J. Math. Univ. Tokushima
Vol.39 (2005), 1-5

Local Switching of Signed Induced Cycles

By

Toru ISHIHARA

Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minamijosanjima, Tokushima 770-8502, JAPAN
e-mail adress: ishihara@ias.tokushima-u.ac.jp
(Received September 30, 2005)

Abstract

A singed cykle is transformed to a line by a sequence of local switchings if and only if its parity is odd. We invesigate induced cycles in a signed graph which are transformed to lines by local switching.

2000 Mathematics Subject Classification. 05C22

Introduction

Local switching of signed graphs is introduced by P. J. Cameron, J.J. Seidel and S. V. Tsaranov in [2]. Signed cycles with odd parity are transformed to lines by a sequence of local switchings, but signed cycles with even parity can not be transformed to lines by no means [5]. What kinds of induced signed cyles are transformed to trees by a sequence of local switchings ? We investigate induced cycles in a signed graph which are transformed into lines by local switching.

We state briefly basic facts about signed graphs. A graph $G = (V, E)$ consists of an n-set V(the vertices) and a set E of unordered pairs from V(the edges). A *signed graph* (G, f) is a graph G with a signing $f : E \to \{1, -1\}$ of the edges. We set $E^+ = f^{-1}(+1)$ and $E^- = f^{-1}(-1)$. For any subset $U \subseteq V$ of vertices, let f_U denote the signing obtained from f by reversing the sign of each edge which has one vertex in U. This defines on the set of signings an equivalence relation, called *switching*. The equivalence classes $\{f_U : U \subseteq V\}$ are the *signed swithing classes* of the graph $G = (V, E)$.

Let $i \in V$ be a vertex of G, and $V(i)$ be the neighbours of i. The *local graph* of (G, f) at i has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of G for which $f(i, j)f(j, k)f(k, i) = -1$. A *rim* of (G, f) at i is any union of connected components of local graph at i. Let J be any rim at i, and let $K = V(i)\backslash J$. *Local switching* of (G, f) with respect to (i, J) is the following operation:(i) delete all edges of G between J and K; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j,k\}$ with sign chosen so that $f(i,j)f(j,k)f(k,i) = -1$; (iii) change the signs of all edges from i to J; (iv) leave all other edges and signs unaltered. Let Ω_n be the set of switching classes of signed graphs of order n. Local switching , applied to any vertex and any rim at the vertex, gives a relation on Ω which is symmetric but not transitive. The equivalence classes of its tansitive closure are called the *clusters* of order n.

A signed graph is said to be *positive* if we can switch all signs of its edges into $+1$. A tree is always considered as a positive signed graph. A tree with only two leaves is said to be *a line* in the present paper.

1. Signed induced cycles

A $k-$cycle $C^k = (V,E)$, where $V = \{a_1, a_2, \cdots, a_k\}$, $E = \{a_1a_2, a_2a_3, \cdots, a_{k-1}a_k, a_ka_1\}$, will be denoted simply $C^k = a_1a_2\cdots a_ka_1$. For signed cycles, there are two switching classes, which are distinguished by the parity or the balance, where the parity of a signed cycle is the parity of the number of its edges which carry a positive sign and the balance is the product of the signs on its edges [2]. We show the follwing in [5].

Theorem . *Let C^k be a $k-$cycle. Then, it is transformed to a tree by a sequence of local switchings if and only if its parity is odd.*

In the present note, we study induced cycles in a signed graph which are transformed into lines by local switching.

An induced cycle with odd (resp. even) parity in a signed graph is called simply an *odd* (resp. *even*) induced cycle in the present paper. Let C be an induced cycle in a signed graph. Let P be a path between vetices a and b which are vertices of C. We call P a *C-path* if P meets C exactly in its ends, and in this case we call it also *a-b path.*

Theorem 1. *Let C be an induced cycle in a signed graph G. For some two its vertices a, b, suppose that there is a C-path P between a and b. Together with two pathes P_1, P_2 between a and b in C, P forms two induced cycles C_1 and C_2. If C is an odd induced cycle, then one of C_1 and C_2 is odd , and the other is even. On the other hand, if C is even, both of C_1 and C_2 are even or odd.*

Proof. Now a parity of a signed path is defined to be the parity of the numbers of positive edges. Firstly, suppose that C is an odd induced cycle. Then, the parity of one of two pathes P_1 and P_2 is odd and the parity of the other is even. We may assume that P_1 is an odd path. Then P_2 is an even path. If P is an odd(resp. even) path, then C_1 is an even(resp. odd) cycle and C_2 is an odd(resp. even) cycle.

Next, suppose that C is an even induced cycle. Then both P_1 and P_2 are odd pathes or even pathes. Suppose that P_1 and P_2 are odd pathes. If P is an odd(resp. even) path, both C_1 and C_2 are even(resp. odd) induced cycles.

When P_1 and P_2 are even pathes, if P is an odd(resp. even) path, both C_1 and C_2 are odd(resp. even) induced cycles.

The following is evident.

Proposition 2. *Let C be an induced cycle in a signed graph G with length k. Take any vertex a of C. Let C' be the induced cycle obtained from C by local switching at a. Then C' has the same parity as C and its length is $k-1$.*

Definition 1. For a given induced cycle C, take any C-path P. As in Theorem 1, we can construct two cycles C_1, C_2. In this case, we say that C *consists of* C_1 and C_2 and that C_1 and C_2 are contained in C. Assume that C is even. If C_1, C_2 are odd, we say that C *consists of odd cycles.* If C_1, C_2 are even but they consist of odd cycles, we say also C *consists of odd cycles.* If C_1, C_2 are even , C_1 consists of odd cycles and C_2 consists of two cycles C_{21}, C_{22} which consist of odd cycles, we say also C *consists of odd cycles* and so on.

Definition 2. Let C be an induced cycle in a signed graph.

We call C a *fundamental odd cycle* if the following two conditions are satisfied. (1) Its parity is odd. (2) If C consists of two induced cycles C_1, C_2 and C_1 is odd, then the even cycle C_2 does not consists of odd cycles. We call C a *fundamental even cycle* if the following tree conditions are satisfied. (1) Its parity is even.(2) It does not consists of odd cycles. (3) For any vertices a, b of C, there is no $a-b$ path whose length is shoter than the lengthes of two $a-b$ pathes on C. The following is evident.

2. Local swithching of signed induced cycles

Lemma 3. *Let C be an induced cycle in a signed graph G with length k. Take a vertex d in the outside of C which is adjacent to vertices a, b of C, where ab is one of the edges of C. Moreover suppose there is no other vertex of C which is adjacent to d. Let C' be the induced cycle obtained from C by local switching at d. Then C' has the same parity as C and its length is $k+1$.*

Let C be a fundamental even cycle in a signed graph G. Take a vertex d in the oudside of C. Assume that the vertex v is adjacent to some vertices of C. The following three cases may occur. (1) There is an edge ab of C and vertices of C which are adjacent to v are just a, b. As C is a fundamental even cycle, there is no pathes which link the vertex v and any other vertice of C. (2) Edges ab and bc are contained in C and vertices of C which are adjacent to v are just a, c or (3) a, b, c. Since C is a fundamental cycle, we have no other cases. When the case (1) occurs, by local switching at d, we get an even cycle C_1 with length one longer than that of C. If there is a C-path which links a and b and makes a fundamental odd cycles with $a-b$ pathes in C, then C consists of odd cycles and is not a fundamental even cycle. Thus, C_1 is a fundamental even cycle. For case (2), by local switching at v, we get two fundamental even cycle $abca$

and the cycle consist of the edge ac and the other $a-c$ path in C. The length of the latter is one less than that of C. In the case (3), suppose that C consists of the path abc and the other $a-c$ path P. By local switching at d, we obtain a fundamental even cycle C_1 which consists of the edge ac and the path P and has the length one less than that of C, or a even cycle C_2 which consists of the path adc and the path P. As C is a fundamental even cycle, C_2 is also a fundamental even cycle with the same length as that of C.

Summing up, we have

Lemma 4. *Let C be a fundamental even cycle in a signed graph G.*

(1) Take any vertex a in C. By local switching at a, we get a fundamental even cycle with lengh one less than that of C. Take a vertex v outside C which is adjacent to some vertices in C. Then the following two cases occur.

(2) Edges ab and bc are contained in C and vertices of C which are adjacent to the vertx v are just a, c or a, b, c. By local swithing at v, we get a fundamental even cycle with lengh one less than that of C or with the same length as that of C.

(3) There is an edge ab of C and vertices of C which are adjacent to v are just a, b. By local switching at d, we get an even induced cycle C_1 with length one longer than that of C. This C_1 may consists of odd cycles. Otherwise, C_1 is a fundamental even cycle.

Let C be a fundamental odd cycle in a signed graph G. Take a vertex d in the outside of C. Assume that the vertex v is adjacent to vertices a, b of C and that ab is an edge of C. As C is a fundamental odd cycle, there are no pathes which link the vertex v with any other vertices of C, By local swithing at v, we obtain a fundamental odd cycle with lengh one longer than that of C. Assume only two vertices a, b of C are adjacent to v and that a, b are not adjacent. Suppose that signs of the edges av and bv are positive. Then, one of $a-b$ path P in C is an even path. The path adb and P make an even cycle. By local switching at v, we get an fundamental even cycle with length less than that of C. Let $a_1, a_2, \cdots, a_k$ be vertices of C which are adjacent to v. We may assume that these vertices are on C in this order and that signs of the edges $va_1, va_2, \cdots, va_k$ are positive. We may also suppose that all pathes $a_1 - a_2$ path, $a_2 - a_3$ path, $\cdots$ $a_{k-1} - a_k$ path are even pathes. By local switching at v, we get at least $k-1$ fundamental even cycles with length less than that of C.

Lemma 5. *Let C be a fundamental odd cycle in a signed graph G.*

(1) Take any vertex a in C. By local switching at a, we get a fundamental odd cycle with lengh one less than that of C.

Take a vertex v outside C which is adjacent to some vertices in C. Then the following two cases occur.

(2) There is an edge ab of C and vertices of C which are adjacent to v are just a, b. By local switching at d, we get a fundamental odd cycle C_1 with length one longer than that of C.

(3) The vertex v is adjacent to some vertices in C any two of which are not adjacent. By local swithing at v, we obtain at least one fundamental even cycle with length less than that of C.

Theorem 6. *We list some facts about transforming cycles into lines by a sequence of local switichings.*

(1) A cycle with length longer than three is not able to be transformed into a line by a local switching. Only an odd cycle with length three can be torans-formed into a line by a local switching.

(2) A fundamental even cycle C can not be transformed into line by a sequence of local switchings when it remains as an even cycle. If there is a vertex v outside C and it is adjacent to only two vertices a, b in C where ab is an edge of C, by local switching at v, we obtain an even cycle with length one longer than that of C. Moreover if this cycle consisits of odd cycles, it may be trannsformed into a line by a sequence of local switchings.

(3) The outside vertex v is adjacent to some vertices in C any two of which are not adjacent. Scince by local switching at v, we obtain some fundamental even cycles, we must avoid this local switching in order to transform the cycle into a line.

Example 1. Let $G = (V, E)$ be a signed graph with $V = \{a_1, a_2, a_3, a_4, a_5\}$ and $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_3a_5, a_4a_1\}$, $E^- = \{a_1a_5\}$. An even cycle $a_1a_2a_3a_4a_1$ consists of two odd cycles $a_1a_2a_3a_5a_1$ and $a_1a_5a_3a_4a_1$. As a_2 and a_4 are regarded inner vertices of the even cycle, by local switching at a_2 or a_4, we obtain an even 3-cycle. On the other hand, by local switching at a_5, we get three odd 3-cycles. By local switching with respect to $(a_1, J = \{a_2\})$, we get an even 3-cycle, because it is regarded as local switching at an inner vertex of an even cycle. On the other hand, by local switching with respect to $(a_1, J = \{a_2, a_4\})$, we obtain odd 3-cycles.

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