J. Math. Univ. Tokushima
Vol.38(2004), 1 − 7

# Local Switching of Some Signed Graphs

By

Toru ISHIHARA

*Faculty of Integrated Arts and Sciences,*
*The University of Tokushima,*
*Minamijosanjima, Tokushima 770-8502, JAPAN*
*e-mail adress: ishihara@ias.tokushima-u.ac.jp*
(Received September 30, 2004)

**Abstract**

Some singed graphs are transformed to trees by a sequence of local switchings.We give some examples of such signed graphs to investigate when signed graphs are transformed to trees by local switching.

2000 Mathematics Subject Classification. 05C22

## Introduction

Local switching of signed graphs is introduced by P. J. Cameron, J.J. Seidel and S. V. Tsaranov in [3]. Some singed Fushimi trees are tansformed to trees by a sequence of local switchings [4]. Signed cycles with odd parity are transformed to trees by a sequence of local switchings, but signed cycles with even parity can not be transformed to trees by no means [5]. What kinds of graphs are transformed to trees by a sequence of local switchings ? It is important and intresting to give examples of signed graphs which are transformed to trees by a sequence of local switchings. In this note, we give rather simple examples of such signed graphs.

We state briefly basic facts about signed graphs. A graph $G = (V, E)$ consists of an n-set $V$(the vertices) and a set $E$ of unordered pairs from $V$(the edges). A *signed graph* $(G, f)$ is a graph $G$ with a signing $f : E \to \{1, -1\}$ of the edges. We set $E^+ = f^{-1}(+1)$ and $E^- = f^{-1}(-1)$. For any subset $U \subseteq V$ of vertices, let $f_U$ denote the signing obtained from $f$ by reversing the sign of each edge which has one vertex in $U$. This defines on the set of signings an equivalence relation, called *switching*. The equivalence classes $\{f_U : U \subseteq V\}$ are the *signed swithing classes* of the graph $G = (V, E)$.

Let $i \in V$ be a vertex of $G$, and $V(i)$ be the neighbours of $i$. The *local graph* of $(G, f)$ at $i$ has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of $G$ for which $f(i,j)f(j,k)f(k,i) = -1$. A *rim* of $(G, f)$ at $i$ is any union of connected components of local graph at $i$. Let $J$ be any rim at $i$, and let $K = V(i)\backslash J$. *Local switching* of $(G, f)$ with respect to $(i, J)$ is the following operation:(i) delete all edges of $G$ between $J$ and $K$; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j, k\}$ with sign chosen so that $f(i,j)f(j,k)f(k,i) = -1$; (iii) change the signs of all edges from $i$ to $J$; (iv) leave all other edges and signs unaltered. Let $\Omega_n$ be the set of switching classes of signed graphs of order $n$. Local switching , applied to any vertex and any rim at the vertex, gives a relation on $\Omega$ which is symmetric but not transitive. The equivalence classes of its tansitive closure are called the *clusters* of order $n$.

## 1. Singed graphs which are transformed into trees by local switching

A connected graph $G = (V, E)$ is called *Fushimi tree* if each block of $G$ is a complete graph. A complete graph is a Fusimi tree of one block. Let $a$ be a cut vertex of a Fushimi tree $G$. If $G$ is divided exactly $m$ connected components when the cut vertex $a$ is deleted, in the present paper, we say that *the Fushimi degree(simply F-degree) of the cut vertex $a$ is $m$* . A connected subgraph of a Fushimi tree $G$ is called a *sub-Fushimi tree* if it consists of some blocks of $G$. A block of Fushimi tree is said to be *pendant* if it has only one cut vertex. It is evident that any Fushimi tree has at least two pendant blocks.

A signed Fushimi tree is called a Fushimi tree *with positive sign* (or simply a *positive* Fushimi tree ) if we can switch all signs of edges into $+1$. A tree is always considered as a Fushimi tree with positive sign. A tree with only two leaves is said to be *a line tree* or simply *a line* in the present paper.

A $k$−cycle $C^k = (V, E)$, where $V = \{a_1, a_2, \cdots, a_k\}$, $E = \{a_1a_2, a_2a_3, \cdots, a_{k-1}a_k, a_ka_1\}$ , will be denoted simply $C^k = a_1a_2 \cdots a_ka_1$. For signed cycles, there are two switching classes, which are distinguished by the parity or the balance, where the parity of a signed cycle is the parity of the number of its edges which carry a positive sign and the balance is the product of the signs on its edges [3]. In the forthcoming paper[5],we will show the following two theorems.

**Theorem 1.** *Let $G$ be a positive Fushimai tree whose any cut vertex has F-degree 2. We can transform $G$ into a line tree by a sequence of local switchings.*

**Theorem 2.** *Let $C^k$ be a $k-$cycle. Then, it is transformed to a tree by a sequence of local switchings if and only if its parity is odd.*

We will show

**Theorem 3.***Let $G = (V, E)$ be a signed graph with $V = \{a_1, a_2, \cdots, a_n, b_2, b_3, \cdots, b_{m-1}\}$ and $E = \{a_1a_2, a_2a_3, \cdots, a_{n-1}a_n, a_na_1, a_1b_2, b_2b_3, \cdots, b_{m-1}a_n$ Consider two cycles $A^n = a_1a_2 \cdots a_na_1$ and $B^m = a_1b_2 \cdots b_{m-1}a_na_1$ .Then, the graph is transformed to a tree by a sequence of local switchings if and only if both parities of $A^n$ and $B^m$ are odd.*

Proof. Assume that the parity of $A^n$ is odd. By a sequence of local swithings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\}), \cdots$, $(a_{n-2}, J = \{a_{n-1}\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\}), \cdots$, $(a_{n-2}, J = \{a_{n-3}\})$, $(a_{n-1}, J = \{a_1\})$, we get a singned graph with edge set $E = \{a_2a_3, \cdots, a_{n-2}a_{n-1}, a_1b_2, b_2b_3, \cdots, b_{m-1}a_n, a_na_{n-1}, a_{n-1}a_1\}$. The parity of the cycle $a_1b_2b_3 \cdots b_{m-1}a_na_{n-1}a_1$ is odd if and only if the parity of $B^m$ is odd. In this case, this cycle is transformed to a tree by a sequence of local swithings. If the parity of $A^n$ is even, by a sequence of local swithings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\}), \cdots$, $(a_{n-2}, J = \{a_{n-1}\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\}, \cdots)$, $(a_{n-2}, J = \{a_{n-1}\})$, we get a singned graph with edge set $E = \{a_1a_2, a_2a_3, \cdots, a_{n-2}a_{n-1}, a_{n-1}a_n, a_{n-2}a_1, a_{n-1}a_1, a_1b_2, b_2b_3, \cdots, b_{m-1}a_n\}$. As the sign of the edge $a_{n-1}a_n$ is $-1$, the cycle $a_1a_{n-1}a_na_1$ can not be transformed to a line.

## 2. Examples of signed graphs which are transformed into trees

For $j = 3, 4, \ldots, 8$, set signed graphs $T_j = (V, E)$ as follows.

$V = \{a_1, a_2, \cdots, a_{j+2}\}$, $E^+ = \{a_ia_{i+1}, a_ia_{i+2} (i = 1, 2, \cdots, j), a_{j+1}a_{j+2}\}$, $E^- = \emptyset$.

Then, we have

**Proposition 4.** *The signed graphs $T_3, T_4, T_5, T_6, T_7$ are transformed to trees by a sequence of local switchings, but $T_8$ can not be transformed to a tree by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3\})$, from $T_3$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_4a_5, a_2a_5\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6\})$, $(a_6, J = \{a_2\})$, from $T_4$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_4a_5, a_5a_6, a_2a_6\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5\})$, $(a_6, J = \{a_2\})$, from $T_5$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_7, a_4a_7, a_7a_6, a_2a_6\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5\})$, $(a_8, J = \{a_7\})$, $(a_6, J = \{a_2\})$, $(a_2, J = \{a_8\})$, from $T_6$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_8, a_8a_2, a_2a_6, a_2a_7, a_7a_4\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5, a_8, a_9\})$, $(a_9, J = \{a_7\})$, $(a_2, J = \{a_9\})$, $(a_4, J = \{a_8\})$, $(a_8, J = \{a_9\})$, from $T_7$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_7, a_7a_9, a_9a_8, a_8a_4, a_8a_2, a_2a_6\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5, a_8\})$, $(a_2, J = \{a_7\})$, from $T_8$, we get a signed graph with edge set $E^+ = \{a_1a_3, a_3a_5, a_5a_7, a_7a_9, a_9a_{10}, a_8a_{10}, a_8a_2, a_2a_7, a_2a_6\}$, $E^- = \{a_4a_7\}$. But this graph can not be transformed to a tree at all.

It is rather difficult to decide that a given signed graph can not be transformed to a tree by a sequence of local switchings. We describe some facts concerning with this point.

**Remark.** A signed cycle with even parity can not be transformed to a tree by a sequence of local switchings. Hence, we do not make a 3-cycle with even parity by local switching. Set $G_1 = (V, E)$ be a signed graph with vertex set $V = \{a_1, a_2, b_1, b_2, c\}$ and edge sets $E^+ = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_1c\}$, $E^- = \{a_2c\}$. By local switching at $b_1$ or $b_2$, we get a 3-cycle with even parity, we can not apply it. By local switching at $a_1$ or $a_2$, if $b_1$ is in $J$ and $b_2$ is in $K$ or if the reverse holds, we get a 3-cycle with even parity. Similarly, set $G_2 = (V, E)$ be a signed graph with vertex set $V = \{a_1, a_2, b_1, \cdots b_n, c\}$ and edge sets $E^+ = \{a_1b_1, \cdots, a_1b_n, a_2b_1, \cdots, a_2b_n, a_1c\}$, $E^- = \{a_2c\}$. We can no do local switching at any $b_i$, $(1 \leq i \leq n)$. If we apply local switching at $a_1$ or $a_2$, all $b_i'$s must be in $J$ or in $K$.

Let $Q_3 = (V, E)$ be a signed graph with $V = \{a_1, a_2, \cdots, a_7, a_8\}$, $E^+ = \{a_1a_2, a_1a_3, a_3a_4, a_3a_5, a_5a_6, a_5a_7, a_7a_8\}$ and $E^- = \{a_2a_4, a_4a_6, a_6a_8\}$. and $Q_4 = (V, E)$ be a signed graph with $V = \{a_1, a_2, \cdots, a_9, a_{10}\}$, $E^+ = \{a_1a_2, a_1a_3, a_3a_4, a_3a_5, a_5a_6, a_5a_7, a_7a_8\ a_7a_9, a_9a_{10}\}$ and $E^- = \{a_2a_4, a_4a_6, a_6a_8, a_8a_{10}\}$.
Now we have

**Proposition 5.** *The graph $Q_3$ is transformed to the graph $T_6$ , and hence to a tree, by a sequence of local switchings. The graph $Q_4$ is transformed to the graph $T_8$ by a sequence of local switchings. Hence this graph can not be transformed to a tree by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_1, J = \{a_2\})$, $(a_7, J = \{a_8\})$, $(a_6, J = \{a_5\})$, $(a_8, J = \{a_7\})$, we get $T_6$ from $Q_3$. Similarly, by a sequence of local switchings, $(a_1, J = \{a_2\})$, $(a_{10}, J = \{a_9\})$, $(a_7, J = \{a_8\})$, $(a_6, J = \{a_5\})$, $(a_9, J = \{a_{10}\})$, $(a_8, J = \{a_7\})$, $(a_{10}, J = \{a_9\})$, we get $T_8$ from $Q_4$.

Define $QH_2$ , $QH_3$ and $QH_4$ as follows;

$QH_2 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, b_1, b_2\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_3b_2\}$, $E^- = \{a_4a_1\}$;

$QH_3 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_2b_2, a_3b_3\}$, $E^- = \{a_4a_1\}$;

$QH_4 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$, $E^- = \{a_4a_1\}$;

We can prove

**Proposition 6.** *The signed graphs $QH_2, QH_3, QH_4$ are transformed to trees by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_4, J = \{a_1\})$, $(a_3, J = \{a_1, b_2\})$, $(b_2, J = \{a_3\})$, from $QH_2$, we get a tree with edge set $E = \{b_1a_1, a_1a_3, a_3b_2, b_2a_2, b_2a_4\}$. By a sequence of local switchings, $(a_4, J = \{a_1\})$, $(a_3, J = \{a_1, b_3\})$, $(b_3, J = \{a_3\})$, from $QH_3$, we get a tree with edge set $E = \{b_1a_1, a_1a_3, a_3b_3, b_3a_2, b_3a_4, a_2b_2\}$. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J =$

$\{a_2, a_4\})$, $(b_3, J = \{a_3\})$, $(b_2, J = \{a_4\})$, $(a_4, J = \{a_3, b_4\})$, $(b_4, J = \{a_4\})$, from $QH_4$, we get a tree with edge set $E = \{b_1a_1, a_1a_3, a_3a_4, a_4b_4, b_4a_3, b_4b_2, b_3a_2\}$.

Set signed graphs $PH_1, PH_2, PH_3(1), PH_3(2), PH_4, PH_5$ as follows;

$PH_1 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1\}$, $E^- = \emptyset$ ;

$PH_2 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2\}$, $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2\}$, $E^- = \emptyset$ ;

$PH_3(1) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$, $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3\}$, $E^- = \emptyset$ ;

$PH_3(2) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$, $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_4b_3\}$ , $E^- = \emptyset$;

$PH_4 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}$, $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$, $E^- = \emptyset$ ;

$PH_5 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5\}$, $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}$, $E^- = \emptyset$ ;

Now we obtain

**Proposition 7.** *The graphs $PH_1, PH_2, PH_3(1), PH_3(2), PH_4$ are transformed to trees, by a sequence of local switchings. The graph $PH_5$ can not be transformed to a tree by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_5\})$, $(a_2, J = \{a_1, a_5\})$, $(a_3, J = \{a_4\})$, from $PH_1$, we get a tree with edge set $E = \{b_1a_1, a_1a_2, a_2a_3, a_2a_5, a_3a_4\}$. By a sequence of local switchings, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_3, a_5\})$, $(a_5, J = \{a_2\})$, from $PH_2$, we get a tree with edge set $E = \{b_1a_1, a_1a_5, a_5a_4, a_2a_5, a_3a_4, a_2b_2\}$. By a sequence of local switchings, $(a_4, J = \{a_5\})$, $(a_5, J = \{a_3\})$, $(a_3, J = \{a_1, b_3\})$, $(b_3, J = \{a_3\})$, from $PH_3(1)$, we get a tree with edge set $E = \{b_1a_1, a_1a_3, a_3b_3, b_3a_2, b_3a_5, a_2b_2, a_5a_4\}$. By a sequence of local switchings, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_2, b_3\})$, $(b_3, J = \{a_4\})$, $(a_5, J = \{a_2\})$, $(b_3, J = \{a_2, a_3\})$, $(a_3, J = \{b_3\})$, from $PH_3(2)$, we get a tree with edge set $E = \{b_1a_1, a_1a_5, a_5a_3, a_3b_3, a_3a_4, b_3a_2, a_2b_2\}$. By a sequence of local switchings, $(a_5, J = \{a_1\})$, $(a_4, J = \{a_3, a_5\})$, $(b_4, J = \{a_4\})$, $(a_3, J = \{a_1, b_3\})$, $(b_3, J = \{a_3\})$, $(a_1, J = \{b_4\})$, $(a_3, J = \{b_4\})$, from $PH_4$, we get a tree with edge set $E = \{b_1a_1, a_1a_4, a_1a_3, a_3b_3, a_3b_4, b_3a_2, a_2b_2, b_4a_5\}$. By a sequence of local switchings, $(a_3, J = \{a_4\})$, $(a_5, J = \{a_4\})$, $(a_4, J = \{a_1, a_3, b_5\})$, $(b_4, J = \{a_4\})$, from $PH_5$, we get a signed graph with edges sets $E^+ = \{b_2a_2, a_2a_4, a_2b_5, b_5b_3, b_5b_4, b_3a_4, b_3a_1, a_1b_1, a_1a_4, a_1b_4, a_4a_5, a_5a_3, a_3b_4\}$, $E^- = \{a_4b_4\}$. We can not apply to this graph local switching at vertices $a_1$, or $a_2$,or $b_3$ , or $b_5$ and can not transform it to a tree.

Define $H_1$, $H_2(1)$, $H_2(2)$, $H_2(3)$, $H_3(1)$, $H_3(2)$, $H_3(3)$ and $H_4(1)$, $H_4(2)$, $H_4(3)$, as follows;

$H_1 = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1\}$, $E^- = \{a_6a_1\}$;

$H_2(1) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2\}$, $E^- = \{a_6a_1\}$;

$H_2(2) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_5b_2\}$, $E^- = \{a_6a_1\}$;

$H_2(3) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_4b_2\}$, $E^- = \{a_6a_1\}$;

$H_3(1) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_3b_3\}$, $E^- = \{a_6a_1\}$;

$H_3(2) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_4b_3\}$, $E^- = \{a_6a_1\}$;

$H_3(3) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_3b_2, a_5b_3\}$, $E^- = \{a_6a_1\}$;

$H_4(1) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_1b_1, a_2b_2, a_3b_3\ a_4b_4\}$, $E^- = \{a_6a_1\}$;

$H_4(2) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_4b_3\ a_5b_4\}$, $E^- = \{a_6a_1\}$;

$H_4(3) = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_3b_2, a_4b_3\ a_5b_4\}$, $E^- = \{a_6a_1\}$;

Now we obtain

**Proposition 8.** *The graphs $H_1, H_2(1), H_2(2), H_2(3), H_3(1), H_3(2), H_4(1)$ are transformed to trees, by a sequence of local switchings. The graphs $H_3(3)$, $H_4(2), H_4(3)$ can not be transformed to trees by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_5\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\})$, $(a_5, J = \{a_1\})$, from $H_1$, we get a tree with edge set $E = \{b_1a_1, a_1a_5, a_5a_6, a_5a_4, a_4a_3\ a_3a_2\}$. By a sequence of local switchings, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_5\})$, $(a_5, J = \{a_6\})$, $(a_4, J = \{a_3\})$, $(a_5, J = \{a_4\})$, $(a_6, J = \{a_2\})$, from $H_2(1)$, we get a tree with edge set $E = \{b_1a_1, a_1a_6, a_6a_5, a_6a_2, a_5a_4\ a_2b_2, a_4a_3\}$. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, $(a_4, J = \{a_5\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\})$, $(a_5, J = \{a_1, b_2\})$, $(b_2, J = \{a_5\})$, from $H_2(2)$, we get a tree with edge set $E = \{b_1a_1, a_1a_5, a_5b_2, b_2a_6, b_2a_4\ a_4a_3, a_3a_2\}$. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_1\})$, $(a_6, J = \{a_5\})$, $(a_5, J = \{a_1\})$, from $H_2(3)$, we get a signed graph with edge sets $E^+ = \{a_1a_3, a_3a_4, a_4a_5, a_1b_1, a_3a_2, a_4b_2, a_5a_6\}$, $E^- = \{a_1a_5\}$ which is isomorphic to the singed graph $QH_4$. Hence, $H_2(3)$ is transformed to a tree, by a sequence of local switchings. By a sequence of local switchings, $(a_6, J = \{a_5\})$, $(a_5, J = \{a_4\})$, $(a_4, J = \{a_3\})$, $(a_5, J = \{a_6\})$, $(a_4, J = \{a_5\})$, $(a_3, J = \{a_1, b_3\})$, $(b_3, J = \{a_3\})$, from $H_3(1)$, we get a tree with edge set $E = \{b_1a_1, a_1a_3, a_3b_3, b_3a_2, a_2b_2, b_3a_4, a_4a_5, a_5a_6\}$. By a sequence of local switchings, $(a_6, J = \{a_5\})$, $(a_5, J = \{a_4\})$, $(a_5, J = \{a_6\})$, $(a_4, J = \{a_1, b_3\})$, $(b_3, J = \{a_4\})$, $(a_3, J = \{a_1\})$, $(a_1, J = \{b_1, b_3\})$, $(b_1, J = \{a_1\})$, from $H_3(2)$, we get a tree with edge set $E = \{b_1a_1, a_1b_3, b_3a_5, a_5a_6, b_1a_4, b_1a_3, a_3a_2, a_2b_2\}$. By a sequence of local switchings, $(a_6, J = \{a_5\})$, $(a_5, J = \{a_4\})$, $(a_5, J = \{a_6\})$, $(a_4, J = \{a_1, b_4\})$, $(b_4, J = \{a_4\})$, $(a_4, J = \{b_4\})$, $(a_1, J = \{a_2, b_4\})$, $(b_1, J = \{a_1\})$, $(a_4, J = \{a_1\})$, $(a_2, J = \{a_1, b_2\})$, $(b_2, J = \{a_2\})$, from $H_4(1)$, we get a tree with edge set $E = \{b_3a_3, a_3a_1, a_1a_2,$

$a_2b_2, b_2a_4, b_2b_1, b_1b_4, b_4a_5\ a_5a_6\}$.

The graphs $H_3(3), H_4(2), H_4(3)$ can not be transformed to trees by any sequences of local switchings at all.

Define $C_{44}$, $C_{45}$, $C_{55}$ and $H_{56}$ as follows;

$C_{44} = (V, E)$, $V = \{a_1, a_2, a_3, a_4, b_1\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1b_1\}$, $E^- = \{a_3b_1\}$;

$C_{45} = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, b_1\}$, $E^+ = \{a_1a_2, a_3a_4, a_4a_5, a_5a_1, a_1b_1\ a_3b_1\}$, $E^- = \{a_2a_3\}$;

$C_{55} = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_6a_1, a_1b_1, a_4b_1\}$, $E^- = \emptyset$;

$C_{56} = (V, E)$, $V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1\}$, $E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_5a_6, a_6a_7, a_1b_1, a_4b_1\}$, $E^- = \{a_4a_5\}$;

Then, we get

**Proposition 9.** *The graphs $C_{44}, C_{45}, C_{55}$ are transformed to trees, by a sequence of local switchings. The graph $C_{56}$ can not be transformed to a tree by a sequence of local switchings.*

Proof. By a sequence of local switchings, $(a_1, J = \{b_1\})$, $(b_1, J = \{a_1, a_3\})$, from $C_{44}$, we get a tree with edge set $E = \{b_1a_1, b_1a_2, b_1a_3,\ b_1a_4\}$. By a sequence of local switchings, $(a_1, J = \{b_1\})$, $(b_1, J = \{a_2, a_5\})$, $(a_5, J = \{a_3\})$, from $C_{45}$, we get a tree with edge set $E = \{b_1a_1, b_1a_2, b_1a_5,\ a_5a_3, a_5a_4\}$. By a sequence of local switchings, $(a_3, J = \{a_4\})$, $(a_5, J = \{a_6\})$, $(a_1, J = \{b_1\})$, $(b_1, J = \{a_2, a_6\})$, $(a_3, J = \{a_4\})$, $(a_5, J = \{a_6\})$, $(a_6, J = \{a_4\})$, $(a_2, J = \{a_4\})$, $(a_4, J = \{b_1\})$, from $C_{55}$, we get a tree with edge set $E = \{b_1a_1, b_1a_4, a_4a_2,\ a_4a_6, a_2a_3\ a_6a_5\}$. The graph $C_{56}$ can not be transformed to a tree by any sequences of local switchings at all.

# References


[1] P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult Line graphs, root systems, and Elliptic geometry, J. Algebra, **43**(1976), 305-327.

[2] D. M. Cvetkovic, M. Doob , I. Gutman and A. Torgasev, Recent results in the theory of graph spectra, Annals of Discrete Mathematics, **36**(1991), North-Holand, Amsterdam.

[3] P. J. Cameron, J. J. Seidel and S. V. Tsaranov, Signed Graphs, Root lattices, and Coxeter groups, J. Algebra, **164**(1994), 173-209.

[4] T. Ishihara, Signed graphs associated with the lattice $A_n$, J. Math. Univ. Tokushima, **36**(2002), 1-6.

[5] T. Ishihara, Local switiching of signed Fushimi tree (in preparation).