J. Math. Univ. Tokushima
Vol. 41 (2007), 1–11

Pandiagonal Constant Sum Matrices

By

Toru ISHIHARA

Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minamijosanjima, Tokushima 770-8502, JAPAN
e-mail address: ishihara@ias.tokushima-u.ac.jp
(Received September 28, 2007)

Abstract

In the present paper, we study square matrices in which the sum of elements in any row, in any column , in any extended diagonal add up to a constant. We call such a matrix a pandiagonal constant sum matrix. We will show that the number of independents elements in a pandiagonal constant sum matrix of order n is $n^2 - 4n + 3$ if n is odd or $n^2 - 4n + 4$ if n is even.

2000 Mathematics Subject Classification. 05B20

Introduction

Let Σ be a a set of n different elements. A latin square of order n is a square matrix with n entries of elements in Σ, none of them occurring twice within any row or column of the matrix. A *matrix of the same number* n is defined to be a square matrix with n^2 entries of n different elements, each appeared exactly n times. A latin square of order n is a matrix of the same number n. A magic square of order n is an arrangement of n^2 consecutive integers in a square, such that the sums of each row each column and each of the main diagonal are the same. If also the sum of each extended diagonal is the same, the magic square is called pandiagonal. Two latin squares $A = (a_{ij}$ and $B = (b_{ij})$ of order n are said to be orthogonal if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) . We can define that two matrices of the same number n are orthogonal similarly. Let $A = (a_{ij})$, $a_{ij} \in R$@be a square matrix of order n. It is called a *constant sum matrix* if the sums of each row and each column are the same. If moreover the sum of each main diagonal is the same , it is called a*diagonal constant sum matrix* and if the sum of each extended diagonal is the same, it is called a *pandiagonal constant sum matrix.* In the present paper, we take $0, 1, \cdots, n-1$ as n consecutive integers and put

$\Sigma = \{0, 1, \cdots, n-1\}$. A pandiagonal latin square on Σ is a pandaiagonal constant sum matrix of the same number n.

Let A and B are orthogonal matrices of the same number n. Put $C = nA + B$. Then it is known [3] that if A and B are diagonal(resp. pandiagonal) constant sum matrices, C is a magic (resp. pandiagonal magic) square.

1. Pandiagonal constant sum matrices

Let $A = (a_{ij}),\ a_{ij} \in R$ be a pandiagonal constant sum matrix of order n. In the present paper, subscripts have the range $0, 1, \cdot, n-1\ \ (mod\ \ n)$. Then we have the following equations.

$$\sum_{i=0}^{n-1} a_{ij} = C,\ \ 0 \le j \le n-1, \tag{1}$$

$$\sum_{j=0}^{n-1} a_{ij} = C,\ \ 1 \le i \le n-1, \tag{2}$$

$$\sum_{i=0}^{n-1} a_{in+j-i} = C,\ \ 1 \le j \le n-1, \tag{3}$$

$$\sum_{i=0}^{n-1} a_{ii+j} = C,\ \ 1 \le j \le n-1, \tag{4}$$

where C is a constant. Notice that from (1) and (2), (3),(4) it follows

$$\sum_{j=0}^{n} a_{0j} = C,$$

$$\sum_{i=0}^{n} a_{in-i} = C,$$

$$\sum_{i=0}^{n} a_{ii} = C.$$

When n is even, that is, $n = 2m$, there is a redundant equation in (3) and (4). In fact, if we set $2j = 2k + 2i\ \ \ (mod\ \ 2m)$, we have

$$\sum_{j=0}^{m-1} \sum_{i=0}^{2m-1} a_{i2m+1+2j-i} = \sum_{j=0}^{m-1} \sum_{i=0}^{2m-1} a_{i1+2j-i} = \sum_{k=0}^{m-1} \sum_{i=0}^{2m-1} a_{i1+2k+i} = mC.$$

Hence, we can consider the equation

$$\sum_{i=0}^{2m-1} a_{i1+i} = C$$

is redundant. Now, when $n = 2m$, we set

$$\begin{aligned}
\sum_{i=0}^{2m-1} a_{ij} &= C, \; 0 \leq j \leq 2m-1, && (5)\\
\sum_{j=0}^{2m-1} a_{ij} &= C, \; 1 \leq i \leq 2m-1, && (6)\\
\sum_{i=0}^{2m-1} a_{i2m+j-i} &= C, \; 1 \leq j \leq 2m-1, && (7)\\
\sum_{i=0}^{2m-1} a_{ii+j} &= C, \; 2 \leq j \leq 2m-1. && (8)
\end{aligned}$$

Theorem 1 When n is an odd number, the equations (1),(2),(3) and (4) are independent, and when $n = 2m$, the equations (5),(6),(7) and (8) are independent.

Proof. Set

$$x_i = a_{0i}, \quad y_i = a_{1i}, \quad z_i = a_{2i}, \quad 0 \leq i \leq n-1.$$

Then we have

$$\begin{aligned}
A_j : x_j + y_j + k_j \sum_{i=3}^{n-1} a_{ij} &= C, \; 0 \leq j \leq n-1,\\
B_j : x_j + y_{j-1} + z_{j-2} + \sum_{i=0}^{n-1} a_{ij-i} &= C, \; 1 \leq j \leq n-1,\\
D_1 : \sum_{j=0}^{n-1} y_j &= C,\\
D_2 : \sum_{j=0}^{n-1} z_j &= C,\\
D_j : \sum_{i=0}^{n-1} a_{ji} &= C, \; 3 \leq j \leq n-1.
\end{aligned}$$

(1) Now suppose that n is odd, that is $n = 2m+1$. Then it holds

$$C_j : x_j + y_{j+1} + z_{j+2} + \sum_{i=3}^{n-1} a_{ii+j} = C, \ \ 1 \le j \le n-1.$$

Now we represent simply the above equations as

$$\begin{aligned}
A_j &= (x_j, y_j, z_j, *), \ \ 0 \le j \le n-1, \\
B_j &= (x_j, y_{j-1}, z_{j-2}, *), \ \ 1 \le j \le n-1, \\
C_j &= (x_j, y_{j+1}, z_{j+2}, *), \ \ 1 \le j \le n-1 \\
D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\
D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\
D_j &= (0,0,0,*), \ \ 3 \le j \le n-1.
\end{aligned}$$

Put

$$\begin{aligned}
B_j(1) &= B_j - A_j = (0, y_{j-1} - y_j, z_{j-2} - z_j, *), \ \ 1 \le j \le n-1 \quad (9) \\
B_{n-1}(2) &= B_{n-1}(1) = (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), \quad (10) \\
B_j(2) &= B_j(1) + B_{j+1}(2) \\
&= (0, y_{j-1} - y_{n-1}, z_{j-2} + z_{j-1} - z_{n-2} - z_{n-1}, *), \quad (11)
\end{aligned}$$

$1 \le j \le n-2$.

Especially, we have

$$B_1(2) = (0, y_0 - y_{n-1}, z_0 - z_{n-2}.*)$$

Next, we get

$$\begin{aligned}
C_j(1) &= C_j - A_j = (0, -y_j + y_{j+1}, -z_j + z_{j+2}, *), \ \ 1 \le j \le n-1, \\
C_j(2) &= C_j(1) + B_{j+1}(1) = (0, 0, z_j - z_{j+1} - z_{j+2} + z_{j+3}, *), \ \ 1 \le j \le n-2, \\
C_{n-1}(2) &= C_{n-1}(1) - B_1(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}$$

Now, set

$$\begin{aligned}
C_j(3) &= C_j(2) + C_{j+1}(2) = (0, 0, z_{j-1} - 2z_{j+1} + z_{j+3}, *), \ \ 1 \le j \le n-2, \\
C_{n-1}(3) &= C_{n-1}(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}$$

Put

$$C_{2m-1}(4) = C_{2m-1}(3), \quad C_{2m-3}(4) = C_{2m-3}(3) + 2C_{2m-1}(4),$$

$$C_{2(m-k)+1}(4) = C_{2(m-k)+1}(3) + 2C_{2(m-k+1)+1}(4) - C_{2(m-k+2)+1}(4), \quad 3 \leq k \leq m.$$

Then we have

$$C_{2(m-k)+1}(4) = (0, 0, z_{2(m-k)} - (k+1)z_{2m} + kz_1, *), \quad 1 \leq k \leq m.$$

Next, we put

$$C_{2m}(4) = C_{2m}(3), \quad C_{2m-2}(4) = C_{2m-2}(3) + 2C_{2m}(4),$$

$$C_{2(m-k)}(4) = C_{2(m-k)}(3) + 2C_{2(m-k+1)}(4) - C_{2(m-k+2)}(4), \quad 2 \leq k \leq m-1.$$

Then, we get

$$C_{2(m-k)}(4) = (0, 0, z_{2(m-k)-1} - (k+1)(z_{2m} - z_1) - z_0, *), \quad 0 \leq k \leq m-1.$$

Set

$$\begin{aligned} C_2(5) &= \frac{1}{2m+1}(C_2(4) + C_1(4)) = (0, 0, z_1 - z_{2m}, *), \\ C_{2(m-k)+1}(5) &= C_{2(m-k)+1}(4) - kC_2(5), \quad 1 \leq k \leq m, \\ C_{2(m-k)}(5) &= C_{2(m-k)}(4) - (k+1)C_2(5) + C_1(5), \quad 0 \leq k \leq m-2. \end{aligned}$$

Thus we obtain

$$C_j(5) = (0, 0, z_{j-1} - z_{2m}, *), \quad 1 \leq j \leq n-1 = 2m.$$

Now the equations

$$\begin{aligned} A_j &= (x_j, y_j, z_j, *), \ \ 0 \leq j \leq n-1, \\ B_j(2) &= (, y_{j-1} - y_{n-1}, z_{j-2} + -z_{j-1} - z_{n-2} - z_{n-1}, *), \ \ 1 \leq j \leq n-2, \\ B_{n-1}(2) &= (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), \\ D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\ C_j(5) &= (0, 0, z_{j-1} - z_{2m}, *), \ \ 1 \leq j \leq n-1, \\ D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\ D_j &= (0, 0, 0, *), \ \ 3 \leq j \leq n-1 \end{aligned}$$

are equivalent to the equations given at first. It is evident that the rank of the coefficient matrix of the equations is $4n-3$. Hence, these equations are independent.

(2) Suppose that n is even, that is, $n = 2m$. By using the similar notations, we consider the following $4n-4$ equations

$$\begin{aligned}
A_j &= (x_j, y_j, z_j, *), \ \ 0 \leq j \leq n-1, \\
B_j &= (x_j, y_{j-1}, z_{j-2}, *), \ \ 1 \leq j \leq n-1, \\
C_j &= (x_{j+1}, y_{j+2}, z_{j+3}, *), \ \ 1 \leq j \leq n-2 \\
.D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\
D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\
D_j &= (0, 0, 0, *), \ \ 3 \leq j \leq n-1.
\end{aligned}$$

We define $B_j(1), B_j(2), \ 1 \leq j \leq n-1$ as similarly as in (9),(10),(11).
We put

$$\begin{aligned}
C_j(1) &= C_j - A_{j+1} = (0, -y_{j+1} + y_{j+2}, -z_{j+1} + z_{j+3}, *), \ \ 1 \leq j \leq n-2, \\
C_j(2) &= C_j(1) + B_{j+2}(1) = (0, 0, z_j - z_{j+1} - z_{j+2} + z_{j+3}, *), \ \ 1 \leq j \leq n-3, \\
C_{n-2}(2) &= C_{n-2}(1) - B_1(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}$$

Now, set

$$\begin{aligned}
C_j(3) &= C_j(2) + C_{j+1}(2) = (0, 0, z_j - 2z_{j+2} + z_{j+4}, *), \ \ 1 \leq j \leq n-2, \\
C_{n-2}(3) &= C_{n-2}(2) = (0, 0, z_{n-2} - z_{n-1} - z_0 + z_1, *).
\end{aligned}$$

Put

$$C_{2m-2}(4) = C_{2m-2}(3), \quad C_{2m-4}(4) = C_{2m-4}(3) + 2C_{2m-2}(4)$$

$$C_{2(m-k)}(4) = C_{2(m-k)}(3) + 2C_{2(m-k+1)}(4) - C_{2(m-k+2)}, \quad 3 \leq j \leq m-1.$$

Then we have

$$C_{2(m-k)}(4) = (0, 0, z_{2(m-k)} - k(z_{2m-1} - z_1) - z_0, *), \quad 1 \leq k \leq m-1.$$

Next, we put

$$C_{2m-3}(4) = C_{2m-3}(3), \quad C_{2m-5}(4) = C_{2m-5}(3) + 2C_{2m-3}(4),$$

$$C_{2(m-k)-1}(4) = C_{2(m-k)-1}(3) + 2C_{2(m-k+1)-1}(4) - C_{2(m-k+2)-1}, \quad 3 \leq k \leq m-1.$$

Then we get

$$C_{2(m-k)-1}(4) = (0, 0, z_{2(m-k)-1} - (k+1)z_{2m-1} + kz_1, *) \quad 1 \le k \le m-1.$$

Set

$$C_1(5) = \frac{1}{m} C_1(4) = (0, 0, z_1 - z_{2m-1}, *),$$

$$C_{2(m-k)}(5) = C_{2(m-k)}(4) - kC_1(5) = (0, 0, z_{2(m-k)} - z_0, *), \quad 1 \le k \le m-1,$$

$$C_{2(m-k)-1}(5) = C_{2(m-k)-1}(4) - kC_1(5) = (0, 0, z_{2(m-k)-1} - z_{2m-1}, *), \quad 1 \le k \le m-1.$$

Now the equations

$$\begin{aligned}
A_j &= (x_j, y_j, z_j, *), \ 0 \le j \le n-1, \\
B_j(2) &= (, y_{j-1} - y_{n-1}, z_{j-2} + -z_{j-1} - z_{n-2} - z_{n-1}, *), \ 1 \le j \le n-2, \\
B_{n-1}(2) &= (0, y_{n-2} - y_{n-1}, z_{n-3} - z_{n-1}, *), \\
D_1 &= (0, \sum_{j=0}^{n-1} y_j, 0, *), \\
C_1(5) &= (0, 0, z_1 - z_{2m-1}, *), \\
C_{2(m-k)}(5) &= (0, 0, z_{2(m-k)} - z_0, *), \quad 1 \le k \le m-1, \\
C_{2(m-k)-1}(5) &= (0, 0, z_{2(m-k)-1} - z_{2m-1}, *), \quad 1 \le k \le m-1, \\
D_2 &= (0, 0, \sum_{j=0}^{n-1} z_j, *), \\
D_j &= (0, 0, 0, *), \ 3 \le j \le n-1
\end{aligned}$$

are equivalent to the equations given at first. It is evident that the rank of the coefficient matrix of the equations is $4n - 4$. Hence, these equations are independent.

2. Pandiagonal zero sum matrices

Let $A = (a_{ij})$, $a_{ij} \in R$ be a pandiagonal constant sum matrix of order n with constant C. In this section, we have from here on , subtracted S/n from every elements in the matrix so that the sum of the elements in any row, column or diagonal will be zero, and we call such modified a *zero-sum matrix.* The results in this section mainly owe to W. R. Andress [1]. We introduce operators R, C such that

$$Ra_{i,j} = a_{i+1,j} \quad Ca_{i,j} = a_{i,j+1}.$$

Set

$$L_n(R) = \sum_{i=0}^{n-1} R^i, \quad D_n(R,C) = \sum_{i=0}^{n-1} R^{n-1-i}C^i.$$

Then we have

$$\begin{aligned} \text{columm} : L_n(R)a_{i,j} &= 0, \\ \text{row} : L_n(C)a_{i,j} &= 0, \\ \text{diagonal} : L_n(RC)a_{i,j} &= 0, \\ \text{diagonal} : D_n(R,C)a_{i,j} &= 0. \end{aligned}$$

Lemma Let $Q_{i,j}$ be elements of a square matrix of order n. If any one of the three conditions (1) $(C-1)Q_{i,j} = 0, \sum_{j=0}^{n-1} Q_{i,j} = 0$, (2) $(R-1)Q_{i,j} = 0, \sum_{i=0}^{n-1} Q_{i,j} = 0$, (3) $(R-C)Q_{i,j} = 0, \sum_{i=0}^{n-1} Q_{i,i} = 0$ holds, it follows that $Q_{i,j} = 0$.

Proof. Assume that the condition (1) holds. Then $Q_{i,j} = 0$ is independent of column. Hence using the second equation of (1), we get $Q_{i,j} = 0$. The other results follow similarly.

From $(L_n(RC) - L_n(R))a_{i,j} = R(C-1)\sum_{i=1}^{n-1} R^{i-1}L_{i-1}(C)a_{i,j} = 0$, using Lemma, we get $\sum_{i=1}^{n-1} R^{i-1}L_{i-1}(C)a_{i,j} = 0$. Since this is true for all values of i, j, it is convenient to suppress the operand $a_{i,j}$ so that

$$\sum_{i=0}^{n-2} R^i L_i(C) = 0. \tag{12}$$

This is a triangle-invariant. We may interchange the operations R, C in the above formula so that

$$\sum_{i=0}^{n-2} C^i L_i(R) = 0. \tag{13}$$

The triangle-invariant (12) remains invariant if we replace R by $1/R$ and multiple R^{n-1} as this merely represents a reflection in a horizontal line, and give

$$\sum_{i=0}^{n-2} R^i L_{n-2-i}(C) = 0. \tag{14}$$

Put

$$S_n = L_n(R)L_n(C).$$

Then, this presents a square of order n. Subtracting (12)-(13), and justifying the removal of the factor $R - C$ by Lemma 1, we obtain the invariant

$$\sum_{i=0}^{\lfloor (n-3)/2 \rfloor} (RC)^i S_{n-2-2i} = 0. \tag{15}$$

Subtracting (14) from (15), adding $C^{-1}D_n$ and multiplying $(RC)^{-1}$, we get the invariant

$$\sum_{i=0}^{\lfloor (n-5)/2 \rfloor} (RC)^i S_{n-3-2i} + R^{n-2}C^{n-2} = 0. \tag{16}$$

Subtracting $\sum_{i=0}^{n-3} R^i L_n(R)$ from (15) and adding $(C^{n-3} + C^{n-2})L_n(R)$ gives the invariant

$$\sum_{i=0}^{\lfloor (n-5)/2 \rfloor} (RC)^i S_{n-4-2i} + (RC)^{n-3} S_2 = 0. \tag{17}$$

From now on, we assume that n is odd, that is, $n = 2m + 1$.
The triangle-invariant (12) remains invariant if we replace C by $1/C$ as this merely represents a reflection in a vertical line, and give

$$\sum_{i=0}^{n-2} R^i L_i(C^{-1}) = 0. \tag{18}$$

Subtracting this from (12) and removing $R(C - C^{\ 1})$, we get

$$\sum_{i=0}^{m-1} R^i \sum_{j=0}^{i} \lfloor (i+2-j)/2 \rfloor (C^j + C^{-j}) + \sum_{i=0}^{m-2} R^i \sum_{j=0}^{m-2-i} \lfloor (m-i-j)/2 \rfloor (C^j + C^{-j}) = 0. \tag{19}$$

Theorem 2 Let $A = (a_{i,j})$ be a pandiagonal constant sum matrix of order n. For an odd n, if $n^2 - 4n + 3$ elements $a_{i,j}, \quad 0 \le i \le n-4, \quad 0 \le j \le n-2$ are given, the other elements decided uniquely. For an even n, if $n^2 - 4n + 4$ elements $a_{i,j}, \quad 0 \le i \le n-4, \quad 0 \le j \le n-2$ and any any one of $a_{n-3,j}, \quad 0 \le j \le n-1$ are given, the other elements decided uniquely.

Proof. Using (12), we can get $a_{i,n-1}, \quad 0 \le i \le n-4$. Assume that n is an odd number. Using (19), we obtain $a_{n-3,j}, \quad 0 \le j \le n-1$ and then $a_{n-2,j}, a_{n-1,j}, \quad 0 \le j \le n-1$. Next, Let n be an even number. Using (16), we can determine $a_{n-2,j}, \quad 0 \le j \le n-1$. Then using (17), from any one of $a_{n-3,j}, \quad 0 \le j \le n-1$, we obtain $a_{n-3,j}, \quad 0 \le j \le n-1$. Now, it is easy to get $a_{n-1,j}, \quad 0 \le j \le n-1$.

3. Orthogonal matrices of the same number

A *matrix of the same number* n is a square matrix with n^2 entries of n different elements, each appeared exactly n times. Two matrices of the same number n $A = (a_{ij}$ and $B = (b_{ij})$ are defined to be orthogonal if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) . It is well known that the largest value of r for which there exist r mutually orthogonal Latin squares of order n is less than n. Now, we have

Theorem 3. Denote by $N(n)$ the largest value of r for which there exist r mutually orthogonal matrices of order n. it holds

$$N(n) \leq n + 1.$$

Proof. Suppose $A_1, \cdots, A_t$ are mutually orthogonal matrices of order n on the symbols $\{0, 1, \cdots, n-1\}$. Take an n-square matrix $S = (s_{i,j})$ whose n^2 positions are labelled $0, 1, \cdots, n^2 - 1$ as follows: $S_{i,j} = ni + j, \quad 0 \leq i \leq j, 0 \leq j \leq n - 1$. Then consider the collection of subsets $B_{r,m}$ defined by

$$B_{r,m} = \{x : x \text{ is the label in} S \text{of aposition in which } A_r \text{ has entry } m\},$$

where $1 \leq r \leq t, \;\; 0 \leq m \leq n-1$. There are tn subsets of size n. It follows from the orthogonality of the A_r that no pair of elements can occur in more than one block. Suppose for example x and y both occur in B_{r_1,m_1} and B_{r_2,m_2}. Then A_{r_1} has the same entry m_1 in x and y, while A_{r_2} has entry m_2 in these positions. Hence the pair (m_1, m_2) occurs twice, contradicting to the orthogonality of A_1 and A_2. Note the number of pairs of elements in the subsets is

$$tn\binom{n}{2} = \frac{1}{2}tn^2(n-1).$$

This number must be more than $\binom{n^2}{2}$. Hence

$$\frac{1}{2}tn^2(n-1) \leq \frac{1}{2}n^2(n^2-1)$$

gives $t \leq n + 1$.

References

[1] W. R. Andress, Basic properties of pandigonal magic squraes, Amer. Math. Monthly 67, 143-152, 1960.

[2] Y. C. Chen and C. M. Fu, Construction and enumeration of pandiagonal magic squares of order n from step method, Ars Combinatoria 48, 233-244, 1998.

[3] W. Proskurowski and A. Proskurowsk, Construction of pandiagonal magic squares from circulant pandiagonal matrices, Ars Combinatoria 15, 153-162,1983.