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Construction of Knut Vik Designs and Orthogonal Knut Vik Designs

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Abstract

Following Euler's method, A. Hedayat constructs some Knut Vik designs. We call them Knut Vik designs of Hedayat in this note. We give Knut Vik designs of Hedayat explicitly and decide when Knut Vik designs of Hedayat are mutually orthogonal.

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Introduction

Let A be a Latin square of order n, that is, an $n \times n$ array in which n distinct symbols are arranged so that each symbol occurs once in each row and column. Index its rows and columns by 1,2,. . . n. By the jth right diagonal of A we mean the following n cell of A:

$$(i, j+i-1); \qquad i = 1, 2, \cdots, n; \qquad (\text{mod } n.)$$

We define also the jth left diagonal of A to the following n cell of A:

$$(i, j-i); \qquad i = 1, 2, \cdots, n; \qquad (\text{mod } n.)$$

Let Σ be a set of n distinct symbols. If we can fill the cells of A by the elements of Σ in such a way that each row, column, right diagonal and left diagonal of A contains all the elements of Σ, we say the resulting structure a Knut Vic design, which we denote by K. It is also called a pandiagonal Latin square [1], [3]. In this paper, we set $\Sigma = \{0, 1, 2, \cdots, n-1\}$. It is well known that Knut Vic designs of order n exists if and only if n is not divisible by 2 or 3. A. Hedayat [3] showed that n is not divisible by 2 or 3 then $K = (k_{ij})$ with $k_{ij} = \lambda i + j (\text{mod } n)$ is a Knut Vic design if $\lambda, \lambda - 1, \lambda + 1$ are relatively prime to n. In this paper, we call these Knut Vic designs as Knut Vic designs of Hedayat, Let n have the prime decompsition

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

Then also he showed that there are

$$N = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} (p_1 - 3)(p_2 - 3) \cdots (p_r - 3).$$

different choices for λ. In the present note, we define a standard way which gives λ satisfing the condition that $\lambda, \lambda - 1, \lambda + 1$ are relatively prime to n. K. Afsarinejad showed that there exist at least $\min(p_i - 3)$, $(i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of order n. We show that there exist at most $\min(p_i - 3)$, $(i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of Hedayat. We also obtain that to each Knut Vik design of Hedayat, there are

$$p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} (p_1 - 4)(p_2 - 4) \cdots (p_r - 4)$$

orthogonal Knut Vik designs of Hedayat.

1. A standard construction of Knut Vik designs of Hedayat

Let n be not divisible by 2 or 3 and have the prime decomposition

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad (3 < p_1 < p_2 < \cdots < p_r).$$

If $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to n, then

$$K = (k_{ij}) \qquad \text{with} \qquad k_{ij} = \lambda i + j \pmod{n}$$

is a Knut Vik design of Hedayat. We decide explicitly when $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to n.
Put

$$m_{i_a}^a = i_a + 1, \qquad 1 \le a \le r, \quad 1 \le i_a \le p_a - 3.$$

From Chinese remainder theorem, we obtain

Lemma 1. *For each $\{i_1, i_2, \cdots, i_r\}$ with $1 \le i_1 \le p_1 - 3, 1 \le i_2 \le p_2 - 3, \cdots,$ $1 \le i_r \le p_r - 3$, there is a positive integer m satisfies*

$$m = m_{i_1}^1 \pmod{p_1}, m = m_{i_2}^2 \pmod{p_2}, \cdots, m = m_{i_r}^r \pmod{p_r}.$$

As this integer is unique on $Z_{(p_1p_2\cdots P_r)}$ *, we denote it by* $m_{i_1i_2\cdots i_r}$.

Now, we obtain

Theorem 2 *Set* $n_1 = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1}$. *For each* $\{t, i_1, i_2, \cdots, i_r\}$ *with* $0 \le t \le n_1 - 1$, $1 \le i_1 \le p_1 - 3, 1 \le i_2 \le p_2 - 3, \cdots, 1 \le i_r \le p_r - 3$, *put*

$$\lambda(t, i_1, i_2, \cdots, i_r) = p_1p_2\cdots p_r t + m_{i_1i_2\cdots i_r},$$

then these λ*'s give* N *different choices for Knut Vik designs of Hedayat.*

Following the proof of Chinese remainder theorem, we construct explicitly integers $m_{i_1i_2\cdots i_r}$ as follows. Put

$$q_1 = p_1^{-1} \pmod{P_2}, q_2 = (p_1p_2)^{-1} \pmod{P_3}, \cdots, q_{r-1} = (p_1p_2\cdots p_{r-1})^{-1} \pmod{P_r}$$

Now we get inductively,

$$\begin{aligned}
m_{i_1i_2\cdots i_r} &= m_{i_1}^1 \pmod{p_1}, \\
m_{i_1i_2\cdots i_r} &= m_{i_1}^1 + p_1s_1 = m_{i_2}^2 \pmod{p_2}, \\
s_1 &= q_1(m_{i_2}^2 - m_{i_1}^1) + s_2p_2, \\
m_{i_1i_2} &= m_{i_1}^1 + p_1q_1(m_{i_2}^2 - m_{i_1}^1), \\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2} + p_1p_2s_2 = m_{i_3}^3 \pmod{p_3}, \\
s_2 &= q_2(m_{i_3}^3 - m_{i_1i_2}) + s_3p_3, \\
m_{i_1i_2i_3} &= m_{i_1i_2} + p_1p_2q_2(m_{i_3}^3 - m_{i_1i_2}), \\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2i_3} + p_1p_2p_3s_3 = m_{i_4}^4 \pmod{p_4}, \\
&\vdots \\
m_{i_1i_2\cdots i_{r-1}} &= m_{i_1i_2\cdots i_{r-2}} + p_1p_2\cdots p_{r-2}q_{r-2}(m_{i_{r-1}}^{r-1} - m_{i_1i_2\cdots i_{r-2}}), \\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2\cdots i_{r-1}} + p_1p_2\cdots p_{r-1}s_{r-1} = m_{i_r}^r \pmod{p_r}, \\
s_{r-1} &= q_{r-1}(m_{i_r}^r - m_{i_1i_2\cdots i_{r-1}}) \pmod{(p_1p_2\cdots p_r)}, \\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2\cdots i_{r-1}} + p_1p_2\cdots p_{r-1}q_{r-1}(m_{i_r}^r - m_{i_1i_2\cdots i_{r-1}})m_{i_1i_2\cdots i_{r-1}} \pmod{(p_1p_2\cdot p_r)}.
\end{aligned}$$

Example 1. Let $n = 5 \times 7$. Then $p_1 = 5, p_2 = 7$ and $q_1 = 3 \pmod{7}$. We have

$$m_1^1 = 2, m_2^2 = 3, m_1^2 = 2, m_2^2 = 3, m_3^2 = 4, m_4^2 = 5.$$

Hence, we get

$$\begin{aligned}
m_{i_1i_2} &= m_{i_1}^1 + 15(m_{i_2}^2 - m_{i_1}^1) \pmod{35}, \\
&= i_1 + 1 + 15(i_2 - i_1) \pmod{35}, \;\; 1 \le i_1 \le 2, \;\; 1 \le i_2 \le 4.
\end{aligned}$$

Thus, we obtain

$m_{11}=2, m_{12}=17, m_{13}=32, m_{14}=12, m_{21}=23, m_{22}=3, m_{23}=18, m_{24}=33 \pmod{35}$.

Example 2. Let $n=5\times 7\times 11$. Then $p_1=5, p_2=7, p_3=11$,,
$q_1=3 \pmod 7), q_2=6 \pmod{11}$. $m^1_{i_1}, m^2_{i_2}$ are the same as in Example 1, and $m^3_{i_3}=i_3+1,\ 1\le i_3\le 8$. It is evident that $m_{i_1i_2}\ 1\le i_1\le 2,\ 1\le i_2\le 4$ are also the same as in Example 1. Now we have

$$m_{i_1i_2i_3}=m_{i_1i_2}+210(m^3_{i_3}-m_{i_1i_2}) \pmod{385}.$$

Now we write simply $m_{i_1i_2*}$ for $(i_{i_1i_2}1, i_{i_1i_2}2, \cdots, i_{i_1i_2}8)$.

$m_{11*}=(2,212,37,247,72,282,107,317),\ m_{12*}=(332,157,367,192,17,227,52,262)$,

$m_{13*}=(277,102,312,137,347,172,382,207),\ m_{14*}=(222,47,257,82,292,117,327,152)$,

$m_{21*}=(233,58,268,93,303,128,338,163),\ m_{22*}=(178,3,213,38,248,73,283,108)$,

$m_{23*}=(123,333,158,368,193,18,228,53),\ m_{24*}=(68,278,103,313,138,348,173,383)$.

Example 3. Let $n=7^2\times 11$. Then $p_1=7, p_2=11,\ q_1=8 \pmod{11}$.
$m^1_{i_1}=i_1+1, m^2_{i_2}=i_2+1,\ 1\le i_1\le 4,\ 1\le i_2\le 8$. Hence, we get

$m_{i_1i_2}=m^1_{i_1}+56(m^2_{i_2}-m^1_{i_1})=i_1+1+56(i_2-i_1) \pmod{77},\ 1\le i_1\le 4,\ 1\le i_2\le 8.$

We write m_{i_1*} for $(i_{i_1}1, i_{i_1}2, \cdots, i_{i_1}8)$.

$$m_{1*}=(2,58,37,16,72,51,30,9),\ m_{2*}=(24,3,59,38,17,73,52,31),$$

$$m_{3*}=(46,25,4,60,39,18,74,53),\ m_{4*}=(68,47,26,5,61,40,19,75).$$

Now we obtain

$$\lambda(t,i_1,i_2)=77t+m_{i_1i_2},\quad 0\le t\le 6,\ 1\le i_1\le 4,\ 1\le i_2\le 8.$$

2. Orthogonal Knut Vik designs of Hedayat

Let K_1 and K_2 be Knut Vik designs of Hedayat of order n. K_1 and K_2 are said to be *orthogonal* if they are orthogonal in the sense of Latin squares. Using the notations in Theorem 2, assume that $K_1 = (k_{ij}^{(1)})$ and $K_2 = (k_{ij}^{(2)})$ are given by

$$k_{ij}^{(1)} = \lambda(t_1, i_1^{(1)}, i_2^{(1)}, \cdots, i_r^{(1)})i + j, \quad k_{ij}^{(2)} = \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \cdots, i_r^{(2)})i + j,$$

where

$$0 \le t_1 \le n_1 - 1, 1 \le i_1^{(1)} \le p_1 - 3, 1 \le i_2^{(1)} \le p_2 - 3, \cdots, 1 \le i_r^{(1)} \le p_r - 3,$$

$$0 \le t_2 \le n_1 - 1, 1 \le i_1^{(2)} \le p_1 - 3, 1 \le i_2^{(2)} \le p_2 - 3, \cdots, 1 \le i_r^{(2)} \le p_r - 3.$$

It is known that K_1 and K_2 are orthogonal if and only if $\lambda(t_1, i_1^{(1)}, i_2^{(1)}, \cdots, i_r^{(1)}) - \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \cdots, i_r^{(2)})$ and n are relatively prime. Hence we have

Lemma 3. *Knut Vik designs of Hedayat K_1 and K_2 are orthogonal if and only if*

$$i_1^{(1)} \ne i_1^{(2)}, i_2^{(1)} \ne i_2^{(2)}, \cdots, i_r^{(1)} \ne i_r^{(2)}.$$

When K_1 and K_2 are orthogonal , we call $(K_1.K_2)$ a *orthogonal pair* in this note. From Lemma 3, it follows

Theorem 4. *For each Knut Vik design of Hedayat, there are*

$$p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1}(p_1-4)(p_2-4)\cdots(p_r-4)$$

Knut Vik designs orthogonal to it. There are

$$\frac{1}{2}p_1^{2\alpha_1-2}p_2^{2\alpha_2-2}\cdots p_r^{2\alpha_r-2}(p_1-3)(p_1-4)(p_2-3)(p_2-4)\cdots(p_r-3)(p_r-4)$$

orthogonal pairs of Knut Vik designs of Hedayat .

Let S be a set of Knut Vik designs of order n . We say S to be a set of mutually orthogonal Knut Vik designs of order n if any two Knut Vik designs in S are orthogonal. It is shown by K. Afsarinejad that there are at least $p_1 - 3$ mutually orthogonal Knut Vik designs of order n. On the other hand, by Lemma 3, there are at most $p_1 - 3$ mutually orthogonal Knut Vik designs of Hedayat. In fact, $p_1 - 3$ mutually orthogonal Knut Vik designs are given by $K_1 = (k_{ij}^{(1)}), K_2 = (k_{ij}^{(2)}), \cdots, K_{p_1-3} = (k_{ij}^{(p_1-3)})$, where

$$k_{ij}^{(1)} = \lambda(t_1, 1, i_2^{(1)}, \cdots, i_r^{(1)})i + j, \quad k_{ij}^{(2)} = \lambda(t_2, 2, i_2^{(2)}, \cdots, i_r^{(2)})i + j, \cdots,$$

$$k_{ij}^{(p_1-3)} = \lambda(t_{p_1-3}, p_1 - 3, i_2^{(p_1-3)}, \cdots, i_r^{(p_1-3)})i + j,$$

and

$$i_2^{(a)} \neq i_2^{(a)}, i_3^{(a)} \neq i_3^{(b)}, \cdots, i_r^{(a)} \neq i_r^{(b)}, \qquad \text{for any } a \neq b.$$

Thus, we obtain

Theorem 5. *The maximum number of mutually orthogonal Knut Vik designs of Hedayat is* $p_1 - 3$. *There are*

$$(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1})^{p_1-3}(p_2-3)(p_2-4)\cdots(p_2-p_1+1)(p_3-3)(p_3-4)\cdots(p_3-p_1+1)$$

$$\cdots\cdots(p_r-3)(p_r-4)\cdots(p_r-p_1+1)$$

sets of $p_1 - 3$ *mutually orthogonal Knut Vik designs of Hedayat.*

Example 4. Let $n = 35$. Then, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 2. The all orthogonal pairs are given by the following pairs of λ's.

$$(\lambda(1,1) = 2, \lambda(2,2) = 3), (\lambda(1,1) = 2, \lambda(2,3) = 18), (\lambda(1,1) = 2, \lambda(2,4) = 33),$$

$$(\lambda(1,2) = 17, \lambda(2,1) = 23), (\lambda(1,2) = 17, \lambda(2,3) = 18), (\lambda(1,2) = 17, \lambda(2,4) = 33),$$

$$(\lambda(1,3) = 32, \lambda(2,1) = 23), (\lambda(1,3) = 32, \lambda(2,2) = 3), (\lambda(1,3) = 32, \lambda(2,4) = 33),$$

$$(\lambda(1,4) = 12, \lambda(2,1) = 23), (\lambda(1,4) = 12, \lambda(2,2) = 3), (\lambda(1,4) = 12, \lambda(2,3) = 18).$$

Let $n = 5 \times 7 \times 11$. Then, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is also 2. There are 672 orthogonal pairs in this case. the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is also 2.
Let $n = 7^2 \times 11$. In this case, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 4. There are $2^4 \times 3 \times 7^3$ orthogonal pairs and $2^4 \times 3 \times 5 \times 7^5$ sets of 4 mutually orthogonal Knut Vik designs of Hedayat.

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On the Diophantine Equation $(x^2+1)(y^2+1)=(z^2+1)^2$

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Abstract

In their paper [4], A. Schinzel and W. Sierpiński have investigated the diophantine equation $(x^2-1)(y^2-1)=(z^2-1)^2$. In this paper, we shall investigate an analogous equation $(x^2+1)(y^2+1)=(z^2+1)^2$.

2000 Mathematics Subject Classification. Primary 11D25; Secondary 11B37, 11D09

Introduction

In their paper [4], A. Schinzel and W. Sierpiński firstly investigated the following diophantine equation

$$(1) \qquad (x^2-1)(y^2-1)=(z^2-1)^2.$$

They have found all the integer solutions for which $x-y=2z$. But they could not find all the integer solutions of (1), and the problem to find all the integer solutions of this diophantine equation still remains as an open problem. In this paper, we shall show the following diophantine equation

$$(2) \qquad (x^2+1)(y^2+1)=(z^2+1)^2$$

has infinitely many integer solutions. Though we could not find all of the integer solutions of this equation, we found all the solutions with the additional condition $x-y=2z$. It is obvious that the above diophantine equation (2) has the solution $|x|=|y|=|z|$. Throughout this paper, we shall call these solutions *trivial* and other solutions *nontrivial*. Without loss of generality, we may restrict ourselves to the nontrivial and nonnegative solutions.

We shall also show the following slightly generalized diophantine equation has infinitely many integer solutions for any fixed positive integer t

$$(x^2+1)(y^2+1)=(z^2+t^2)^2. \tag{3}$$

The equation $(x^2+1)(y^2+1)=[((x-y)/2)^2+1]^2$

In this section, we shall show the diophantine equation (2) has infinitely many integer solutions with $x-y=2z$.
The left hand side of the equation (2) can be written as

$$(x^2+1)(y^2+1)=(xy+1)^2+(x-y)^2.$$

Since $z=(x-y)/2$, the right hand side of the equation (2) is

$$z^2+1=\left(\frac{x-y}{2}\right)^2+1=\frac{(x-y)^2+4}{4}.$$

Thus we have

$$16(xy+1)^2+16(x-y)^2=(x-y)^4+8(x-y)^2+16,$$

and then

$$16(xy+1)^2=(x-y)^4-8(x-y)^2+16=((x-y)^2-4)^2.$$

Therefore we have

$$(x-y)^2-4=\pm 4(xy+1)$$
$$\iff \begin{cases} (x-y)^2-4=x^2-2xy+y^2-4=-4xy-4 \\ \text{or} \\ (x-y)^2-4=x^2-2xy+y^2-4=4xy+4. \end{cases}$$

We note that
$x^2-2xy+y^2-4=-4xy-4$
if and only if
$x^2+2xy+y^2=(x+y)^2=0.$
Hence we have $y=-x$ and then $z=(x-y)/2=x$. Now we see these solutions are trivial. Let us consider another case $x^2-2xy+y^2-4=4xy+4$. Then we have $x^2-6xy+y^2=8$. Since $x^2-6xy+y^2=(x-3y)^2-8y^2$, we can conclude $x-3y$ must be divisible by 4. Put $w=(x-3y)/4$. Then we have

$16w^2 - 8y^2 = 8$, that is, $y^2 - 2w^2 = -1$. Let us denote $\epsilon = 1 + \sqrt{2}$ and $\bar{\epsilon} = 1 - \sqrt{2}$. Define the binary recurrence sequences $\{t_n\}$ and $\{s_n\}$ by putting

$$\begin{cases} t_n = (\varepsilon^n + \bar{\varepsilon}^n)/2, \\ s_n = (\varepsilon^n - \bar{\varepsilon}^n)/2\sqrt{2}. \end{cases}$$

Then $\{t_n\}$ and $\{s_n\}$ satisfy

$$t_{n+1} = 2t_n + t_{n-1}, \quad s_{n+1} = 2s_n + s_{n-1},$$

and

$$t_n^2 - 2s_n^2 = (-1)^n.$$

Combining the fact that $y^2 - 2w^2 = -1$ and y is nonnegative, we see $y = t_{2n-1}$ and $|w| = s_{2n-1}$ for some positive integer n. Then we have

$$w = \frac{x - 3y}{4} = \pm s_{2n-1} \Longleftrightarrow x = 3t_{2n-1} \pm 4s_{2n-1}.$$

From the fact that $\varepsilon^{2n-1} = t_{2n-1} + s_{2n-1}\sqrt{2}$ and $\varepsilon^2 = 3 + 2\sqrt{2}$, we have

$$\begin{aligned} \varepsilon^{2n+1} &= t_{2n+1} + s_{2n+1}\sqrt{2} = (t_{2n-1} + s_{2n-1}\sqrt{2})(3 + 2\sqrt{2}) \\ &= 3t_{2n-1} + 4s_{2n-1} + (3s_{2n-1} + 2t_{2n-1})\sqrt{2}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{2n-3} &= t_{2n-3} + s_{2n-3}\sqrt{2} = (t_{2n-1} + s_{2n-1}\sqrt{2})(3 - 2\sqrt{2}) \\ &= 3t_{2n-1} - 4s_{2n-1} + (3s_{2n-1} - 2t_{2n-1})\sqrt{2}). \end{aligned}$$

Hence we have verified $3t_{2n-1} + 4s_{2n-1} = t_{2n+1}$ and $3t_{2n-1} - 4s_{2n-1} = t_{2n-3}$. Thus we have $x = t_{2n+1}$ or t_{2n-3}. In the case $x = t_{2n-3}$, we have $x = t_{2n-3} < y = t_{2n-1}$, which contradicts to the assumption $2z = x - y \geq 0$. Hence any nonnegative solution of (2) can be written as $x = t_{2n+1}$ and $y = t_{2n-1}$ for some positive integer n. From the recurrence relation $t_{2n+1} = 2t_{2n} + t_{2n-1}$, we have

$$z = \frac{x - y}{2} = \frac{t_{2n+1} - t_{2n-1}}{2} = t_{2n}.$$

Thus we have shown the following theorem.

Theorem 1. *With the above notation, the diophantine equation* (2) *has infinitely many positive integer solutions. Moreover any positive integer solution* (x, y, z) *which satisfies* $2z = x - y$ *can be written as* $x = t_{2n+1}, y = t_{2n-1}, z = t_{2n}$ *for some positive integer* n.

Generalization

Let t be a positive integer. Then we shall generalize the above results to the following diophantine equation (3)

$$(x^2 + 1)(y^2 + 1) = (z^2 + t^2)^2.$$

We note that t^2+1 is not square for any $t \geq 1$ and $\sqrt{t^2+1} \notin \boldsymbol{Q}$. Let us denote $\eta = t + \sqrt{t^2+1}$ and $\bar{\eta} = t - \sqrt{t^2+1}$. Now we shall define binary recurrence sequences by putting

$$\begin{cases} v_n = (\eta^n + \bar{\eta}^n)/2, \\ u_n = (\eta^n - \bar{\eta}^n)/2\sqrt{t^2+1}. \end{cases}$$

Then we have $\{u_n\}$ and $\{v_n\}$ satisfy

$$\begin{cases} u_0 = 0, \quad u_1 = 1, \quad u_2 = 2t, \ldots\ldots\ldots, \quad u_{n+1} = 2tu_n + u_{n-1}, \\ v_0 = 1, \quad v_1 = t, \quad v_2 = 2t^2+1, \ldots, \quad v_{n+1} = 2tv_n + v_{n-1}, \end{cases}$$

and

$$\begin{cases} v_{2n-1}^2 + 1 = u_{2n-1}^2(t^2+1), \\ v_{2n+1}^2 + 1 = u_{2n+1}^2(t^2+1). \end{cases}$$

Then we obtain

$$(v_{2n+1}^2+1)(v_{2n-1}^2+1) = [(t^2+1)u_{2n+1}u_{2n-1}]^2.$$

Here we see

$$(t^2+1)u_{2n+1}u_{2n-1} = \frac{\eta^{2n+1} - \bar{\eta}^{2n+1}}{2} \cdot \frac{\eta^{2n-1} - \bar{\eta}^{2n-1}}{2}$$

$$= \frac{1}{2}\left(\frac{\eta^{4n} + \bar{\eta}^{4n} + \eta^2 + \bar{\eta}^2}{2}\right) = \frac{1}{2}(v_{4n} + 2t^2 + 1).$$

On the other hand we have

$$v_{2n}^2 = \left(\frac{\eta^{2n} + \bar{\eta}^{2n}}{2}\right)^2 = \frac{\eta^{4n} + \bar{\eta}^{4n} + 2}{4} = \frac{1}{2}(v_{4n} + 1).$$

Thus we have shown

$$(t^2+1)u_{2n+1}u_{2n-1} = v_{2n}^2 + t^2 = (t^2+1)(u_{2n}^2+1).$$

Hence we have

$$(v_{2n+1}^2+1)(v_{2n-1}^2+1) = (v_{2n}^2 + t^2)^2 = [(t^2+1)(u_{2n}^2+1)]^2.$$

Therefore we have obtained the following theorem.

Theorem 2. *With the above notation, the diophantine equation* (3) *has infinitely many positive integer solutions* $x = v_{2n+1}, y = v_{2n-1}, z = v_{2n}$ *with some positive integer* n.

From the above argument, we have the following corollary.

Corollary. *The diophantine equation*

$$(x^2+1)(y^2+1) = [(t^2+1)(z^2+1)]^2$$

has infinitely many parameterized positive integer solutions $(x, y, z, t) = (v_{2n+1}, v_{2n-1}, u_{2n}, t)$.

Concluding remarks

Finally, we shall recall the classical results on Shinzel-Sierpiński equation (1) with $x - y = 2z$. We have $(t_{2n+2}^2 - 1)(t_{2n}^2 - 1) = (2s_{2n+2}s_{2n})^2$.
Here

$$2s_{2n+2}s_{2n} = \frac{1}{4}(\varepsilon^{2n+2} - \bar{\varepsilon}^{2n+2})(\varepsilon^{2n} - \bar{\varepsilon}^{2n})$$

$$= \frac{1}{4}(\varepsilon^{4n+2} + \bar{\varepsilon}^{4n+2} - \varepsilon^2 - \bar{\varepsilon}^2) = \frac{1}{4}(\varepsilon^{4n+2} - \bar{\varepsilon}^{4n+2} - 6)$$

$$= \frac{\varepsilon^{4n+2} + \bar{\varepsilon}^{4n+2} - 2}{4} - 1 = \left(\frac{\varepsilon^{2n+1} + \bar{\varepsilon}^{2n+1}}{2}\right)^2 - 1$$

$$= t_{2n+1}^2 - 1 = \left(\frac{t_{2n+2} - t_{2n}}{2}\right)^2 - 1.$$

Thus we have recalled the elementary fact that any positive integer solution of

$$(x^2-1)(y^2-1) = (z^2-1)^2 \quad \text{with } 2z = x - y,$$

is given by $(x, y, z) = (t_{2n+2}, t_{2n}, t_{2n+1})$.

Here we shall combine this classical result and Theorem 1 as follows. Put $e = \pm 1$. Then the following diophantine equation

$$(x^2+e)(y^2+e) = (z^2+e)^2. \tag{4}$$

with $2z = x - y$ has infinitely many positive integer solutions $(x, y, z) = (t_{n+2}, t_n, t_{n+1})$. Here, n is even for the case $e = -1$ and n is odd for the case $e = 1$.

In [2], the positive integer solutions of (1) not of the form $(t_{2n+2}, t_{2n}, t_{2n+1})$ are called *sporadic* solutions. For example, there are several sporadic solutions $(31, 4, 11), (97, 2, 13), (48049, 155, 2729)$ quoted by Szymiczek. On the contrary, it seems rare that our equation (2) has sporadic solutions. We have verified the only positive integer solutions with $0 < y < z < 300000$ are

$$\begin{aligned}(x,y,z) &= (7,1,3),\\ &\quad (41,7,17),\\ &\quad (239,41,99),\\ &\quad (1393,239,577),\\ &\quad (8119,1393,3363),\\ &\quad (47321,8119,19601),\\ &\quad (275867,47321,114243).\end{aligned}$$

Hence, there is no sporadic solution for $0 < y < z < 300000$.

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A Note on a Heat Invariant and the Ricci Flow on Surfaces

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Abstract

In this short note, we consider the monotonicity of the heat invariant $a_2(g)$ for a Riemannian metric g under the normalized Ricci flow on a closed surface. We show that $a_2(g(t))$ is decreasing under the normalized Ricci flow $g(t)$ in the space of metrics of non-positive curvature.

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Introduction

Let M be an n dimensional compact C^∞ manifold without boundary. Given a C^∞ Riemannian metric g on M. Then, we have the Laplace-Beltrami operator $\Delta = \Delta_g$ acting on C^∞ functions on M, whose spectrum consists of non-negative eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \uparrow \infty.$$

It is well known concerning $\{\lambda_k\}$ that we have the asymptotic expansion

$$\sum_{k=0}^{\infty} e^{-\lambda_k s} \underset{s\downarrow 0}{\sim} (4\pi s)^{-n/2}\{a_0(g) + a_1(g)s + \cdots + a_j(g)s^j + \cdots\},$$

where the coefficients $a_j(g)$ are the heat invariants, first three of which are given by

$$\begin{aligned} a_0(g) &= \mathrm{Vol}(M,g), \quad a_1(g) = \frac{1}{6}\int_M S\, dV_g, \\ a_2(g) &= \frac{1}{360}\int_M [2|R|^2 - 2|\mathrm{Ric}|^2 + 5S^2] dV_g. \end{aligned} \tag{0.1}$$

Here, $S = S(g)$ is the scalar curvature, $\mathrm{Ric} = \mathrm{Ric}(g)$ is the Ricci tensor and $R = R(g)$ is the Riemannian curvature tensor of (M, g). In the case where M is a closed surface $a_2(g)$ reduces to

$$a_2(g) = \frac{1}{60}\int_M S^2\, dV_g. \tag{0.2}$$

By noticing the Schwarz inequality and the Gauss-Bonnet theorem, we have

$$\int_M S^2 dV_g \geq \Big(\int_M S\,dV_g\Big)^2 \Big/ \mathrm{Vol}(M,g) = \{4\pi\chi(M)\}^2/\mathrm{Vol}(M,g) \tag{0.3}$$

($\chi(M)$ being the Euler characteristic). Thus, the functional $a_2(g)$ attains its minimum at the metric of constant curvature in the space of C^∞ metrics on M with fixed volume.

In the present note we consider the behavior of the invariant $a_2(g)$ given by (0.2) under the normalized Ricci flow on closed surfaces. The normalized Ricci flow (introduced by Hamilton [3]) on a closed surface M is a one-parameter C^∞ family $g(t)$ $(t \geq 0)$ of C^∞ metrics on M which is evolved by the equation

$$\frac{\partial}{\partial t} g_{ij} = (s - S) g_{ij},$$

where s denotes the average scalar curvature given by

$$s := \Big(\int_M S\,dV_g\Big) \Big/ \mathrm{Vol}\,(M,g).$$

Here $\mathrm{Vol}(M,g) = \mathrm{Vol}(M,g(t))$ is constant along $g(t)$, and s is also constant, in fact

$$s = 4\pi\chi(M)/\mathrm{Vol}(M,g(0)).$$

Note that the stationary points of the normalized Ricci flow are the metrics of constant curvature s.

The main result of this note is the following.

Theorem. *Suppose M is a closed surface with $\chi(M) < 0$. Let $g(t)$ $(t \geq 0)$ be a normalized Ricci flow on M such that*

$$S(g(0)) \leq 0. \tag{0.4}$$

Then, $a_2(g(t))$ is monotonously decreasing in t, namely

$$\frac{d}{dt} a_2(g(t)) \leq 0,$$

and the equality holds if and only if $g(t)$ is a metric of (negative) constant curvature s.

Remark. Let $g(t)$ $(t \geq 0)$ be any Ricci flow on a surface M with $\chi(M) < 0$. Then, it has been shown by Hamilton [3, Theorems 4.6 and 4.9] that there exists t_0 such that $S(g(t)) \leq 0$ for $\forall t \geq t_0$, and that $g(t)$ converges to a metric of constant negative curvature s as $t \to \infty$.

1. Proof of the theorem

Let $g(t)$ is a one-parameter C^∞ family of C^∞ metrics on the closed surface M. We put

$$h_{ij} = h_{ij}(t) := \frac{\partial}{\partial t} g_{ij}, \quad h^{ij} := g^{ik} h_{kj} \big(= \sum_k g^{ik} h_{kj} \big),$$

which are symmetric 2-tensor fields on M. Put

$$F(t) := \int_M \{S(g(t))\}^2 \, dV_{g(t)} \; (= 60 \, a_2(g(t)) \,).$$

Then, we have the following.

Lemma. *The derivative of $F(t)$ is given by*

$$\frac{dF}{dt} = \int_M \big[2(\nabla_i \nabla_j S) h^{ij} + \big(2\Delta S - \frac{1}{2} S^2\big) h^j_j \big] \, dV_g \quad (h^j_j := h_{ij} g^{ij}). \tag{1.1}$$

If $g(t)$ is a normalized Ricci flow, then

$$\frac{dF}{dt} = \int_M \big[-2S\Delta S + S^2(S - s) \big] \, dV_g. \tag{1.2}$$

Proof. We easily obtain the formulas (1.1) by straightforward calculations following the variation formulas for the volume element, the Levi-Civita connection and its associated curvature tensors (given in [2], for example). The formula (1.2) is derived from (1.1) for $h_{ij} = (s - S) g_{ij}$. □

Put $\widetilde{S} = S - s$. Then, we have

$$\int_M \widetilde{S} \, dV_g = 0, \tag{1.3}$$

and the formula (1.2) is rewritten as

$$\begin{aligned} \frac{dF}{dt} &= \int_M \big[-2\widetilde{S}\Delta\widetilde{S} + \widetilde{S}^2 S + sS^2 - s^2 S \big] dV_g \\ &= -2 \int_M \widetilde{S}\Delta\widetilde{S} \, dV_g + \int_M \widetilde{S}^2 S \, dV_g + s \int_M S^2 dV_g - s^3 \mathrm{Vol}(M, g). \end{aligned}$$

By virtue of (1.3) we have

$$\int_M \widetilde{S}\Delta\widetilde{S} \, dV_g \geq \lambda_1 \int_M \widetilde{S}^2 dV_g$$

for the non-zero first eigenvalue $\lambda_1 = \lambda_1(g)$ of $\Delta = \Delta_g$. Moreover, by virtue of (0.3) we have

$$\int_M S^2 dV_g \geq s^2 \mathrm{Vol}(M, g).$$

Hence, we get

$$\frac{dF}{dt} \le -2\lambda_1 \int_M \widetilde{S}^2 dV_g + \int_M \widetilde{S}^2 S\, dV_g = -\int_M (2\lambda_1 - S)\widetilde{S}^2 dV_g \qquad (1.4)$$

if $s \le 0$ (which means $\chi(M) \le 0$). Thus, we have the following.

Proposition. *Suppose M is a closed surface with $\chi(M) \le 0$. Let $g(t)$ be a normalized Ricci flow on M. If*

$$\int_M \{2\lambda_1(g(t)) - S(g(t))\}\left(\widetilde{S}(g(t))\right)^2 dV_{g(t)} \ge 0,$$

then, we have

$$\frac{d}{dt} a_2(g(t)) \le 0\,.$$

Proof of Theorem. By applying the maximum principle to the evolution equation of $S(g(t))$, we get $S(g(t)) \le 0$ for $\forall t \ge 0$ if $S(g(0)) \le 0$ ([3, Theorem 3.2]). Moreover, we notice that in (1.4) the equality holds if and only if $\widetilde{S} = 0$. Thus, we obtain Theorem as a corollary of Proposition. □

It is reserved for future discussions to clarify what occurs if the condition (0.4) is removed.

Concluding Remarks

1. The functional $F(g) = \int_M S^2 dV_g$ (called "Calabi energy") was first considered by Calabi [1], and it was shown that in the case where M is a closed surface the critical points of F are metrics of constant curvature (which attain the minimum) and that F is decreasing under the "Calabi flow".

2. As a recent result related to this note we refer to [4], in which it was proved that the determinant of the Laplacian on a closed surface monotonously increases under the normalized Ricci flow.

3. The heat invariant $a_2(g)$ given (0.1) for n dimensional compact Riemannian manifolds has been considered in "Spectral Geometry," particularly to characterize the flat metrics by the spectrum (see [5], [6]).

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