemma 2.3 *Suppose that the initial data $v_0 \in W^{1,1}(\mathbb{R}^2)$ and $v_1 \in L^1(\mathbb{R}^2)$. Then, the solution $v(t)$ of* (2.1) *satisfies that*

$$\|v(t)\|_{L^1(\mathbb{R}^2)} \leq C(\|v_0\|_{W^{1,1}(\mathbb{R}^2)} + \|v_1\|_{L^1(\mathbb{R}^2)})$$

for $t \geq 0$.

3 L^1 estimate

In this Section we will derive the L^1 and H^1 estimates for the solution of (1.1) combining the so-called cut-off technique with Lemmas 2.1–2.3.

Theorem 3.1 *Under the assumption of Theorem 1.1, the solution* $u(t)$ *of* (1.1) *satisfies that*

$$\|u(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t)\,, \tag{3.1}$$
$$\|u(t)\|_{L^1(\Omega)} \le Cd_2\log(2+t) \tag{3.2}$$

for $t \ge 0$, *where* d_2 *is the quantity given by* (1.7).

Theorem 3.1 will be deduced from the following Propositions 3.2 and 3.3 together with

$$\|u(t)\|_X \le \|u_\chi(t)\|_X + \|u(t) - u_\chi(t)\|_X \tag{3.3}$$

for $X = H^1(\Omega)$ or $L^1(\Omega)$, where $u_\chi(t)$ is the solution of (3.4).

Let $r > r_0$. As cut-off functions in $\mathbb{R}^2$, we take smooth functions $\chi_1(x)$ and $\chi_2(x)$ such that $0 \le \chi_1(x),\ \chi_2(x) \le 1$,

$$\chi(x) = \chi_1(x) = \begin{cases} 0 & \text{if } |x| \le r \\ 1 & \text{if } |x| \ge r+1 \end{cases} \quad \text{and} \quad \chi_2(x) = \begin{cases} 0 & \text{if } |x| \le r+2 \\ 1 & \text{if } |x| \ge r+3\,. \end{cases}$$

First we study on the solution $u_\chi(t)$ to the initial-boundary value problem of the dissipative wave equation with the initial data $(\chi u_0, \chi u_1)$:

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)u_\chi = 0 \quad \text{in } \Omega \times (0,\infty) \\ (u_\chi, \partial_t u_\chi)\big|_{t=0} = (\chi u_0, \chi u_1) \quad \text{and} \quad u_\chi\big|_{\partial\Omega} = 0\,. \end{cases} \tag{3.4}$$

We can expect that $u_\chi(t)$ behavior like the solution $u(t)$ of (1.1) if $|x|$ is large.

Proposition 3.2 *Under the assumption of the Theorem 1.1, the solution* $u_\chi(t)$ *of* (3.4) *satisfies that*

$$\|u_\chi(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t)\,, \tag{3.5}$$
$$\|u_\chi(t)\|_{L^1(\Omega)} \le Cd_2\log(2+t) \tag{3.6}$$

for $t \ge 0$, *where* d_2 *is the quantity given by* (1.7).

Proof. These estimates will be derived by using Lemmas 2.1, 2.1, and 2.3 together with

$$\|u_\chi(t)\|_X \le \|\chi v(t)\|_X + \|u_\chi(t) - \chi v(t)\|_X \tag{3.7}$$

for $X = H^1(\Omega)$ or $L^1(\Omega)$, where $v(t)$ is the solution to the Cauchy problem :

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)v = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (v, \partial_t v)\big|_{t=0} = (\overline{u}_0, \overline{u}_1)\,, \end{cases} \tag{3.8}$$

where $\overline{f}$ is a function in $\mathbb{R}^2$ such that $\overline{f}(x) = f(x)$ in $x \in \Omega$ and $\overline{f}(x) = 0$ in $x \notin \Omega$. It is easy to see from Lemma 2.2 and Lemma 2.3 that

$$\|v(t)\|_{H^1(\Omega)} \le Cd_1(1+t)^{-1/2} \quad \text{and} \quad \|v(t)\|_{L^1(\Omega)} \le Cd_1 \tag{3.9}$$

for $t \ge 0$, respectively.

Then, we see that the function $\chi v(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi v) = g \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\chi v, \partial_t \chi v)\big|_{t=0} = (\chi\overline{u}_0, \chi\overline{u}_1)\,, \end{cases}$$

where $g = -2\nabla\chi \cdot \nabla v - \Delta\chi \cdot v$ with $\operatorname{supp} g \subset \{x \in \mathbb{R}^2 \,|\, r \le |x| \le r+1\}$, and hence, as a function in $\Omega \times (0,\infty)$,

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi v) = g \quad \text{in } \Omega \times (0,\infty) \\ (\chi v, \partial_t \chi v)\big|_{t=0} = (\chi u_0, \chi u_1) \quad \text{and} \quad (\chi v)\big|_{\partial\Omega} = 0\,. \end{cases}$$

Moreover, we observe that the function $w(t) = u_\chi(t) - \chi v(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)w = -g \quad \text{in } \Omega \times (0,\infty) \\ (w, \partial_t w)\big|_{t=0} = (0,0) \quad \text{and} \quad w\big|_{\partial\Omega} = 0\,. \end{cases}$$

Here, we denote the solution to the initial-boundary value problem of (1.1) with the initial data (u_0, u_1) by $S(t;\{u_0,u_1\})$, and then, by the Duhamel principle (e.g. [4]), we see that

$$w(t) = \int_0^t S(t-s;\{0,-g(s)\})\,ds\,.$$

Since it follows from Lemma 2.2 and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} \|g(t)\|_{L^2(\mathbb{R}^2)} &= \|g(t)\|_{L^2(B_{r+1}\setminus B_r)} \le C\|\nabla v(t)\|_{L^2(\mathbb{R}^2)} + C\|v(t)\|_{L^\infty(\mathbb{R}^2)} \\ &\le Cd_2(1+t)^{-1}\,, \end{aligned}$$

applying Lemma 2.1 to the function $w(t)$ in the domain $\Omega_{r+3} = \Omega \cap B_{r+3}$, we have that

$$\begin{aligned} \|w(t)\|_{H^1(\Omega_{r+3})} &\le C\int_0^t (1+(t-s)(\log(t-s))^2)^{-1}\|g(s)\|_{L^2(\mathbb{R}^2)}\,ds \\ &\le Cd_2\int_0^t (1+(t-s)(\log(t-s))^2)^{-1}(1+s)^{-1}\,ds \\ &\le Cd_2(1+t)^{-1}\,, \end{aligned} \tag{3.10}$$

and also,

$$\|w(t)\|_{L^1(\Omega_{r+3})} \le C\|w(t)\|_{H^1(\Omega_{r+3})} \le Cd_2(1+t)^{-1}\,, \tag{3.11}$$

where we use the fact that $\int_0^\infty (1+t(\log t)^2)^{-1}\,dt \le C + \int_1^\infty e^s/(1+e^s s^2)\,ds$ $\le C + \int_1^\infty 1/s^2\,ds \le C$ with $t = e^s$.

On the other hand, the function $\overline{w}(t) = \overline{u_\chi}(t) - \chi v(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)\overline{w} = -g & \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\overline{w}, \partial_t \overline{w})\big|_{t=0} = (0,0)\,, \end{cases}$$

and then, $\chi_2\overline{w}(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi_2\overline{w}) = h & \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\chi_2\overline{w}, \partial_t \chi_2\overline{w})\big|_{t=0} = (0,0)\,, \end{cases}$$

where $h = -2\nabla\chi_2 \cdot \nabla\overline{w} - \Delta\chi_2 \cdot \overline{w}$ with $\operatorname{supp} h \subset \{x \in \mathbb{R}^2 \mid r+2 \le |x| \le r+3\}$.

Here, we denote the solution to the Cauchy problem of (2.1) with the initial data (v_0, v_1) by $\tilde{S}(t; \{v_0, v_1\})$, and then, by the Duhamel principle, we see that

$$\chi_2\overline{w}(t) = \int_0^t \tilde{S}(t-s; \{0, h(s)\})\,ds\,.$$

Applying Lemma 2.2 to the function $\chi_2\overline{w}(t)$, we have from (3.10) that

$$\begin{aligned} \|w(t)\|_{H^1(\Omega_{r+3}^c)} &\le \|\chi_2\overline{w}(t)\|_{H^1(\mathbb{R}^2)} \le \int_0^t \|\tilde{S}(t-s; \{0, h(s)\})\|_{H^1(\mathbb{R}^2)}\,ds \\ &\le C\int_0^t (1+t-s)^{-1/2}(\|h(s)\|_{L^2(\mathbb{R}^2)} + \|h(s)\|_{L^1(\mathbb{R}^2)})\,ds \\ &\le C\int_0^t (1+t-s)^{-1/2}\|w(s)\|_{H^1(\Omega_{r+3})}\,ds \\ &\le Cd_2\int_0^t (1+t-s)^{-1/2}(1+s)^{-1}\,ds \\ &\le Cd_2(1+t)^{-1/2}\log(2+t)\,. \end{aligned} \tag{3.12}$$

Therefore, from (3.7), (3.9), (3.10), and (3.12) we obtain

$$\begin{aligned} \|u_\chi(t)\|_{H^1(\Omega)} &\le C\|v(t)\|_{H^1(\mathbb{R}^2)} + C\|w(t)\|_{H^1(\Omega_{r+3})} + C\|w(t)\|_{H^1(\Omega_{r+3}^c)} \\ &\le Cd_2(1+t)^{-1/2}\log(2+t)\,, \end{aligned}$$

which is the desired estimate (3.5).

By the similar way, applying Lemma 2.3 to the function $\chi_3\overline{w}(t)$, we have from (3.10) that

$$\begin{aligned}\|w(t)\|_{L^1(\Omega^c_{r+3})} &\le \|\chi_2\overline{w}(t)\|_{L^1(\mathbb{R}^2)} \le C\int_0^t \|\tilde{S}(t-s;\{0,h(s)\})\|_{L^1(\mathbb{R}^2)}\,ds \\ &\le C\int_0^t \|h(s)\|_{L^1(\mathbb{R}^2)}\,ds \le C\int_0^t \|w(s)\|_{H^1(\Omega_{r+3})}\,ds \\ &\le Cd_2\int_0^t (1+s)^{-1}\,ds \le Cd_2\log(2+t)\,. \end{aligned} \tag{3.13}$$

Therefore, from (3.7), (3.9), (3.11), and (3.13) we obtain

$$\begin{aligned}\|u_\chi(t)\|_{L^1(\Omega)} &\le \|v(t)\|_{L^1(\mathbb{R}^2)} + C\|w(t)\|_{L^1(\Omega_{r+3})} + C\|w(t)\|_{L^1(\Omega^c_{r+3})} \\ &\le Cd_2\log(2+t)\,,\end{aligned}$$

which is the desired estimate (3.6). □

Proposition 3.3 *Under the assumption of Theorem 1.1, the function* $U(t) = u(t) - u_\chi(t)$ *satisfies*

$$\|U(t)\|_{H^1(\Omega)} = \|u(t)-u_\chi(t)\|_{H^1(\Omega)} \le Cd_1(1+t)^{-1/2}\,, \tag{3.14}$$

$$\|U(t)\|_{L^1(\Omega)} = \|u(t)-u_\chi(t)\|_{L^1(\Omega)} \le Cd_1(1+t)^{-1/2} \tag{3.15}$$

for $t \ge 0$, *where* d_1 *is the quantity given by* (1.2).

Proof. It is easy to see that the function $U(t) = u(t) - u_\chi(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)U = 0 \quad \text{in } \Omega\times(0,\infty) \\ (U,\partial_t U)\big|_{t=0} = ((1-\chi)u_0,(1-\chi)u_1) \quad \text{and} \quad U\big|_{\partial\Omega} = 0\,, \end{cases}$$

and then, by Lemma 2.1 again, we observe that

$$\|U(t)\|_{H^1(\Omega_{r+3})} \le C(1+t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})\,, \tag{3.16}$$

and also,

$$\|U(t)\|_{L^1(\Omega_{r+3})} \le C(1+t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})\,. \tag{3.17}$$

Moreover, we see that the function $\chi_2\overline{U}(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi_2\overline{U}) = f \quad \text{in } \mathbb{R}^2\times(0,\infty) \\ (\chi_2\overline{U},\partial_t\chi_2\overline{U})\big|_{t=0} = (0,0)\,, \end{cases}$$

where $f = -2\nabla\chi_2 \cdot \nabla\overline{U} - \Delta\chi_2 \cdot \overline{U}$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^2 \,|\, r+2 \leq |x| \leq r+3\}$, and also, it follows

$$\chi_2\overline{U}(t) = \int_0^t \tilde{S}(t-s;\{0,f(s)\})\,ds\,.$$

Applying Lemma 2.2 to the function $\chi_2\overline{U}(t)$, we have from (3.16) that

$$\begin{aligned}
\|U(t)\|_{H^1(\Omega_{r+3}^c)} &\leq \|\chi_2\overline{U}(t)\|_{H^1(\mathbb{R}^2)} \leq \int_0^t \|\tilde{S}(t-s;\{0,f(s)\})\|_{H^1(\mathbb{R}^2)}\,ds \\
&\leq C\int_0^t (1+t-s)^{-1/2}(\|f(s)\|_{L^2(\mathbb{R}^2)} + \|f(s)\|_{L^1(\mathbb{R}^2)})\,ds \\
&\leq C\int_0^t (1+t-s)^{-1/2}\|f(s)\|_{L^2(B_{r+3}\setminus B_{r+2})}\,ds \\
&\leq C\int_0^t (1+t-s)^{-1/2}\|U(s)\|_{H^1(\Omega_{r+3})}\,ds \\
&\leq C\int_0^t (1+t-s)^{-1/2}(1+s(\log s)^2)^{-1}\,ds \leq Cd_1(1+t)^{-1/2}\,. \qquad (3.18)
\end{aligned}$$

Therefore, we know that (3.14) follows from (3.16) and (3.18).

By the similar way, applying Lemma 2.3 to the function $\chi_2\overline{U}(t)$, we have from (3.16) that

$$\begin{aligned}
\|U(t)\|_{L^1(\Omega_{r+3}^c)} &\leq \|\chi_2\overline{U}(t)\|_{L^1(\mathbb{R}^2)} \leq \int_0^t \|\tilde{S}(t-s;\{0,f(s)\})\|_{L^1(\mathbb{R}^2)}\,ds \\
&\leq C\int_0^t \|f(s)\|_{L^1(\mathbb{R}^2)}\,ds \leq Cd_1(1+t)^{-1/2}\,. \qquad (3.19)
\end{aligned}$$

Therefore, we know that (3.15) follows from (3.17) and (3.19). □

Proof of Theorem 3.1. Summing up the above estimates (3.5), (3.14), and (3.6), (3.15) together with (3.3), we obtain (3.1) and (3.2), respectively. □

4 Energy estimates

In this section we will derive the energy and second energy estimates for (1.1) by using the energy method. For simplicity, we often use $\|\cdot\|$ as the L^2 norm, that is, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Theorem 4.1 *Using the assumption of Theorem 1.1, the solution $u(t)$ of (1.1) satisfies that*

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leq Cd_2(1+t)^{-1}\log(2+t) \qquad (4.1)$$

for $t \geq 0$, where d_2 is the quantity given by (1.7).

Proof. We denote the total energy for (1.1) by

$$E(t) = E_1(t) \equiv \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 ,$$

which has the energy identity

$$\frac{d}{dt}E(t) + \|\partial_t u(t)\|^2 = 0 \tag{4.2}$$

or

$$E(t) + \int_0^t \|\partial_t u(s)\|^2 \, ds = E(0) . \tag{4.3}$$

Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2 + (u(t), \partial_t u(t))\right) + \|\nabla u(t)\|^2 - \|\partial_t u(t)\|^2 = 0 , \tag{4.4}$$

and then, integrating it in time,

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds \\
&\le \frac{1}{2}\|u_0\|^2 + \|u_0\|\|u_1\| + \|u(t)\|\|\partial_t u(t)\| + \int_0^t \|\partial_t u(s)\|^2 \, ds \\
&\le C d_0^2 + \frac{1}{4}\|u(t)\|^2 + \|\partial_t u(t)\|^2 + \int_0^t \|\partial_t u(s)\|^2 \, ds
\end{aligned}$$

with $d_0 = \|u_0\|_{H^1(\Omega)} + \|u_1\|$, and hence, from (4.3) we obtain

$$\begin{aligned}
&\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds \\
&\qquad \le C d_0^2 + C\|\partial_t u(t)\|^2 + C \int_0^t \|\partial_t u(s)\|^2 \, ds \le C d_0^2 .
\end{aligned} \tag{4.5}$$

Thus, from (4.3) and (4.5) we have

$$\int_0^t E(s) \, ds \le C d_0^2 . \tag{4.6}$$

For $m \ge 1$, we observe from (4.3) and (4.4) that

$$\frac{d}{dt} t^m E(t) + t^m \|\partial_t u(t)\|^2 = m t^{m-1} E(t) \tag{4.7}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}t^m\|u(t)\|^2 + t^m(u(t), \partial_t u(t))\right) + t^m\|\nabla u(t)\|^2$$
$$= \frac{m}{2}t^{m-1}\|u(t)\|^2 + mt^{m-1}(u(t), \partial_t u(t)) + t^m\|\partial_t u(t)\|^2 , \qquad (4.8)$$

respectively, and moreover, integrating (4.7) and (4.8) in time, we have that

$$t^m E(t) + \int_0^t s^m\|\partial_t u(s)\|^2\, ds = m\int_0^t s^{m-1}E(s)\, ds \qquad (4.9)$$

and

$$\frac{1}{2}t^m\|u(t)\|^2 + \int_0^t s^m\|\nabla u(s)\|^2\, ds$$
$$\leq \frac{1}{4}t^m\|u(s)\|^2 + t^m\|\partial_t u(t)\|^2 + m\int_0^t s^{m-1}\|u(s)\|^2\, ds$$
$$+ \int_0^t (ms^{m-1} + s^m)\|\partial_t u(s)\|^2\, ds , \qquad (4.10)$$

respectively, where we used the Young inequality at the last inequality.

Then, we obtain from (4.9) for $m = 1$ together with (4.6) that

$$tE(t) + \int_0^t s\|\partial_t u(s)\|^2\, ds = \int_0^t E(s)\, ds \leq Cd_0^2 , \qquad (4.11)$$

and from (4.10) for $m = 1$ together with (4.11) and (4.3),

$$t\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\, ds$$
$$\leq Ct\|\partial_t u(t)\|^2 + C\int_0^t \|u(s)\|^2\, ds + C\int_0^t (1+s)\|\partial_t u(s)\|^2\, ds$$
$$\leq Cd_0^2 + C\int_0^t \|u(s)\|^2\, ds \leq Cd_2^2(\log(2+t))^3 , \qquad (4.12)$$

where we used (3.1) at the last inequality. Thus, from (4.11) and (4.12) we have

$$\int_0^t sE(s)\, ds \leq Cd_2^2(\log(2+t))^3 . \qquad (4.13)$$

Moreover, from (4.9) for $m = 2$ together with (4.13) we have

$$t^2E(t) + \int_0^t s^2\|\partial_t u(s)\|^2\, ds = 2\int_0^t sE(s)\, ds \leq Cd_2^2(\log(2+t))^3 , \qquad (4.14)$$

and from (4.10) for $m = 2$ together with (4.14), (4.11), and (3.1),

$$\begin{aligned} &t^2\|u(t)\|^2 + \int_0^t s^2\|\nabla u(s)\|^2\,ds \\ &\le Ct^2\|\partial_t u(t)\|^2 + C\int_0^t s\|u(s)\|^2\,ds + \int_0^t (s+s^2)\|\partial_t u(s)\|^2\,ds \\ &\le Cd_2^2(\log(2+t))^3 + C\int_0^t s\|u(s)\|^2\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,, \end{aligned} \tag{4.15}$$

where we used (3.1) at the last inequality, and hence, from (4.14) and (4.15) we have

$$\int_0^t s^2 E(s)\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,. \tag{4.16}$$

Thus, from (4.9) for $m = 3$ together with (4.16) we have

$$t^3E(t) + \int_0^t s^3\|\partial_t u(s)\|^2\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,, \tag{4.17}$$

and hence, the desired decay estimate (4.1) follows from (4.3) and (4.17). □

Moreover, by using the energy method again, we have the following estimates.

Theorem 4.2 *Using the assumption of Theorem 1.1, the solution $u(t)$ of* (1.1) *satisfies that*

$$\|\partial_t^2 u(t)\|_{L^2(\Omega)} + \|\partial_t\nabla u(t)\|_{L^2(\Omega)} \le Cd_2(1+t)^{-2}\log(2+t)\,, \tag{4.18}$$

$$\|\partial_t u(t)\|_{H^1(\Omega)} + \|\nabla u(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1}\log(2+t)\,, \tag{4.19}$$

$$\|u(t)\|_{H^2(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t) \tag{4.20}$$

for $t \ge 0$, where d_2 is the quantity given by (1.7).

Proof. We will carry out the similar way as the proof the Theorem 4.1. Put $V(t) = \partial_t u(t)$ and

$$E_2(t) \equiv \frac{1}{2}\|\partial_t V(t)\|^2 + \frac{1}{2}\|\nabla V(t)\|^2 = \frac{1}{2}\|\partial_t^2 u(t)\|^2 + \frac{1}{2}\|\partial_t\nabla u(t)\|^2\,.$$

Then, we see that the function $V(t)$ satisfies that

$$(\partial_t^2 + \partial_t - \Delta)V = 0 \quad \text{in } \Omega\times(0,\infty)$$

with $V\big|_{\partial\Omega} = \partial_t u\big|_{\partial\Omega} = 0$, and

$$\frac{d}{dt}E_2(t) + \|\partial_t V(t)\|^2 = 0 \tag{4.21}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\|V(t)\|^2 + (V(t), \partial_t V(t))\right) + \|\nabla V(t)\|^2 - \|\partial_t V(t)\|^2 = 0\,. \tag{4.22}$$

Thus, from (4.21) and (4.22) we have that

$$E_2(t) + \int_0^t \|\partial_t V(s)\|^2\, ds = E_2(0) \quad \text{and} \quad \|V(t)\|^2 + \int_0^t \|\nabla V(s)\|^2\, ds \le Cd_2^2\,,$$

respectively, and hence, we obtain

$$\int_0^t E_2(s)\, ds \le Cd_2^2\,. \tag{4.23}$$

For $m \ge 1$, we observe from (4.21) and (4.22) that

$$\frac{d}{dt} t^m E_2(t) + t^m \|\partial_t V(t)\|^2 = m t^{m-1} E_2(t) \tag{4.24}$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2} t^m \|V(t)\|^2 + t^m (V(t), \partial_t V(t))\right) + t^m \|\nabla V(t)\|^2 \\ &\quad = \frac{m}{2} t^{m-1} \|V(t)\|^2 + m t^{m-1} (V(t), \partial_t V(t)) + t^m \|\partial_t V(t)\|^2\,, \end{aligned} \tag{4.25}$$

respectively, and moreover, integrating (4.24) and (4.25) in time like (4.9) and (4.10), we have

$$t^m E_2(t) + \int_0^t s^m \|\partial_t V(s)\|^2\, ds = m \int_0^t s^{m-1} E_2(s)\, ds \tag{4.26}$$

and

$$\begin{aligned} \frac{1}{2} t^m \|V(t)\|^2 + \int_0^t s^m \|\nabla V(s)\|^2\, ds &\le \frac{1}{4} t^m \|V(t)\|^2 + t^m \|\partial_t V(t)\|^2 \\ &\quad + m \int_0^t s^{m-1} \|V(s)\|^2\, ds + \int_0^t (m s^{m-1} + s^m) \|\partial_t V(s)\|^2\, ds \end{aligned}$$

or

$$\begin{aligned} &t^m \|V(t)\|^2 + \int_0^t s^m \|\nabla V(s)\|^2\, ds \\ &\quad \le C \int_0^t (1+s)^{m-1} E_2(s)\, ds + C \int_0^t s^{m-1} \|\partial_t u(s)\|^2\, ds\,, \end{aligned} \tag{4.27}$$

respectively. Thus, from (4.26) and (4.27) for $m = 1$ together with (4.23) and (4.3) we observe that

$$
\begin{aligned}
&tE_2(t) + \int_0^t s\|\partial_t V(s)\|^2\,ds \le Cd_2^2\,,\\
&t\|V(t)\|^2 + \int_0^t s\|\nabla V(s)\|^2\,ds \le Cd_2^2 + C\int_0^t \|\partial_t u(s)\|^2\,ds \le Cd_2^2\,,
\end{aligned}
$$

and

$$
\int_0^t sE_2(s)\,ds \le Cd_2^2\,. \tag{4.28}
$$

From (4.26) and (4.27) for $m = 2$ together with (4.23), (4.28), and (4.11) we observe that

$$
\begin{aligned}
&t^2E_2(t) + \int_0^t s^2\|\partial_t V(s)\|^2\,ds \le Cd_2^2\,,\\
&t^2\|V(t)\|^2 + \int_0^t s^2\|\nabla V(s)\|^2\,ds\\
&\qquad \le Cd_2^2 + C\int_0^t s\|\partial_t u(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,
\end{aligned}
$$

and

$$
\int_0^t s^2E_2(s)\,ds \le Cd_2^2(\log(2+t))^3\,. \tag{4.29}
$$

From (4.26) and (4.27) for $m = 3$ together with (4.23), (4.29), and (4.14) we observe that

$$
\begin{aligned}
&t^3E_2(t) + \int_0^t s^3\|\partial_t V(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,\\
&t^3\|V(t)\|^2 + \int_0^t s^3\|\nabla V(s)\|^2\,ds\\
&\qquad \le Cd_2^2(\log(2+t))^3 + C\int_0^t s^2\|\partial_t u(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,
\end{aligned}
$$

and

$$
\int_0^t s^3E_2(s)\,ds \le Cd_2^2(\log(2+t))^3\,. \tag{4.30}
$$

From (4.26) and (4.27) for $m = 4$ together with (4.23), (4.30), and (4.17) we observe that

$$t^4 E_2(t) + \int_0^t s^4 \|\partial_t V(s)\|^2\, ds \le C d_2^2 (\log(2+t))^3\,,$$

$$\begin{aligned} t^4 \|V(t)\|^2 &+ \int_0^t s^4 \|\nabla V(s)\|^2\, ds \\ &\le C d_2^2 (\log(2+t))^3 + C \int_0^t s^3 \|\partial_t u(s)\|^2\, ds \le C d_2^2 (1+t)(\log(2+t))^2\,, \end{aligned}$$

and

$$\int_0^t s^4 E_2(s)\, ds \le C d_2^2 (1+t)(\log(2+t))^2\,. \tag{4.31}$$

Therefore, from (4.26) for $m = 5$ and (4.31) we obtain that

$$t^5 E_2(t) + \int_0^t s^5 \|\partial_t V(s)\|^2\, ds \le C d_2^2 (1+t)(\log(2+t))^2$$

or

$$\|\partial_t^2 u(t)\| + \|\partial_t \nabla u(t)\| \le C d_2 (1+t)^{-2} \log(2+t)\,. \tag{4.32}$$

Moreover, by the elliptic regularity theorem in exterior domains (see [5], [22]) together with (1.1), (4.1), and (4.32) that

$$\begin{aligned} \|\nabla u(t)\|_{H^1(\Omega)} &\le C\|\Delta u(t)\| + C\|\nabla u(t)\| \\ &\le C\|\partial_t u(t)\| + C\|\partial_t^2 u(t)\| + C\|\nabla u(t)\| \\ &\le C d_2 (1+t)^{-1} \log(2+t)\,, \end{aligned} \tag{4.33}$$

and hence, the desired estimates (4.18)–(4.20) follows from (3.1), (4.32), and (4.33). □

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