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Construction of Knut Vik Designs and Orthogonal Knut Vik Designs

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Abstract

Following Euler's method, A. Hedayat constructs some Knut Vik designs. We call them Knut Vik designs of Hedayat in this note. We give Knut Vik designs of Hedayat explicitly and decide when Knut Vik designs of Hedayat are mutually orthogonal.

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Introduction

Let A be a Latin square of order n, that is, an $n \times n$ array in which n distinct symbols are arranged so that each symbol occurs once in each row and column. Index its rows and columns by 1,2,. . . n. By the jth right diagonal of A we mean the following n cell of A:

$$(i, j+i-1); \qquad i = 1, 2, \cdots, n; \qquad (\text{mod } n.)$$

We define also the jth left diagonal of A to the following n cell of A:

$$(i, j-i); \qquad i = 1, 2, \cdots, n; \qquad (\text{mod } n.)$$

Let Σ be a set of n distinct symbols. If we can fill the cells of A by the elements of Σ in such a way that each row, column, right diagonal and left diagonal of A contains all the elements of Σ, we say the resulting structure a Knut Vic design, which we denote by K. It is also called a pandiagonal Latin square [1], [3]. In this paper, we set $\Sigma = \{0, 1, 2, \cdots, n-1\}$. It is well known that Knut Vic designs of order n exists if and only if n is not divisible by 2 or 3. A. Hedayat [3] showed that n is not divisible by 2 or 3 then $K = (k_{ij})$ with $k_{ij} = \lambda i + j(\text{mod } n)$ is a Knut Vic design if $\lambda, \lambda - 1, \lambda + 1$ are relatively prime to n. In this paper, we call these Knut Vic designs as Knut Vic designs of Hedayat, Let n have the prime decompsition

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

Then also he showed that there are

$$N = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} (p_1 - 3)(p_2 - 3) \cdots (p_r - 3).$$

different choices for λ. In the present note, we define a standard way which gives λ satisfing the condition that $\lambda, \lambda - 1, \lambda + 1$ are relatively prime to n. K. Afsarinejad showed that there exist at least $\min(p_i - 3)$, $(i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of order n. We show that there exist at most $\min(p_i - 3)$, $(i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of Hedayat. We also obtain that to each Knut Vik design of Hedayat, there are

$$p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} (p_1 - 4)(p_2 - 4) \cdots (p_r - 4)$$

orthogonal Knut Vik designs of Hedayat.

1. A standard construction of Knut Vik designs of Hedayat

Let n be not divisible by 2 or 3 and have the prime decomposition

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad (3 < p_1 < p_2 < \cdots < p_r).$$

If $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to n, then

$$K = (k_{ij}) \qquad \text{with} \qquad k_{ij} = \lambda i + j \pmod{n}$$

is a Knut Vik design of Hedayat. We decide explicitly when $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to n.
Put

$$m_{i_a}^a \;=\; i_a + 1, \qquad 1 \le a \le r, \quad 1 \le i_a \le p_a - 3.$$

From Chinese remainder theorem, we obtain

Lemma 1. *For each* $\{i_1, i_2, \cdots, i_r\}$ *with* $1 \le i_1 \le p_1 - 3, 1 \le i_2 \le p_2 - 3, \cdots,$ $1 \le i_r \le p_r - 3$, *there is a positive integer* m *satisfies*

$$m = m_{i_1}^1 \pmod{p_1}, m = m_{i_2}^2 \pmod{p_2}, \cdots, m = m_{i_r}^r \pmod{p_r}.$$

As this integer is unique on $Z_{(p_1p_2\cdots P_r)}$, we denote it by $m_{i_1i_2\cdots i_r}$.

Now, we obtain

Theorem 2 *Set $n_1 = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1}$. For each $\{t, i_1, i_2, \cdots, i_r\}$ with $0 \le t \le n_1 - 1$, $1 \le i_1 \le p_1 - 3, 1 \le i_2 \le p_2 - 3, \cdots, 1 \le i_r \le p_r - 3$, put*

$$\lambda(t, i_1, i_2, \cdots, i_r) = p_1p_2\cdots p_rt + m_{i_1i_2\cdots i_r},$$

then these λ's give N different choices for Knut Vik designs of Hedayat.

Following the proof of Chinese remainder theorem, we construct explicitly integers $m_{i_1i_2\cdots i_r}$ as follows. Put

$q_1 = p_1^{-1} \pmod{P_2}, q_2 = (p_1p_2)^{-1} \pmod{P_3}, \cdots, q_{r-1} = (p_1p_2\cdots p_{r-1})^{-1} \pmod{P_r}$

Now we get inductively,

$$\begin{aligned}
m_{i_1i_2\cdots i_r} &= m_{i_1}^1 \pmod{p_1},\\
m_{i_1i_2\cdots i_r} &= m_{i_1}^1 + p_1s_1 = m_{i_2}^2 \pmod{p_2},\\
s_1 &= q_1(m_{i_2}^2 - m_{i_1}^1) + s_2p_2,\\
m_{i_1i_2} &= m_{i_1}^1 + p_1q_1(m_{i_2}^2 - m_{i_1}^1),\\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2} + p_1p_2s_2 = m_{i_3}^3 \pmod{p_3},\\
s_2 &= q_2(m_{i_3}^3 - m_{i_1i_2}) + s_3p_3,\\
m_{i_1i_2i_3} &= m_{i_1i_2} + p_1p_2q_2(m_{i_3}^3 - m_{i_1i_2}),\\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2i_3} + p_1p_2p_3s_3 = m_{i_4}^4 \pmod{p_4},\\
&\vdots\\
m_{i_1i_2\cdots i_{r-1}} &= m_{i_1i_2\cdots i_{r-2}} + p_1p_2\cdots p_{r-2}q_{r-2}(m_{i_{r-1}}^{r-1} - m_{i_1i_2\cdots i_{r-2}}),\\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2\cdots i_{r-1}} + p_1p_2\cdots p_{r-1}s_{r-1} = m_{i_r}^r \pmod{p_r},\\
s_{r-1} &= q_{r-1}(m_{i_r}^r - m_{i_1i_2\cdots i_{r-1}}) \pmod{(p_1p_2\cdots p_r)},\\
m_{i_1i_2\cdots i_r} &= m_{i_1i_2\cdots i_{r-1}} + p_1p_2\cdots p_{r-1}q_{r-1}(m_{i_r}^r - m_{i_1i_2\cdots i_{r-1}})m_{i_1i_2\cdots i_{r-1}} \pmod{(p_1p_2\cdot p_r)}.
\end{aligned}$$

Example 1. Let $n = 5 \times 7$. Then $p_1 = 5, p_2 = 7$ and $q_1 = 3 \pmod 7$. We have

$$m_1^1 = 2, m_2^2 = 3, m_1^2 = 2, m_2^2 = 3, m_3^2 = 4, m_4^2 = 5.$$

Hence, we get

$$\begin{aligned}
m_{i_1i_2} &= m_{i_1}^1 + 15(m_{i_2}^2 - m_{i_1}^1) \pmod{35},\\
&= i_1 + 1 + 15(i_2 - i_1) \pmod{35}, \ \ 1 \le i_1 \le 2, \ \ 1 \le i_2 \le 4.
\end{aligned}$$

Thus, we obtain

$m_{11} = 2, m_{12} = 17, m_{13} = 32, m_{14} = 12, m_{21} = 23, m_{22} = 3, m_{23} = 18, m_{24} = 33 \pmod{35}.$

Example 2. Let $n = 5 \times 7 \times 11$. Then $p_1 = 5, p_2 = 7, p_3 = 11$,, $q_1 = 3 \pmod{7}, q_2 = 6 \pmod{11}$. $m^1_{i_1}, m^2_{i_2}$ are the same as in Example 1, and $m^3_{i_3} = i_3 + 1,\ 1 \le i_3 \le 8$. It is evident that $m_{i_1 i_2}\ 1 \le i_1 \le 2,\ 1 \le i_2 \le 4$ are also the same as in Example 1. Now we have

$$m_{i_1 i_2 i_3} = m_{i_1 i_2} + 210(m^3_{i_3} - m_{i_1 i_2}) \pmod{385}.$$

Now we write simply $m_{i_1 i_2 *}$ for $(i_{i_1 i_2 1}, i_{i_1 i_2 2}, \cdots, i_{i_1 i_2 8})$.

$m_{11*} = (2, 212, 37, 247, 72, 282, 107, 317),\ m_{12*} = (332, 157, 367, 192, 17, 227, 52, 262),$

$m_{13*} = (277, 102, 312, 137, 347, 172, 382, 207),\ m_{14*} = (222, 47, 257, 82, 292, 117, 327, 152),$

$m_{21*} = (233, 58, 268, 93, 303, 128, 338, 163),\ m_{22*} = (178, 3, 213, 38, 248, 73, 283, 108),$

$m_{23*} = (123, 333, 158, 368, 193, 18, 228, 53),\ m_{24*} = (68, 278, 103, 313, 138, 348, 173, 383).$

Example 3. Let $n = 7^2 \times 11$. Then $p_1 = 7, p_2 = 11,\ q_1 = 8 \pmod{11}$. $m^1_{i_1} = i_1 + 1, m^2_{i_2} = i_2 + 1,\ 1 \le i_1 \le 4,\ 1 \le i_2 \le 8$. Hence, we get

$m_{i_1 i_2} = m^1_{i_1} + 56(m^2_{i_2} - m^1_{i_1}) = i_1 + 1 + 56(i_2 - i_1) \pmod{77},\ 1 \le i_1 \le 4,\ 1 \le i_2 \le 8.$

We write $m_{i_1 *}$ for $(i_{i_1 1}, i_{i_1 2}, \cdots, i_{i_1 8})$.

$$m_{1*} = (2, 58, 37, 16, 72, 51, 30, 9),\ m_{2*} = (24, 3, 59, 38, 17, 73, 52, 31),$$

$$m_{3*} = (46, 25, 4, 60, 39, 18, 74, 53),\ m_{4*} = (68, 47, 26, 5, 61, 40, 19, 75).$$

Now we obtain

$$\lambda(t, i_1, i_2) = 77t + m_{i_1 i_2},\quad 0 \le t \le 6,\ 1 \le i_1 \le 4,\ 1 \le i_2 \le 8.$$

2. Orthogonal Knut Vik designs of Hedayat

Let K_1 and K_2 be Knut Vik designs of Hedayat of order n. K_1 and K_2 are said to be *orthogonal* if they are orthogonal in the sense of Latin squares. Using the notations in Theorem 2, assume that $K_1 = (k_{ij}^{(1)})$ and $K_2 = (k_{ij}^{(2)})$ are given by

$$k_{ij}^{(1)} = \lambda(t_1, i_1^{(1)}, i_2^{(1)}, \cdots, i_r^{(1)})i + j, \quad k_{ij}^{(2)} = \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \cdots, i_r^{(2)})i + j,$$

where

$$0 \le t_1 \le n_1 - 1, 1 \le i_1^{(1)} \le p_1 - 3, 1 \le i_2^{(1)} \le p_2 - 3, \cdots, 1 \le i_r^{(1)} \le p_r - 3,$$

$$0 \le t_2 \le n_1 - 1, 1 \le i_1^{(2)} \le p_1 - 3, 1 \le i_2^{(2)} \le p_2 - 3, \cdots, 1 \le i_r^{(2)} \le p_r - 3.$$

It is known that K_1 and K_2 are orthogonal if and only if $\lambda(t_1, i_1^{(1)}, i_2^{(1)}, \cdots, i_r^{(1)}) - \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \cdots, i_r^{(2)})$ and n are relatively prime. Hence we have

Lemma 3. *Knut Vik designs of Hedayat K_1 and K_2 are orthogonal if and only if*

$$i_1^{(1)} \ne i_1^{(2)}, i_2^{(1)} \ne i_2^{(2)}, \cdots, i_r^{(1)} \ne i_r^{(2)}.$$

When K_1 and K_2 are orthogonal , we call $(K_1.K_2)$ a *orthogonal pair* in this note. From Lemma 3, it follows

Theorem 4. *For each Knut Vik design of Hedayat, there are*

$$p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1}(p_1-4)(p_2-4)\cdots(p_r-4)$$

Knut Vik designs orthogonal to it. There are

$$\frac{1}{2}p_1^{2\alpha_1-2}p_2^{2\alpha_2-2}\cdots p_r^{2\alpha_r-2}(p_1-3)(p_1-4)(p_2-3)(p_2-4)\cdots(p_r-3)(p_r-4)$$

orthogonal pairs of Knut Vik designs of Hedayat .

Let S be a set of Knut Vik designs of order n . We say S to be a set of mutually orthogonal Knut Vik designs of order n if any two Knut Vik designs in S are orthogonal. It is shown by K. Afsarinejad that there are at least $p_1 - 3$ mutually orthogonal Knut Vik designs of order n. On the other hand, by Lemma 3, there are at most $p_1 - 3$ mutually orthogonal Knut Vik designs of Hedayat. In fact, $p_1 - 3$ mutually orthogonal Knut Vik designs are given by $K_1 = (k_{ij}^{(1)}), K_2 = (k_{ij}^{(2)}), \cdots, K_{p_1-3} = (k_{ij}^{(p_1-3)})$, where

$$k_{ij}^{(1)} = \lambda(t_1, 1, i_2^{(1)}, \cdots, i_r^{(1)})i + j, \quad k_{ij}^{(2)} = \lambda(t_2, 2, i_2^{(2)}, \cdots, i_r^{(2)})i + j, \cdots,$$

$$k_{ij}^{(p_1-3)} = \lambda(t_{p_1-3}, p_1 - 3, i_2^{(p_1-3)}, \cdots, i_r^{(p_1-3)})i + j,$$

and

$$i_2^{(a)} \neq i_2^{(a)}, i_3^{(a)} \neq i_3^{(b)}, \cdots, i_r^{(a)} \neq i_r^{(b)}, \quad \text{for any } a \neq b.$$

Thus, we obtain

Theorem 5. *The maximum number of mutually orthogonal Knut Vik designs of Hedayat is $p_1 - 3$. There are*

$$(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1})^{p_1-3}(p_2-3)(p_2-4)\cdots(p_2-p_1+1)(p_3-3)(p_3-4)\cdots(p_3-p_1+1)$$

$$\cdots\cdots(p_r - 3)(p_r - 4)\cdots(p_r - p_1 + 1)$$

sets of $p_1 - 3$ mutually orthogonal Knut Vik designs of Hedayat.

Example 4. Let $n = 35$. Then, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 2. The all orthogonal pairs are given by the following pairs of λ's.

$$(\lambda(1,1) = 2, \lambda(2,2) = 3), (\lambda(1,1) = 2, \lambda(2,3) = 18), (\lambda(1,1) = 2, \lambda(2,4) = 33),$$

$$(\lambda(1,2) = 17, \lambda(2,1) = 23), (\lambda(1,2) = 17, \lambda(2,3) = 18), (\lambda(1,2) = 17, \lambda(2,4) = 33),$$

$$(\lambda(1,3) = 32, \lambda(2,1) = 23), (\lambda(1,3) = 32, \lambda(2,2) = 3), (\lambda(1,3) = 32, \lambda(2,4) = 33),$$

$$(\lambda(1,4) = 12, \lambda(2,1) = 23), (\lambda(1,4) = 12, \lambda(2,2) = 3), (\lambda(1,4) = 12, \lambda(2,3) = 18).$$

Let $n = 5 \times 7 \times 11$. Then, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is also 2. There are 672 orthogonal pairs in this case. the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is also 2.
Let $n = 7^2 \times 11$. In this case, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 4. There are $2^4 \times 3 \times 7^3$ orthogonal pairs and $2^4 \times 3 \times 5 \times 7^5$ sets of 4 mutually orthogonal Knut Vik designs of Hedayat.

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