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L^1 Estimate for the Dissipative Wave Equation in a Two Dimensional Exterior Domain

Dedicated to Professor Toru Ishihara on his 65th birthday

By

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Abstract

We consider the initial–boundary value problem in a two dimensional exterior domain for the dissipative wave equation $(\partial_t^2 + \partial_t - \Delta)u = 0$ with the homogeneous Dirichlet boundary condition. Using the so-called cut-off technique together with the local energy estimate and L^1 and L^2 estimates in the whole space, we derive the L^p estimates with $1 \le p \le \infty$ for the solution.

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1 Introduction and Results

Let Ω be an exterior domain in 2-dimensional Euclidean space $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and its complement $\Omega^c = \mathbb{R}^2 \setminus \Omega$ will be contained in the ball $B_{r_0} = \{x \in \mathbb{R}^2 \,|\, |x| < r_0\}$ with some $r_0 > 0$. We never impose any geometric condition on the domain Ω.

We investigate L^p estimates with $p \ge 1$ of the solution to the initial-boundary value problem for the dissipative wave equation :

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)u = 0\,, \quad u = u(x,t)\,, \quad \text{in } \Omega \times (0,\infty) \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \quad \text{and} \quad u\big|_{\partial\Omega} = 0\,, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$ and $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial_{x_j}^2$ is the Laplacian.

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In previous paper [18], for 2-dimensional case, we have already studied the following decay estimates of the solution of (1.1) :

$$\|u(t)\|_{L^p(\Omega)} \leq Cd_1(1+t)^{-(1-1/p)+\delta}$$

for $1 \leq p < \infty$ and

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leq d_1(1+t)^{-1+\delta}$$

for $t \geq 0$ with any small $\delta > 0$, where d_1 is the quantity given by

$$d_1 = \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{W^{1,1}(\Omega)} + \|u_1\|_{L^1(\Omega)}\,, \tag{1.2}$$

under the initial data $u_0 \in H_0^1(\Omega) \cap W^{1,1}(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^1(\Omega)$.

The purpose of this paper is an improvement of these estimates.

Our main result is as follows.

Theorem 1.1 *Let Ω be an exterior domain in $\mathbb{R}^2$. Suppose that the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap W^{1,1}(\Omega)$ and $u_1 \in H_0^1(\Omega) \cap L^1(\Omega)$. Then, the solution $u(t)$ of* (1.1) *satisfies that*

$$\|u(t)\|_{L^p(\Omega)} \leq Cd_2(1+t)^{-(1-1/p)} \log(2+t) \tag{1.3}$$

for $1 \leq p \leq \infty$ and

$$\|\partial_t^2 u(t)\|_{L^2(\Omega)} + \|\partial_t \nabla u(t)\|_{L^2(\Omega)} \leq Cd_2(1+t)^{-2} \log(2+t)\,, \tag{1.4}$$

$$\|\partial_t u(t)\|_{H^1(\Omega)} + \|\nabla u(t)\|_{H^1(\Omega)} \leq Cd_2(1+t)^{-1} \log(2+t)\,, \tag{1.5}$$

$$\|u(t)\|_{H^2(\Omega)} \leq Cd_2(1+t)^{-1/2} \log(2+t) \tag{1.6}$$

for $t \geq 0$, where d_2 is the quantity given by

$$d_2 = \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|u_0\|_{W^{1,1}(\Omega)} + \|u_1\|_{L^1(\Omega)}\,. \tag{1.7}$$

Theorem 1.1 follows from Theorems 3.1, 4.1, and 4.2, immediately.

We note that under the initial data belonging to some weighted energy space, the L^2 estimate $\|u(t)\|_{L^2(\Omega)} \leq C(1+t)^{-1/2}$ has been given by Ikehata and Matsuyama [8] (also, see Saeki and Ikehata [23] for the energy estimate, Ikehata [7], Nakao [12], [13] and the references cited therein).

On the other hand, for N-dimensional cases $\Omega \subset \mathbb{R}^N$ for $N \geq 3$, in previous paper [22], we have given the L^p estimates of the solutions

$$\|u(t)\|_{L^p(\Omega)} \leq C(1+t)^{-(N/2)(1-1/p)}$$

for $1 \leq p \leq 2$, and the L^2 estimates of the derivatives (see [18] for $N \leq 3$).

This paper is organized as follows. In Section 2, we prepare some Propositions for the proof of Theorem 1.1. In Section 3, we derive the L^1 estimate and the L^2 estimate of the solution. In Section 4, we give the energy and second energy estimates for (1.1).

We use only familiar functional spaces and omit the definitions. Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

In this Section, for the proof of Theorem 1.1, we will state some known results for the solution of (1.1).

First we state the result on the local energy decay estimate for (1.1) in 2-dimensional case, which was proved by W. Kawashita (W. Dan) in [2]. (Also, see Dan and Shibata [3], Shibata and Tsutsumi [24], Ono [22].)

Lemma 2.1 *Let Ω be an exterior domain in $\mathbb{R}^2$ and let $r > r_0$. Suppose that that initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ and*

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega_r ,$$

where $\Omega_r = \Omega \cap B_r$. Then, the solution $u(t)$ of (1.1) satisfies that

$$\|u(t)\|_{H^1(\Omega_r)} + \|\partial_t u(t)\|_{L^2(\Omega_r)} \leq C(1 + t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})$$

for $t \geq 0$.

Next, we state the estimates of the solution and its derivatives to the Cauchy problem in the whole space $\mathbb{R}^2$:

$$\left\{ \begin{array}{ll} (\partial_t^2 + \partial_t - \Delta)v = 0\,, \quad v = v(x,t)\,, \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1)\,. \end{array} \right. \tag{2.1}$$

The following L^2 estimates are well-known (see Matsumura [10], and also Kawashima, Nakao and Ono [9]).

Lemma 2.2 *Let $m \geq 0$ be a non-negative integer. Suppose that the initial data $v_0 \in H^m(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $v_1 \in H^{m-1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then, the solution $v(t)$ of (2.1) satisfies that for $0 \leq k + b \leq m$,*

$$\begin{aligned} \|\partial_t^k \nabla^b v(t)\|_{L^2(\mathbb{R}^2)} &\leq C(1+t)^{-k-b/2-1/2} \\ &\quad \times (\|v_0\|_{H^m(\mathbb{R}^2)} + \|v_1\|_{H^{m-1}(\mathbb{R}^2)} + \|v_0\|_{L^1(\mathbb{R}^2)} + \|v_1\|_{L^1(\mathbb{R}^2)}) \end{aligned}$$

for $t \geq 0$.

By using the representation formula of the solution $v(t)$ of (2.1) (see Courant and Hilbert [1]), we have derived the L^1 estimate in previous papers [16], [17], [19]. (Also, see Nishihara [14], [15], Ono [20], [21]. Cf. Hosono and Ogawa [6], Milani and Han [11].)

Lemma 2.3 *Suppose that the initial data $v_0 \in W^{1,1}(\mathbb{R}^2)$ and $v_1 \in L^1(\mathbb{R}^2)$. Then, the solution $v(t)$ of (2.1) satisfies that*

$$\|v(t)\|_{L^1(\mathbb{R}^2)} \leq C(\|v_0\|_{W^{1,1}(\mathbb{R}^2)} + \|v_1\|_{L^1(\mathbb{R}^2)})$$

for $t \geq 0$.

3 L^1 estimate

In this Section we will derive the L^1 and H^1 estimates for the solution of (1.1) combining the so-called cut-off technique with Lemmas 2.1–2.3.

Theorem 3.1 *Under the assumption of Theorem 1.1, the solution* $u(t)$ *of* (1.1) *satisfies that*

$$\|u(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t)\,, \tag{3.1}$$
$$\|u(t)\|_{L^1(\Omega)} \le Cd_2\log(2+t) \tag{3.2}$$

for $t \ge 0$*, where* d_2 *is the quantity given by* (1.7).

Theorem 3.1 will be deduced from the following Propositions 3.2 and 3.3 together with

$$\|u(t)\|_X \le \|u_\chi(t)\|_X + \|u(t) - u_\chi(t)\|_X \tag{3.3}$$

for $X = H^1(\Omega)$ or $L^1(\Omega)$, where $u_\chi(t)$ is the solution of (3.4).

Let $r > r_0$. As cut-off functions in $\mathbb{R}^2$, we take smooth functions $\chi_1(x)$ and $\chi_2(x)$ such that $0 \le \chi_1(x),\ \chi_2(x) \le 1$,

$$\chi(x) = \chi_1(x) = \begin{cases} 0 & \text{if } |x| \le r \\ 1 & \text{if } |x| \ge r+1 \end{cases} \quad \text{and} \quad \chi_2(x) = \begin{cases} 0 & \text{if } |x| \le r+2 \\ 1 & \text{if } |x| \ge r+3\,. \end{cases}$$

First we study on the solution $u_\chi(t)$ to the initial-boundary value problem of the dissipative wave equation with the initial data $(\chi u_0, \chi u_1)$:

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)u_\chi = 0 \quad \text{in } \Omega \times (0,\infty) \\ (u_\chi, \partial_t u_\chi)\big|_{t=0} = (\chi u_0, \chi u_1) \quad \text{and} \quad u_\chi\big|_{\partial\Omega} = 0\,. \end{cases} \tag{3.4}$$

We can expect that $u_\chi(t)$ behavior like the solution $u(t)$ of (1.1) if $|x|$ is large.

Proposition 3.2 *Under the assumption of the Theorem 1.1, the solution* $u_\chi(t)$ *of* (3.4) *satisfies that*

$$\|u_\chi(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t)\,, \tag{3.5}$$
$$\|u_\chi(t)\|_{L^1(\Omega)} \le Cd_2\log(2+t) \tag{3.6}$$

for $t \ge 0$*, where* d_2 *is the quantity given by* (1.7).

Proof. These estimates will be derived by using Lemmas 2.1, 2.1, and 2.3 together with

$$\|u_\chi(t)\|_X \le \|\chi v(t)\|_X + \|u_\chi(t) - \chi v(t)\|_X \tag{3.7}$$

for $X = H^1(\Omega)$ or $L^1(\Omega)$, where $v(t)$ is the solution to the Cauchy problem :

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)v = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (v, \partial_t v)\big|_{t=0} = (\overline{u}_0, \overline{u}_1)\,, \end{cases} \tag{3.8}$$

where $\overline{f}$ is a function in $\mathbb{R}^2$ such that $\overline{f}(x) = f(x)$ in $x \in \Omega$ and $\overline{f}(x) = 0$ in $x \notin \Omega$. It is easy to see from Lemma 2.2 and Lemma 2.3 that

$$\|v(t)\|_{H^1(\Omega)} \le C d_1 (1+t)^{-1/2} \quad \text{and} \quad \|v(t)\|_{L^1(\Omega)} \le C d_1 \tag{3.9}$$

for $t \ge 0$, respectively.

Then, we see that the function $\chi v(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi v) = g \quad \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\chi v, \partial_t \chi v)\big|_{t=0} = (\chi\overline{u}_0, \chi\overline{u}_1)\,, \end{cases}$$

where $g = -2\nabla\chi \cdot \nabla v - \Delta\chi \cdot v$ with $\operatorname{supp} g \subset \{x \in \mathbb{R}^2 \,|\, r \le |x| \le r+1\}$, and hence, as a function in $\Omega \times (0,\infty)$,

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi v) = g \quad \text{in } \Omega \times (0,\infty) \\ (\chi v, \partial_t \chi v)\big|_{t=0} = (\chi u_0, \chi u_1) \quad \text{and} \quad (\chi v)\big|_{\partial\Omega} = 0\,. \end{cases}$$

Moreover, we observe that the function $w(t) = u_\chi(t) - \chi v(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)w = -g \quad \text{in } \Omega \times (0,\infty) \\ (w, \partial_t w)\big|_{t=0} = (0,0) \quad \text{and} \quad w\big|_{\partial\Omega} = 0\,. \end{cases}$$

Here, we denote the solution to the initial-boundary value problem of (1.1) with the initial data (u_0, u_1) by $S(t; \{u_0, u_1\})$, and then, by the Duhamel principle (e.g. [4]), we see that

$$w(t) = \int_0^t S(t-s; \{0, -g(s)\})\, ds\,.$$

Since it follows from Lemma 2.2 and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} \|g(t)\|_{L^2(\mathbb{R}^2)} = \|g(t)\|_{L^2(B_{r+1}\setminus B_r)} &\le C\|\nabla v(t)\|_{L^2(\mathbb{R}^2)} + C\|v(t)\|_{L^\infty(\mathbb{R}^2)} \\ &\le C d_2 (1+t)^{-1}\,, \end{aligned}$$

applying Lemma 2.1 to the function $w(t)$ in the domain $\Omega_{r+3} = \Omega \cap B_{r+3}$, we have that

$$\begin{aligned} \|w(t)\|_{H^1(\Omega_{r+3})} &\le C \int_0^t (1 + (t-s)(\log(t-s))^2)^{-1} \|g(s)\|_{L^2(\mathbb{R}^2)}\, ds \\ &\le C d_2 \int_0^t (1 + (t-s)(\log(t-s))^2)^{-1} (1+s)^{-1}\, ds \\ &\le C d_2 (1+t)^{-1}\,, \end{aligned} \tag{3.10}$$

and also,

$$\|w(t)\|_{L^1(\Omega_{r+3})} \le C\|w(t)\|_{H^1(\Omega_{r+3})} \le Cd_2(1+t)^{-1}, \tag{3.11}$$

where we use the fact that $\int_0^\infty (1+t(\log t)^2)^{-1}\,dt \le C + \int_1^\infty e^s/(1+e^s s^2)\,ds \le C + \int_1^\infty 1/s^2\,ds \le C$ with $t = e^s$.

On the other hand, the function $\overline{w}(t) = \overline{u_\chi}(t) - \chi v(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)\overline{w} = -g & \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\overline{w}, \partial_t \overline{w})\big|_{t=0} = (0,0), \end{cases}$$

and then, $\chi_2 \overline{w}(t)$ satisfies that

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi_2\overline{w}) = h & \text{in } \mathbb{R}^2 \times (0,\infty) \\ (\chi_2\overline{w}, \partial_t \chi_2\overline{w})\big|_{t=0} = (0,0), \end{cases}$$

where $h = -2\nabla\chi_2 \cdot \nabla\overline{w} - \Delta\chi_2 \cdot \overline{w}$ with $\operatorname{supp} h \subset \{x \in \mathbb{R}^2 \mid r+2 \le |x| \le r+3\}$.

Here, we denote the solution to the Cauchy problem of (2.1) with the initial data (v_0, v_1) by $\tilde{S}(t; \{v_0, v_1\})$, and then, by the Duhamel principle, we see that

$$\chi_2\overline{w}(t) = \int_0^t \tilde{S}(t-s; \{0, h(s)\})\,ds.$$

Applying Lemma 2.2 to the function $\chi_2\overline{w}(t)$, we have from (3.10) that

$$\begin{aligned} \|w(t)\|_{H^1(\Omega_{r+3}^c)} &\le \|\chi_2\overline{w}(t)\|_{H^1(\mathbb{R}^2)} \le \int_0^t \|\tilde{S}(t-s;\{0,h(s)\})\|_{H^1(\mathbb{R}^2)}\,ds \\ &\le C\int_0^t (1+t-s)^{-1/2}(\|h(s)\|_{L^2(\mathbb{R}^2)} + \|h(s)\|_{L^1(\mathbb{R}^2)})\,ds \\ &\le C\int_0^t (1+t-s)^{-1/2}\|w(s)\|_{H^1(\Omega_{r+3})}\,ds \\ &\le Cd_2\int_0^t (1+t-s)^{-1/2}(1+s)^{-1}\,ds \\ &\le Cd_2(1+t)^{-1/2}\log(2+t). \end{aligned} \tag{3.12}$$

Therefore, from (3.7), (3.9), (3.10), and (3.12) we obtain

$$\begin{aligned} \|u_\chi(t)\|_{H^1(\Omega)} &\le C\|v(t)\|_{H^1(\mathbb{R}^2)} + C\|w(t)\|_{H^1(\Omega_{r+3})} + C\|w(t)\|_{H^1(\Omega_{r+3}^c)} \\ &\le Cd_2(1+t)^{-1/2}\log(2+t), \end{aligned}$$

which is the desired estimate (3.5).

By the similar way, applying Lemma 2.3 to the function $\chi_3\overline{w}(t)$, we have from (3.10) that

$$\begin{aligned}\|w(t)\|_{L^1(\Omega^c_{r+3})} &\le \|\chi_2\overline{w}(t)\|_{L^1(\mathbb{R}^2)} \le C\int_0^t \|\tilde{S}(t-s;\{0,h(s)\})\|_{L^1(\mathbb{R}^2)}\,ds \\ &\le C\int_0^t \|h(s)\|_{L^1(\mathbb{R}^2)}\,ds \le C\int_0^t \|w(s)\|_{H^1(\Omega_{r+3})}\,ds \\ &\le Cd_2\int_0^t (1+s)^{-1}\,ds \le Cd_2\log(2+t)\,. \end{aligned} \tag{3.13}$$

Therefore, from (3.7), (3.9), (3.11), and (3.13) we obtain

$$\begin{aligned}\|u_\chi(t)\|_{L^1(\Omega)} &\le \|v(t)\|_{L^1(\mathbb{R}^2)} + C\|w(t)\|_{L^1(\Omega_{r+3})} + C\|w(t)\|_{L^1(\Omega^c_{r+3})} \\ &\le Cd_2\log(2+t)\,,\end{aligned}$$

which is the desired estimate (3.6). □

Proposition 3.3 *Under the assumption of Theorem 1.1, the function* $U(t) = u(t) - u_\chi(t)$ *satisfies*

$$\|U(t)\|_{H^1(\Omega)} = \|u(t)-u_\chi(t)\|_{H^1(\Omega)} \le Cd_1(1+t)^{-1/2}\,, \tag{3.14}$$

$$\|U(t)\|_{L^1(\Omega)} = \|u(t)-u_\chi(t)\|_{L^1(\Omega)} \le Cd_1(1+t)^{-1/2} \tag{3.15}$$

for $t \ge 0$, *where* d_1 *is the quantity given by* (1.2).

Proof. It is easy to see that the function $U(t) = u(t) - u_\chi(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)U = 0 \quad \text{in } \Omega\times(0,\infty) \\ (U,\partial_t U)\big|_{t=0} = ((1-\chi)u_0,(1-\chi)u_1) \quad \text{and} \quad U\big|_{\partial\Omega} = 0\,, \end{cases}$$

and then, by Lemma 2.1 again, we observe that

$$\|U(t)\|_{H^1(\Omega_{r+3})} \le C(1+t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})\,, \tag{3.16}$$

and also,

$$\|U(t)\|_{L^1(\Omega_{r+3})} \le C(1+t(\log t)^2)^{-1}(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)})\,. \tag{3.17}$$

Moreover, we see that the function $\chi_2\overline{U}(t)$ satisfies

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)(\chi_2\overline{U}) = f \quad \text{in } \mathbb{R}^2\times(0,\infty) \\ (\chi_2\overline{U},\partial_t\chi_2\overline{U})\big|_{t=0} = (0,0)\,, \end{cases}$$

where $f = -2\nabla\chi_2 \cdot \nabla\overline{U} - \Delta\chi_2 \cdot \overline{U}$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^2 \,|\, r+2 \le |x| \le r+3\}$, and also, it follows

$$\chi_2\overline{U}(t) = \int_0^t \tilde{S}(t-s; \{0, f(s)\})\, ds\,.$$

Applying Lemma 2.2 to the function $\chi_2\overline{U}(t)$, we have from (3.16) that

$$\begin{aligned}
\|U(t)\|_{H^1(\Omega^c_{r+3})} &\le \|\chi_2\overline{U}(t)\|_{H^1(\mathbb{R}^2)} \le \int_0^t \|\tilde{S}(t-s;\{0,f(s)\})\|_{H^1(\mathbb{R}^2)}\, ds \\
&\le C\int_0^t (1+t-s)^{-1/2}(\|f(s)\|_{L^2(\mathbb{R}^2)} + \|f(s)\|_{L^1(\mathbb{R}^2)})\, ds \\
&\le C\int_0^t (1+t-s)^{-1/2}\|f(s)\|_{L^2(B_{r+3}\setminus B_{r+2})}\, ds \\
&\le C\int_0^t (1+t-s)^{-1/2}\|U(s)\|_{H^1(\Omega_{r+3})}\, ds \\
&\le C\int_0^t (1+t-s)^{-1/2}(1+s(\log s)^2)^{-1}\, ds \le Cd_1(1+t)^{-1/2}\,. \qquad (3.18)
\end{aligned}$$

Therefore, we know that (3.14) follows from (3.16) and (3.18).

By the similar way, applying Lemma 2.3 to the function $\chi_2\overline{U}(t)$, we have from (3.16) that

$$\begin{aligned}
\|U(t)\|_{L^1(\Omega^c_{r+3})} &\le \|\chi_2\overline{U}(t)\|_{L^1(\mathbb{R}^2)} \le \int_0^t \|\tilde{S}(t-s;\{0,f(s)\})\|_{L^1(\mathbb{R}^2)}\, ds \\
&\le C\int_0^t \|f(s)\|_{L^1(\mathbb{R}^2)}\, ds \le Cd_1(1+t)^{-1/2}\,. \qquad (3.19)
\end{aligned}$$

Therefore, we know that (3.15) follows from (3.17) and (3.19). □

Proof of Theorem 3.1. Summing up the above estimates (3.5), (3.14), and (3.6), (3.15) together with (3.3), we obtain (3.1) and (3.2), respectively. □

4 Energy estimates

In this section we will derive the energy and second energy estimates for (1.1) by using the energy method. For simplicity, we often use $\|\cdot\|$ as the L^2 norm, that is, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Theorem 4.1 *Using the assumption of Theorem 1.1, the solution $u(t)$ of (1.1) satisfies that*

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \le Cd_2(1+t)^{-1}\log(2+t) \qquad (4.1)$$

for $t \ge 0$, where d_2 is the quantity given by (1.7).

Proof. We denote the total energy for (1.1) by

$$E(t) = E_1(t) \equiv \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 ,$$

which has the energy identity

$$\frac{d}{dt}E(t) + \|\partial_t u(t)\|^2 = 0 \tag{4.2}$$

or

$$E(t) + \int_0^t \|\partial_t u(s)\|^2 \, ds = E(0) . \tag{4.3}$$

Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2 + (u(t), \partial_t u(t))\right) + \|\nabla u(t)\|^2 - \|\partial_t u(t)\|^2 = 0 , \tag{4.4}$$

and then, integrating it in time,

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds \\
&\leq \frac{1}{2}\|u_0\|^2 + \|u_0\|\|u_1\| + \|u(t)\|\|\partial_t u(t)\| + \int_0^t \|\partial_t u(s)\|^2 \, ds \\
&\leq Cd_0^2 + \frac{1}{4}\|u(t)\|^2 + \|\partial_t u(t)\|^2 + \int_0^t \|\partial_t u(s)\|^2 \, ds
\end{aligned}$$

with $d_0 = \|u_0\|_{H^1(\Omega)} + \|u_1\|$, and hence, from (4.3) we obtain

$$\begin{aligned}
&\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds \\
&\leq Cd_0^2 + C\|\partial_t u(t)\|^2 + C\int_0^t \|\partial_t u(s)\|^2 \, ds \leq Cd_0^2 .
\end{aligned} \tag{4.5}$$

Thus, from (4.3) and (4.5) we have

$$\int_0^t E(s) \, ds \leq Cd_0^2 . \tag{4.6}$$

For $m \geq 1$, we observe from (4.3) and (4.4) that

$$\frac{d}{dt}t^m E(t) + t^m\|\partial_t u(t)\|^2 = mt^{m-1}E(t) \tag{4.7}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}t^m\|u(t)\|^2 + t^m(u(t), \partial_t u(t))\right) + t^m\|\nabla u(t)\|^2$$
$$= \frac{m}{2}t^{m-1}\|u(t)\|^2 + mt^{m-1}(u(t), \partial_t u(t)) + t^m\|\partial_t u(t)\|^2\,, \qquad (4.8)$$

respectively, and moreover, integrating (4.7) and (4.8) in time, we have that

$$t^m E(t) + \int_0^t s^m\|\partial_t u(s)\|^2\,ds = m\int_0^t s^{m-1}E(s)\,ds \qquad (4.9)$$

and

$$\frac{1}{2}t^m\|u(t)\|^2 + \int_0^t s^m\|\nabla u(s)\|^2\,ds$$
$$\leq \frac{1}{4}t^m\|u(s)\|^2 + t^m\|\partial_t u(t)\|^2 + m\int_0^t s^{m-1}\|u(s)\|^2\,ds$$
$$+ \int_0^t (ms^{m-1} + s^m)\|\partial_t u(s)\|^2\,ds\,, \qquad (4.10)$$

respectively, where we used the Young inequality at the last inequality.

Then, we obtain from (4.9) for $m = 1$ together with (4.6) that

$$tE(t) + \int_0^t s\|\partial_t u(s)\|^2\,ds = \int_0^t E(s)\,ds \leq Cd_0^2\,, \qquad (4.11)$$

and from (4.10) for $m = 1$ together with (4.11) and (4.3),

$$t\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\,ds$$
$$\leq Ct\|\partial_t u(t)\|^2 + C\int_0^t \|u(s)\|^2\,ds + C\int_0^t (1+s)\|\partial_t u(s)\|^2\,ds$$
$$\leq Cd_0^2 + C\int_0^t \|u(s)\|^2\,ds \leq Cd_2^2(\log(2+t))^3\,, \qquad (4.12)$$

where we used (3.1) at the last inequality. Thus, from (4.11) and (4.12) we have

$$\int_0^t sE(s)\,ds \leq Cd_2^2(\log(2+t))^3\,. \qquad (4.13)$$

Moreover, from (4.9) for $m = 2$ together with (4.13) we have

$$t^2E(t) + \int_0^t s^2\|\partial_t u(s)\|^2\,ds = 2\int_0^t sE(s)\,ds \leq Cd_2^2(\log(2+t))^3\,, \qquad (4.14)$$

and from (4.10) for $m = 2$ together with (4.14), (4.11), and (3.1),

$$\begin{aligned} & t^2\|u(t)\|^2 + \int_0^t s^2\|\nabla u(s)\|^2\,ds \\ & \le Ct^2\|\partial_t u(t)\|^2 + C\int_0^t s\|u(s)\|^2\,ds + \int_0^t (s+s^2)\|\partial_t u(s)\|^2\,ds \\ & \le Cd_2^2(\log(2+t))^3 + C\int_0^t s\|u(s)\|^2\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,, \end{aligned} \tag{4.15}$$

where we used (3.1) at the last inequality, and hence, from (4.14) and (4.15) we have

$$\int_0^t s^2 E(s)\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,. \tag{4.16}$$

Thus, from (4.9) for $m = 3$ together with (4.16) we have

$$t^3E(t) + \int_0^t s^3\|\partial_t u(s)\|^2\,ds \le Cd_2^2(1+t)(\log(2+t))^2\,, \tag{4.17}$$

and hence, the desired decay estimate (4.1) follows from (4.3) and (4.17). □

Moreover, by using the energy method again, we have the following estimates.

Theorem 4.2 *Using the assumption of Theorem 1.1, the solution $u(t)$ of (1.1) satisfies that*

$$\|\partial_t^2 u(t)\|_{L^2(\Omega)} + \|\partial_t\nabla u(t)\|_{L^2(\Omega)} \le Cd_2(1+t)^{-2}\log(2+t)\,, \tag{4.18}$$

$$\|\partial_t u(t)\|_{H^1(\Omega)} + \|\nabla u(t)\|_{H^1(\Omega)} \le Cd_2(1+t)^{-1}\log(2+t)\,, \tag{4.19}$$

$$\|u(t)\|_{H^2(\Omega)} \le Cd_2(1+t)^{-1/2}\log(2+t) \tag{4.20}$$

for $t \ge 0$, where d_2 is the quantity given by (1.7).

Proof. We will carry out the similar way as the proof the Theorem 4.1. Put $V(t) = \partial_t u(t)$ and

$$E_2(t) \equiv \frac{1}{2}\|\partial_t V(t)\|^2 + \frac{1}{2}\|\nabla V(t)\|^2 = \frac{1}{2}\|\partial_t^2 u(t)\|^2 + \frac{1}{2}\|\partial_t\nabla u(t)\|^2\,.$$

Then, we see that the function $V(t)$ satisfies that

$$(\partial_t^2 + \partial_t - \Delta)V = 0 \quad \text{in } \Omega\times(0,\infty)$$

with $V\big|_{\partial\Omega} = \partial_t u\big|_{\partial\Omega} = 0$, and

$$\frac{d}{dt}E_2(t) + \|\partial_t V(t)\|^2 = 0 \tag{4.21}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\|V(t)\|^2 + (V(t), \partial_t V(t))\right) + \|\nabla V(t)\|^2 - \|\partial_t V(t)\|^2 = 0\,. \tag{4.22}$$

Thus, from (4.21) and (4.22) we have that

$$E_2(t) + \int_0^t \|\partial_t V(s)\|^2\, ds = E_2(0) \quad \text{and} \quad \|V(t)\|^2 + \int_0^t \|\nabla V(s)\|^2\, ds \le C d_2^2\,,$$

respectively, and hence, we obtain

$$\int_0^t E_2(s)\, ds \le C d_2^2\,. \tag{4.23}$$

For $m \ge 1$, we observe from (4.21) and (4.22) that

$$\frac{d}{dt} t^m E_2(t) + t^m \|\partial_t V(t)\|^2 = m t^{m-1} E_2(t) \tag{4.24}$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2} t^m \|V(t)\|^2 + t^m (V(t), \partial_t V(t))\right) + t^m \|\nabla V(t)\|^2 \\ &\quad = \frac{m}{2} t^{m-1} \|V(t)\|^2 + m t^{m-1} (V(t), \partial_t V(t)) + t^m \|\partial_t V(t)\|^2\,, \end{aligned} \tag{4.25}$$

respectively, and moreover, integrating (4.24) and (4.25) in time like (4.9) and (4.10), we have

$$t^m E_2(t) + \int_0^t s^m \|\partial_t V(s)\|^2\, ds = m \int_0^t s^{m-1} E_2(s)\, ds \tag{4.26}$$

and

$$\begin{aligned} \frac{1}{2} t^m \|V(t)\|^2 + \int_0^t s^m \|\nabla V(s)\|^2\, ds &\le \frac{1}{4} t^m \|V(t)\|^2 + t^m \|\partial_t V(t)\|^2 \\ &\quad + m \int_0^t s^{m-1} \|V(s)\|^2\, ds + \int_0^t (m s^{m-1} + s^m) \|\partial_t V(s)\|^2\, ds \end{aligned}$$

or

$$\begin{aligned} &t^m \|V(t)\|^2 + \int_0^t s^m \|\nabla V(s)\|^2\, ds \\ &\quad \le C \int_0^t (1+s)^{m-1} E_2(s)\, ds + C \int_0^t s^{m-1} \|\partial_t u(s)\|^2\, ds\,, \end{aligned} \tag{4.27}$$

respectively. Thus, from (4.26) and (4.27) for $m = 1$ together with (4.23) and (4.3) we observe that

$$
\begin{aligned}
&tE_2(t) + \int_0^t s\|\partial_t V(s)\|^2\,ds \le Cd_2^2\,,\\
&t\|V(t)\|^2 + \int_0^t s\|\nabla V(s)\|^2\,ds \le Cd_2^2 + C\int_0^t \|\partial_t u(s)\|^2\,ds \le Cd_2^2\,,
\end{aligned}
$$

and

$$
\int_0^t sE_2(s)\,ds \le Cd_2^2\,. \tag{4.28}
$$

From (4.26) and (4.27) for $m = 2$ together with (4.23), (4.28), and (4.11) we observe that

$$
\begin{aligned}
&t^2E_2(t) + \int_0^t s^2\|\partial_t V(s)\|^2\,ds \le Cd_2^2\,,\\
&t^2\|V(t)\|^2 + \int_0^t s^2\|\nabla V(s)\|^2\,ds\\
&\qquad \le Cd_2^2 + C\int_0^t s\|\partial_t u(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,
\end{aligned}
$$

and

$$
\int_0^t s^2E_2(s)\,ds \le Cd_2^2(\log(2+t))^3\,. \tag{4.29}
$$

From (4.26) and (4.27) for $m = 3$ together with (4.23), (4.29), and (4.14) we observe that

$$
\begin{aligned}
&t^3E_2(t) + \int_0^t s^3\|\partial_t V(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,\\
&t^3\|V(t)\|^2 + \int_0^t s^3\|\nabla V(s)\|^2\,ds\\
&\qquad \le Cd_2^2(\log(2+t))^3 + C\int_0^t s^2\|\partial_t u(s)\|^2\,ds \le Cd_2^2(\log(2+t))^3\,,
\end{aligned}
$$

and

$$
\int_0^t s^3E_2(s)\,ds \le Cd_2^2(\log(2+t))^3\,. \tag{4.30}
$$

From (4.26) and (4.27) for $m = 4$ together with (4.23), (4.30), and (4.17) we observe that

$$t^4 E_2(t) + \int_0^t s^4 \|\partial_t V(s)\|^2 \, ds \le C d_2^2 (\log(2+t))^3 \,,$$

$$\begin{aligned} & t^4 \|V(t)\|^2 + \int_0^t s^4 \|\nabla V(s)\|^2 \, ds \\ & \qquad \le C d_2^2 (\log(2+t))^3 + C \int_0^t s^3 \|\partial_t u(s)\|^2 \, ds \le C d_2^2 (1+t)(\log(2+t))^2 \,, \end{aligned}$$

and

$$\int_0^t s^4 E_2(s) \, ds \le C d_2^2 (1+t)(\log(2+t))^2 \,. \tag{4.31}$$

Therefore, from (4.26) for $m = 5$ and (4.31) we obtain that

$$t^5 E_2(t) + \int_0^t s^5 \|\partial_t V(s)\|^2 \, ds \le C d_2^2 (1+t)(\log(2+t))^2$$

or

$$\|\partial_t^2 u(t)\| + \|\partial_t \nabla u(t)\| \le C d_2 (1+t)^{-2} \log(2+t) \,. \tag{4.32}$$

Moreover, by the elliptic regularity theorem in exterior domains (see [5], [22]) together with (1.1), (4.1), and (4.32) that

$$\begin{aligned} \|\nabla u(t)\|_{H^1(\Omega)} & \le C \|\Delta u(t)\| + C \|\nabla u(t)\| \\ & \le C \|\partial_t u(t)\| + C \|\partial_t^2 u(t)\| + C \|\nabla u(t)\| \\ & \le C d_2 (1+t)^{-1} \log(2+t) \,, \end{aligned} \tag{4.33}$$

and hence, the desired estimates (4.18)–(4.20) follows from (3.1), (4.32), and (4.33). □

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