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Theory of Infraexponential Holomorphic Functions

By

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Abstract

In this paper, we define the notion of $\tilde{\mathcal{O}}$-pseudoconvex open sets without using the technical conditions of Kawai[13], [14]. Thereby $\widetilde{\boldsymbol{C}}^n$ can be considered as an $\tilde{\mathcal{O}}$-pseudoconvex open set. Further it is proved that an $\tilde{\mathcal{O}}$-pseudoconvex open set is an open set in $\widetilde{\boldsymbol{C}}^n$ whose finite part is pseudoconvex open set in $\boldsymbol{C}^n$. We prove that a domain of $\tilde{\mathcal{O}}$-holomorphy is an $\tilde{\mathcal{O}}$-pseudoconvex open set. But the converse is an open problem.

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Introduction

In 1969 and 1970, Kawai defined the sheaf $\tilde{\mathcal{O}}$ of infraexponential holomorphic functions and proved the Oka-Cartan-Kawai Theorem B for $\tilde{\mathcal{O}}$-pseudoconvex open sets in his papers on Fourier hyperfunctions[13], [14]. In his definition of $\tilde{\mathcal{O}}$-pseudoconvex domains, one technical condition is assumed. Thereby $\widetilde{\boldsymbol{C}}^n$ is not $\tilde{\mathcal{O}}$-pseudoconvex. Many authors follow him[4], [5], [6], [7], [8], [9], [10], [11], [12].

In this paper, we define the notion of $\tilde{\mathcal{O}}$-pseudoconvex open sets without this technical condition. Thereby $\widetilde{\boldsymbol{C}}^n$ can be considered an $\tilde{\mathcal{O}}$-pseudoconvex open set. Further we clarify the relation of $\tilde{\mathcal{O}}$-pseudoconvex open sets and pseudoconvex open sets in $\boldsymbol{C}^n$. Namely, an $\tilde{\mathcal{O}}$-pseudoconvex open set Ω is nothingelse but an open set in $\widetilde{\boldsymbol{C}}^n$ such that $\Omega \cap \boldsymbol{C}^n$ is a pseudoconvex open set in $\boldsymbol{C}^n$. Therefore, when we merely add an pseudoconvex open set in $\boldsymbol{C}^n$ the corresponding points at infinity, it does not become an $\tilde{\mathcal{O}}$-pseudoconvex open set unless it is an open set in $\widetilde{C}^n$. In this point there exists a difference between the theory of holomorphic functions on C^n and the theory of infraexponential holomorphic functions on $\widetilde{C}^n$. We know that domains of $\tilde{\mathcal{O}}$-holomorphy on $\widetilde{C}^n$ are $\tilde{\mathcal{O}}$-pseudoconvex open sets. But the converse is an open problem. It is also an open problem wether we can prove the Oka-Cartan-Kawai Theorem B for arbitrary $\tilde{\mathcal{O}}$-pseudoconvex open sets without Kawai's technical condition or not.

We improve the proof of a key Lemma for the proof of Runge's Theorem for rapidly decreasing holomorphic functions. Using this, we prove Runge's Theorem.

Note. While I was writing this paper, I obtained Berenstein and Struppa's paper[1]. There they also tried to generalize the notion of $\tilde{\mathcal{O}}$-pseudoconvex open sets. But their definition is different from ours.

1. The sheaves $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$

At first we remember the notion of holomorphic functions. Let C^n be the n-dimensional complex Euclidean space and Ω an open set in C^n. A smooth function $f(z)$ on Ω is said to be holomorphic if it satisfies the Cauchy-Riemann equation $\overline{\partial} f = 0$ on Ω. We denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions on Ω. We define the sheaf $\mathcal{O}$ of holomorphic functions over C^n to be the sheaf $\{\mathcal{O}(\Omega); \Omega$ is an open set in $C^n\}$. For every $f(z) \in \mathcal{O}(\Omega)$, $\sup\{|f(z)|; z \in K\} < \infty$ holds for every compact subset K of Ω. If we define a seminorm $\| f \|_K$ of $\mathcal{O}(\Omega)$ by the relation $\| f \|_K = \sup\{|f(z)|; z \in K\}$, $\mathcal{O}(\Omega)$ becomes a Fréchet space with respect to the topology defined by the family of seminorms $\{\| f \|_K; K$ is a compact set in $\Omega\}$.

We denote by $C(\Omega)$ the space of all continuous functions on Ω.

Next we remember the definition of radial compactification D^n of the n-dimensional real Euclidean space R^n following Kawai[14], Definition 1.1.1, where $n \geq 1$.

Definition 1.1(Kawai). We denote by D^n the radial compactification $R^n \sqcup S_\infty^{n-1}$ which denotes the disjoint union of R^n and the $(n-1)$-dimensional sphere S_∞^{n-1} at infinity. When x is a vector in $R^n \setminus \{0\}$, we denote by $x\infty$ the point in S_∞^{n-1} whose representative is x in the identification of S_∞^{n-1} with $(R^n \setminus \{0\})/R^+$. Here R^+ denotes the set of all positive real numbers. Each element in R^+ is considered as a multiplication operator on $R^n \setminus \{0\}$. The space D^n is endowed with the following natural topology. Namely, (i) if a point x of D^n belongs to R^n, a fundamental system of neighborhoods of x is given by the family of all open spheres in R^n including x. (ii) If a point x of D^n belongs to S_∞^{n-1}, a fundamental system of neighborhoods of $x(= y\infty)$ is given by the family $\{(C + a) \cup C_\infty; C_\infty \ni y\infty\}$. Here a runs through all points in $\boldsymbol{R}^n$ and C runs through all open cones in $\boldsymbol{R}^n$ with the vertex at the origin which contains $y \in \boldsymbol{R}^n \setminus \{0\}$ and C_∞ denotes the set $\{z\infty; z \in C\}$.

We denote by $\widetilde{\boldsymbol{C}}^n$ the space $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ endowed with the direct product topology. $\boldsymbol{D}^n$ and S_∞^{n-1} are identified with the subsets of $\widetilde{\boldsymbol{C}}^n$ by the relations $\boldsymbol{D}^n \simeq \boldsymbol{D}^n \times \sqrt{-1}\{0\} \hookrightarrow \widetilde{\boldsymbol{C}}^n$ and $S_\infty^{n-1} \simeq S_\infty^{n-1} \times \sqrt{-1}\{0\} \hookrightarrow \widetilde{\boldsymbol{C}}^n$. For a subset E of $\widetilde{\boldsymbol{C}}^n$, we denote by $\mathrm{int}(E)$ its interior and by $\mathrm{cl}(E) = E^{\mathrm{cl}}$ its closure with respect to the topology of $\widetilde{\boldsymbol{C}}^n$. For $n = 1$, we put $\boldsymbol{D} = \boldsymbol{D}^1$ and $\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{C}}^1$.

Definition 1.2(the sheaf $\widetilde{\mathcal{O}}$ of slowly increasing holomorphic functions). We define the sheaf $\widetilde{\mathcal{O}}$ over $\widetilde{\boldsymbol{C}}^n$ to be the sheaf $\{\widetilde{\mathcal{O}}(\Omega); \Omega$ is an open set in $\widetilde{\boldsymbol{C}}^n\}$, where the section module $\widetilde{\mathcal{O}}(\Omega)$ on an open set Ω in $\widetilde{\boldsymbol{C}}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^n$ such that, for any positive number ε and for any compact set K in Ω, the estimate $\sup\{|f(z)|e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^n\} < \infty$ holds. Here $e(t)$ denotes the exponential function $e^t = \exp(t)$ of $t \in \boldsymbol{C}$. A function $f \in \widetilde{\mathcal{O}}(\Omega)$ is also said to be an infraexponential holomorphic function.

If we define a seminorm $\|f\|_{K,\varepsilon}$ of $\widetilde{\mathcal{O}}(\Omega)$ by the relation

$$\|f\|_{K,\varepsilon} = \sup\{|f|e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^n\},$$

the space $\widetilde{\mathcal{O}}(\Omega)$ becomes an FS-space with respect to the topology defined by the family of seminorms $\{\|f\|_{K,\varepsilon}; K$ is a compact set in Ω and ε is a positive number$\}$. As to the notion of FS-spaces, we refer to Komatsu[15], [16].

Definition 1.3(the sheaf $\underset{\sim}{\mathcal{O}}$ of rapidly decreasing holomorphic functions). We define the sheaf $\underset{\sim}{\mathcal{O}}$ over $\widetilde{\boldsymbol{C}}^n$ to be the sheaf $\{\underset{\sim}{\mathcal{O}}(\Omega); \Omega$ is an open set in $\widetilde{\boldsymbol{C}}^n\}$, where the section module $\underset{\sim}{\mathcal{O}}(\Omega)$ on an open set Ω in $\widetilde{\boldsymbol{C}}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^n$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{|f(z)|e(\delta|z|); z \in K \cap \boldsymbol{C}^n\} < \infty$ holds.

Definition 1.4(definition of the space $\mathcal{O}_b^\eta(U)$). Let U be an open set in $\widetilde{\boldsymbol{C}}^n$. For $\eta \in \boldsymbol{R}$, the Banach space $\mathcal{O}_b^\eta(U)$ is defined to be the space

$$\mathcal{O}_b^\eta(U) = \{f \in C(U^{\mathrm{cl}} \cap \boldsymbol{C}^n; f|_{U \cap \boldsymbol{C}^n} \in \mathcal{O}(U \cap \boldsymbol{C}^n),$$

$$\sup\{|f(z)|e(-\eta|z|); z \in U^{\mathrm{cl}} \cap \boldsymbol{C}^n\} < \infty\}.$$

Let K be a compact set in $\widetilde{\boldsymbol{C}}^n$. Let $\underset{\sim}{\mathcal{O}}(K)$ be the space of all rapidly decreasing holomorphic functions on a certain neighborhood of K.

Let $\{U_m\}_{m \geq 1}$ be a fundamental system of neighborhoods of K such that $U_{m+1} \subset\subset U_m$, which means that U_{m+1} has a compact neighborhood in U_m with respect to the topology of $\widetilde{\boldsymbol{C}}^n$. Then we have the isomorphism $\underset{\sim}{\mathcal{O}}(K) \cong \lim\limits_{\substack{\longrightarrow \\ m}} \mathrm{ind}\, \mathcal{O}_b^{-1/m}(U_m)$.

Then $\underset{\sim}{\mathcal{O}}(K)$ becomes a DFS-space. As to the notion of DFS-spaces, we refer to Komatsu[15], [16].

Let Ω be an open set in $\widetilde{C}^n$ and $\{K_m\}_{m\geq 1}$ be an exhausting family of compact subsets of Ω such that $K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots \subset \Omega$ and $\cup_m K_m = \Omega$ holds. Then we have an isomorphism

$$\underset{\sim}{\mathcal{O}}(\Omega) \cong \lim_{m} \operatorname{proj} \underset{\sim}{\mathcal{O}}(K_m).$$

Then $\underset{\sim}{\mathcal{O}}(\Omega)$ becomes an FS-space with respect to the projective limit topology by Lemma A in Ito[7], p.262.

It is easy to see that $\widetilde{\mathcal{O}}|_{C^n} = \underset{\sim}{\mathcal{O}}|_{C^n} = \mathcal{O}$.

2. $\widetilde{\mathcal{O}}$-subharmonic functions

We recall that a C^2-function h in an open set Ω in C is called harmonic if $\Delta h = 4\partial^2 h/\partial z \partial \overline{z} = 0$ in Ω.

Let Ω be an open set in $\widetilde{C}$. A C^2-function h on $\Omega \cap C$ is called $\widetilde{\mathcal{O}}$-harmonic if the following (i) and (ii) hold:

(i) h is harmonic on $\Omega \cap C$.

(ii) For every compact set L in $\widetilde{C}$, $h(z)$ is bounded on $L \cap C$.

Definition 2.1. Let Ω be an open set in C. A function u defined on Ω and with values in $[-\infty, +\infty)$ is called subharmonic if

(a) u is upper semicontinuous, that is, $\{z; z \in \Omega, u(z) < s\}$ is open for every real number s.

(b) For every compact set K in Ω and every continuous function h on K which is harmonic in the interior of K and is $\geq u$ on the boundary of K we have $u \leq h$ in K.

By our definition the function which is $-\infty$ identically is subharmonic. But sometimes this is excluded in the definition.

Definition 2.2. Let Ω be an open set in $\widetilde{C}$. A function u defined on $\Omega \cap C$ and with values in $[-\infty, +\infty)$ is called $\widetilde{\mathcal{O}}$-subharmonic if

(i) u is subharmonic in $\Omega \cap C$.

(ii) For every compact set L in Ω, u is bounded on $L \cap C$.

Theorem 2.3. *Let Ω be an open set in $\widetilde{C}$. Then we have the following.*

(1) *If u is $\widetilde{\mathcal{O}}$-subharmonic in Ω and $c > 0$, it follows that cu is $\widetilde{\mathcal{O}}$-subharmonic in Ω.*

(2) *If u_α, $(\alpha \in A)$ is a family of $\widetilde{\mathcal{O}}$-subharmonic functions in Ω, then $u = \sup_\alpha u_\alpha$ is $\widetilde{\mathcal{O}}$-subharmonic if u is upper semicontinuous and bounded on $L \cap C$ for every compact set L in Ω, which is always the case if A is finite.*

(3) *If $u_1, u_2, \cdots$ is a decreasing sequence of $\widetilde{\mathcal{O}}$-subharmonic functions in Ω, then $u = \lim_{j\to\infty} u_j$ is also $\widetilde{\mathcal{O}}$-subharmonic in Ω.*

Proof. This follows from Hörmander[3], Theorem 1.6.2, p.16 and the condition (ii) of Definition 2.2. Q.E.D.

Corollary 2.4. *Let Ω be an open set in $\widetilde{C}$. $u_1 + u_2$ is $\widetilde{\mathcal{O}}$-subharmonic in Ω if u_1 and u_2 are $\widetilde{\mathcal{O}}$-subharmonic in Ω.*

Corollary 2.5. *Let Ω be an open set in $\widetilde{C}$. A function u defined in Ω is $\widetilde{\mathcal{O}}$-subharmonic if every point in Ω has a neighborhood where u is $\widetilde{\mathcal{O}}$-subharmonic.*

Theorem 2.6. *Let Ω be an open set in $\widetilde{C}$. Let φ be a convex increasing function on $\boldsymbol{R}$ and set $\varphi(-\infty) = \lim_{x\to-\infty} \varphi(x)$. Then $\varphi(u)$ is $\widetilde{\mathcal{O}}$-subharmonic in Ω if u is $\widetilde{\mathcal{O}}$-subharmonic in Ω.*

Corollary 2.7. *Let Ω be an open set in $\widetilde{C}$. Let $u_1, u_2 \geq 0$ and assume that $\log u_j$ is $\widetilde{\mathcal{O}}$-subharmonic in $\Omega (j = 1, 2; \log 0 = -\infty)$. Then $\log(u_1 + u_2)$ is $\widetilde{\mathcal{O}}$-subharmonic in Ω.*

Theorem 2.8. *Let Ω be an open set in $\widetilde{C}$. Let u be $\widetilde{\mathcal{O}}$-subharmonic in Ω and not $-\infty$ identically in any component of Ω. Then u is integrable on all compact subset of $\Omega \cap C$ (we write $u|_{\Omega\cap C} \in L^1_{\text{loc}}(\Omega \cap C)$), which implies that $u > -\infty$ almost everywhere.*

Theorem 2.9. *Let Ω be an open set in $\widetilde{C}$. If u is $\widetilde{\mathcal{O}}$-subharmonic in Ω and not $-\infty$ identically in any component of Ω, we have*

$$\int u \Delta v d\lambda \geq 0 \tag{2.1}$$

if $v \in C^2_0(\Omega \cap C)$ and $v \geq 0$. Here $d\lambda$ denotes the Lebesgue measure.

Theorem 2.10. *Let Ω be an open set in $\widetilde{C}$. Let $u \in L^1_{\text{loc}}(\Omega \cap C)$ and assume that* (2.1) *holds. Further assume that* $\operatorname{ess\,sup}\{u; z \in L \cap C\} < \infty$ *for every compact subset L of Ω. Then there is one and only one $\widetilde{\mathcal{O}}$-subharmonic function U in Ω which is equal to u almost everywhere. If φ is an integrable non-negative function of $|z|$ with compact support, we have, for every $z \in \Omega \cap C$,*

$$U(z) = \lim_{\delta\to 0} \frac{\displaystyle\int u(z - \delta z')\varphi(z')d\lambda(z')}{\displaystyle\int \varphi(z')d\lambda(z')}. \tag{2.2}$$

Proof. Since U is $\widetilde{\mathcal{O}}$-subharmonic, U is subharmonic in $\Omega \cap C$. For small δ, we have

$$U(z) \leq \int U(z - \delta z')\varphi(z')d\lambda(z') / \int \varphi(z')d\lambda(z').$$

U is semicontinuous from above. The upper limit of the right hand side when $\delta \longrightarrow 0$ is $\leq U(z)$. Hence (2.2) must hold if $u = U$ almost everywhere.

To prove the theorem we first assume that $u \in C^2(\Omega \cap C)$ such that $\sup\{u; z \in K \cap C\} < \infty$ for every compact subset K of Ω. Then (2.1) can be integrated by parts and therefore equivalent to $\Delta u \geq 0$. Hence

$$\int \left(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial \theta^2} \right) u(z + re^{i\theta})d\theta \geq 0.$$

If we write $M(r) = (2\pi)^{-1} \int_0^{2\pi} u(z+re^{i\theta})d\theta$, it follows that $M''(r)+r^{-1}M'(r) \geq 0$, that is, $rM'(r)$ is increasing. Since $rM'(r) \longrightarrow 0$ when $r \longrightarrow 0$, we get $M'(r) \geq 0$. Hence $M(0) \leq M(r)$ which proves that u is $\tilde{\mathcal{O}}$-subharmonic.

Now choose a function $\varphi \in C_0^\infty(C)$ with support in the unit disc so that $\varphi \geq 0$ and φ depends only on $|z|$. Then

$$u_\delta(z) = \int u(z - \delta z')\varphi(z')d\lambda(z') / \int \varphi(z')d\lambda(z')$$

is in $C^\infty((\Omega \cap C)_\delta)$ and $u_\delta \longrightarrow u$ in L^1-norm on compact subsets of Ω when $\delta \longrightarrow 0$. For sufficiently small δ, ess $\sup\{u_\delta(z); z \in K \cap C\} < \infty$ for every compact subset K of Ω_δ. It is immediately verified that (2.1) holds in Ω_δ with u replaced by u_δ. Hence the first part of the proof shows that u_δ is $\tilde{\mathcal{O}}$-subharmonic, which implies that

$$\int u_\delta(z - \varepsilon z')\varphi(z')d\lambda(z') / \int \varphi(z')d\lambda(z')$$

decreases when $\varepsilon \downarrow 0$. If we let $\delta \longrightarrow 0$, we conclude that $u_\varepsilon(z)$ decreases when $\varepsilon \downarrow 0$. Hence $U(z) = \lim_{\varepsilon \to 0} u_\varepsilon(z)$ exists and is $\tilde{\mathcal{O}}$-subharmonic by Hörmander[3], Theorem 1.6.2, p.16. Since $u_\varepsilon \longrightarrow u$ in $L^1_{\mathrm{loc}}(\Omega)$ we conclude that $U = u$ almost everywhere, which completes the proof. Q.E.D.

We have thus proved that a function $u \in C^2$ is $\tilde{\mathcal{O}}$-subharmonic in Ω if and only if $\Delta u \geq 0$ in $\Omega \cap C$ and the condition (ii) of Definition 2.2 holds. In the above, when $\Delta u > 0$ we shall say that u is strictly $\tilde{\mathcal{O}}$-subharmonic in Ω.

Theorem 2.11. *Let Ω be an open set in $\tilde{C}$. If $0 \leq f \in C^2$ and $\log f$ is $\tilde{\mathcal{O}}$-subharmonic in Ω, the function $\log(1+f)$ is strictly $\tilde{\mathcal{O}}$-subharmonic in Ω except where $\operatorname{grad} f = \Delta f = 0$.*

3. $\tilde{\mathcal{O}}$-plurisubharmonic functions

Definition 3.1. Let Ω be an open set in $\tilde{C}^n$. We call a function φ on Ω $\tilde{\mathcal{O}}$-plurisubharmonic if the following two conditions are satisfied:

(i) φ is a plurisubharmonic function on $\Omega \cap C^n$.

(ii) For every compact subset L in Ω, φ is bounded on $L \cap C^n$.

Theorem 3.2. *Let Ω be an open set in $\tilde{C}^n$. Then we have the following:*

(1) *If u is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω and $c > 0$, it follows that cu is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω.*

(2) *If $u_\alpha, (\alpha \in A)$ is a family of $\tilde{\mathcal{O}}$-plurisubharmonic functions in Ω, then $u = \sup_\alpha u_\alpha$ is $\tilde{\mathcal{O}}$-plurisubharmonic if u is upper semicontinuous and bounded on $L \cap C^n$ for every compact set L in Ω, which is always the case if A is finite.*

(3) *If $u_1, u_2, \cdots$ is a decreasing sequence of $\tilde{\mathcal{O}}$-plurisubharmonic functions in Ω, then $u = \lim_{j\to\infty} u_j$ is also $\tilde{\mathcal{O}}$-plurisubharmonic in Ω.*

Proof. It goes in a similar way to Theorem 2.3. Q.E.D.

Corollary 3.3. *Let Ω be an open set in $\tilde{C}^n$. u_1+u_2 is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω if u_1 and u_2 are $\tilde{\mathcal{O}}$-plurisubharmonic in Ω.*

Corollary 3.4. *Let Ω be an open set in $\tilde{C}^n$. A function u defined in Ω is $\tilde{\mathcal{O}}$-plurisubharmonic if every point in Ω has a neighborhood where u is $\tilde{\mathcal{O}}$-plurisubharmonic.*

Theorem 3.5. *Let Ω be an open set in $\tilde{C}^n$. Let φ be a convex increasing function on $\boldsymbol{R}$ and set $\varphi(-\infty) = \lim_{x\to-\infty} \varphi(x)$. Then $\varphi(u)$ is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω if u is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω.*

Corollary 3.6. *Let Ω be an open set in $\tilde{C}^n$. Let $u_1, u_2 \geq 0$ and assume that $\log u_j$ is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω $(j = 1, 2; \log 0 = -\infty)$. Then $\log(u_1 + u_2)$ is $\tilde{\mathcal{O}}$-plurisubharmonic in Ω.*

Theorem 3.7. *Assume that $0 \leq \varphi \in C_0^\infty(\boldsymbol{C}^n), \varphi = 0, (|z| > 1), \varphi$ depends only on $|z_1|, \cdots, |z_n|$ and $\int \varphi(z)d\lambda(z) = 1$, where $d\lambda$ is the Lebesgue measure. Let u be $\tilde{\mathcal{O}}$-plurisubharmonic in Ω. Put*

$$u_\varepsilon(z) = \int u(z - \varepsilon\zeta)\varphi(\zeta)d\lambda(\zeta).$$

Then $u_\varepsilon(z)$ is $\tilde{\mathcal{O}}$-plurisubharmonic and $u_\varepsilon \in C^\infty$ where $d(z, (C\Omega) \cap \boldsymbol{C}^n) > \varepsilon$, and $u_\varepsilon \downarrow u$ where $\varepsilon \downarrow 0$ (we asuume that u is not identically $-\infty$).

Proof. In Theorem 2.10, that u_ε decreases when $\varepsilon \downarrow 0$ was proved in the case $n = 1$.

Iteration of this result shows that u_ε is decreasing also if $n > 1$, and from the case $n = 1$ we also immediately find that $u \leq u_\varepsilon$. Since $\lim_{\varepsilon\to 0} u_\varepsilon \leq u$ in view of the upper semicontinuity of u, we conclude that $u_\varepsilon \downarrow u$ when $\varepsilon \downarrow 0$.

That u_ε is $\tilde{\mathcal{O}}$-plurisubharmonic follows immediately from the case $n = 1$.

Conversely Theorem 3.2 shows immediately that the limit of a decreasing sequence of $\tilde{\mathcal{O}}$-plurisubharmonic functions is $\tilde{\mathcal{O}}$-plurisubharmonic. Q.E.D.

4. Domains of $\tilde{\mathcal{O}}$-holomorphy

At first we remember the notion of domains of holomorphy following Definition 2.5.1 of Hörmander[3], p.36 and some of their properties.

Definition 4.1. An open set Ω in $\boldsymbol{C}^n$ is called a domain of holomorphy if there exist no open sets Ω_1 and Ω_2 in $\boldsymbol{C}^n$ with the following properties:

(a) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

(b) Ω_2 is connected and not contained in Ω.

(c) For every $u \in \mathcal{O}(\Omega)$ there is a function $u_2 \in \mathcal{O}(\Omega_2)$ (necessarily uniquely determined) such that $u = u_2$ in Ω_1.

Definition 4.2. Let Ω be an open set in $\boldsymbol{C}^n$ and K a compact set in Ω. Then $\hat{K}_\Omega$ is said to be an $\mathcal{O}(\Omega)$-hull of K and defined to be the set $\hat{K}_\Omega = \{z; z \in \Omega, |f(z)| \leq \sup|f| \text{ if } f \in \mathcal{O}(\Omega)\}$.

Let D be an open polydisc in $\boldsymbol{C}^n$ with center at 0. Put $\Delta_\Omega^D(z) = \sup\{r; \{z\}+ rD \subset \Omega\}$. Let $f \in \mathcal{O}(\Omega)$ and assume $|f(z)| \leq \Delta_\Omega^D(z), (z \in K)$. Let $u \in \mathcal{O}(\Omega)$. Then, for every $\zeta \in \hat{K}_\Omega, \sum_\alpha (z-\zeta)^\alpha \partial^\alpha u(\zeta)/\alpha!$ converges where z belongs to the polydisc $\{\zeta\} + |f(\zeta)|D$.

Let δ be an arbitrary continuous function in $\boldsymbol{C}^n$ such that $\delta > 0$ except 0 and $\delta(tz) = |t|\delta(z), (t \in \boldsymbol{C}, z \in \boldsymbol{C}^n)$. We put $\delta(z, C\Omega) = \inf_{w \in C\Omega} \delta(z-w)$. Then $\delta(z, C\Omega)$ is a continuous function of z.

Theorem 4.3. *Let Ω be a domain of holomorphy. Let $f \in \mathcal{O}(\Omega)$ and assume $|f(z)| \leq \delta(z, C\Omega), (z \in K)$, where K is a compact subset of Ω . Then we have $|f(z)| \leq \delta(z, C\Omega), (z \in \hat{K}_\Omega)$. In particular, if f is a constant, we have*

$$\inf_{z \in K, w \in C\Omega} \delta(z-w) = \inf_{z \in \hat{K}_\Omega, w \in C\Omega} \delta(z-w).$$

Theorem 4.4. *Let Ω be an open set in $\boldsymbol{C}^n$. Then the following* (1)$\sim$(4) *are equivalent:*

(1) *Let Ω be a domain of holomorphy.*

(2) *If $K \subset\subset \Omega$, we have $\hat{K}_\Omega \subset\subset \Omega$ and*

$$\sup_{z \in K} |f(z)|/\delta(z, C\Omega) = \sup_{z \in \hat{K}_\Omega} |f(z)|/\delta(z, C\Omega), (f \in \mathcal{O}(\Omega))$$

(3) *If $K \subset\subset \Omega$, we have $\hat{K}_\Omega \subset\subset \Omega$.*

(4) *There exists a function $f \in \mathcal{O}(\Omega)$ which cannot be continued analytically beyond Ω, that is, it is not possible to find Ω_1 and Ω_2 satisfying the following* (a) *and* (b):

(a) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

(b) *Ω_2 is connected and not contained in Ω, and $f_2 \in \mathcal{O}(\Omega_2)$ so that $f = f_2$ in Ω_1.*

Now we give the definition of domains of $\tilde{\mathcal{O}}$-holomorphy.

Definition 4.5. An open set in $\tilde{\boldsymbol{C}}^n$ is said to be a domain of $\tilde{\mathcal{O}}$-holomorphy if the following two condetions (i) and (ii) are satisfied:

(i) $\Omega \cap \boldsymbol{C}^n$ is a domain of holomorphy in $\boldsymbol{C}^n$.

(ii) There are no open sets Ω_1 and Ω_2 in $\tilde{\boldsymbol{C}}^n$ with the following three properties:

(a) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

(b) Ω_2 is connected and not contained in Ω.

(c) For every $u \in \tilde{\mathcal{O}}(\Omega)$ there is a function $u_2 \in \tilde{\mathcal{O}}(\Omega_2)$ (necessarily uniquely deteremined) such that $u = u_2$ in Ω_1.

5. $\widetilde{\mathcal{O}}$-pseudoconvex domains

Let Ω be a domain of $\widetilde{\mathcal{O}}$-holomorphy. Then $\Omega \cap \boldsymbol{C}^n$ is a domain of holomorphy. Let $z_0 \in \Omega \cap \boldsymbol{C}^n$ and $w \in \boldsymbol{C}^n$. Choose r so small that $D = \{z_0 + \tau w; \tau \in \boldsymbol{C}, |\tau| \leq r\} \subset \Omega \cap \boldsymbol{C}^n$.

Let $f(\tau)$ be an analytic polynomial such that $-\log \delta(z_0+\tau w, C\Omega) \leq \mathrm{Re} f(\tau)$, $|\tau| = r$. Then there exists an analytic polynomial F in $\boldsymbol{C}^n$ such that

$$F(z_0 + \tau w) = f(\tau).$$

Then our hypothesis can be written

$$|e^{-F(z)}| \leq \delta(z, (C\Omega) \cap \boldsymbol{C}^n), (z \in \partial D).$$

Since $\mathcal{O}(\Omega \cap \boldsymbol{C}^n)$-hull of ∂D contains D by the maximum principle, then, by Theorem 4.3, we have

$$|e^{-F(z)}| \leq \delta(z, (C\Omega) \cap \boldsymbol{C}^n), (z \in D).$$

That is,

$$-\log \delta(z_0 + \tau w, (C\Omega) \cap \boldsymbol{C}^n) \leq \mathrm{Re} f(\tau), |\tau| \leq r.$$

The same conclusion is obvious if $w = 0$. Hence $-\log \delta(z + \tau w, (C\Omega) \cap \boldsymbol{C}^n)$ is, for fixed $z \in \boldsymbol{C}^n$ and $w \in \boldsymbol{C}^n$, a subharmonic function of τ where it is defined.

Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$. Then we have the assertion:

> $\sup(-\log \delta(z, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}\, z|^2)$ is $\widetilde{\mathcal{O}}$-plurisubharmonic in Ω if and only if Ω is $\widetilde{\mathcal{O}}$-pseudoconvex in $\widetilde{\boldsymbol{C}}^n$.

In the sequel, we show this fact.

We denote by $\widetilde{P}(\Omega)$ the set of all $\widetilde{\mathcal{O}}$-plurisubharmonic functions on Ω.

Let K be a compact subset of Ω. We define $\widetilde{P}(\Omega)$-hull of K by the set $\hat{K}_\Omega^{\widetilde{P}}$:

$$\hat{K}_\Omega^{\widetilde{P}} = \{z \in \Omega \cap \boldsymbol{C}^n; u(z) \leq \sup_K u, u \in \widetilde{P}(\Omega)\}.$$

Definition 5.1. Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$. Then Ω is called an $\widetilde{\mathcal{O}}$-pseudoconvex open set if there exists a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function u in Ω such that

$$\Omega_c = \{z; z \in \Omega \cap \boldsymbol{C}^n, u(z) < c\} \subset\subset \Omega$$

for every $c \in \boldsymbol{R}$. We also say such an open set Ω an $\widetilde{\mathcal{O}}$-pseudoconvex domain.

For example, the open sets $\widetilde{\boldsymbol{C}}^n$ and $\boldsymbol{D}^n \times \sqrt{-1}\{y \in \boldsymbol{R}^n; |y| < a\}$ are $\widetilde{\mathcal{O}}$-pseudoconvex open sets $(a > 0)$. The latter is also $\widetilde{\mathcal{O}}$-pseudoconvex in the sense of Kawai [13], [14].

Theorem 5.2. *Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$. Then the following* (1) $\sim$ (4) *are equivalent.*

(1) Ω *is* $\widetilde{\mathcal{O}}$*-pseudoconvex.*

(2) $\sup\{-\log\delta(z,(C\Omega)\cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}$ *is* $\widetilde{\mathcal{O}}$*-plurisubharmonic in* Ω *if we put* $\delta(z,(C\Omega)\cap \boldsymbol{C}^n) = \inf_{w\in(C\Omega)\cap \boldsymbol{C}^n}\delta(z-w)$, *where* $\delta(z)$ *is a continuous function in* $\boldsymbol{C}^n$ *such that* $\delta(tz) = |t|\delta(z), (t \in \boldsymbol{C}, z \in \boldsymbol{C}^n)$, *and* $\delta(0) = 0$, $\delta(z) > 0, (z \neq 0)$.

(3) *There exists a continuous* $\widetilde{\mathcal{O}}$*-plurisubharmonic function* u *in* Ω *such that* $\Omega_c = \mathrm{int}(\{z; z \in \Omega\cap \boldsymbol{C}^n, u(z) < c\}^{\mathrm{cl}}) \subset\subset \Omega$ *for every* $c \in \boldsymbol{R}$.

(4) $\hat{K}_\Omega^{\tilde{P}} \subset\subset \Omega$ *if* $K \subset\subset \Omega$.

Proof. By Definition 5.1, (1) and (3) are equivalent. Now we prove the equivalence of conditions (2), (3) and (4). If (2) is fulfilled, we only have to set $u(z) = \sup\{-\log\delta(z,(C\Omega)\cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}$ to get a function satisfying (3). That (3) implies (4) is obvious, so we need only prove that (4) implies (2).

Let $z_0 \in \Omega\cap \boldsymbol{C}^n, 0 \neq w \in \boldsymbol{C}^n$. Choose $r > 0$ so that $D = \{z_0 + \tau w; |\tau| \leq r\} \subset \Omega \cap \boldsymbol{C}^n$. Let $f(\tau)$ be an analytic polynomial such that

$$\sup\{-\log\delta(z_0+\tau w,(C\Omega)\cap \boldsymbol{C}^n), |\mathrm{Im}(z_0+\tau w)|^2\} \leq \mathrm{Re} f(\tau), |\tau| = r.$$

Then we show that

$$\inf\{\delta(z_0+\tau w,(C\Omega)\cap \boldsymbol{C}^n), e^{-|\mathrm{Im}(z_0+\tau w)|^2}\} \geq |e^{-f(\tau)}| \qquad (5.1)$$

for $|\tau| \leq r$. That is, we show that

$$\delta(z_0+\tau w,(C\Omega)\cap \boldsymbol{C}^n) \geq |e^{-f(\tau)}|,$$

$$e^{-|\mathrm{Im}(z_0+\tau w)|^2} \geq |e^{-f(\tau)}|.$$

To do so, we take any vector $a \in \boldsymbol{C}^n, \delta(a) < 1$, and consider for $0 \leq \lambda \leq 1$ the mapping

$$\tau \to z_0 + \tau w + \lambda a e^{-f(\tau)}, |\tau| \leq r.$$

We denote by D_λ its range.

Put $B_r = \{z_0 + \tau w; |\mathrm{Im}(z_0+\tau w)|^2 \leq \mathrm{Re} f(\tau), |\tau| \leq r\}$. Then we have $B_r \cap D_0 = D$. If we put $\Lambda = \{\lambda; 0 \leq \lambda \leq 1, D_\lambda \cap B_r \subset \Omega\}$, Λ is an open subset of [0,1]. In order to show $\Lambda = [0,1]$, we show that Λ is closed. Put $K = \{z_0 + \tau w + \lambda a e^{-f(\tau)}; |\tau| = r, 0 \leq \lambda \leq 1\} \cap B_r$. Then K is a compact sebset of $\Omega\cap \boldsymbol{C}^n$. Let $u \in P(\Omega\cap \boldsymbol{C}^n)$ and $\lambda \in \Lambda$. Here $P(\Omega\cap \boldsymbol{C}^n)$ denotes the set of all plurisubharmonic functions on $\Omega \cap \boldsymbol{C}^n$. Then $\tau \longrightarrow u(z_0 + \tau w + \lambda a e^{-f(\tau)})$ is subharmonic in a neighborhood of the disc $|\tau| \leq r$. Then we have $u(z_0 + \tau w + \lambda a e^{-f(\tau)}) \leq \sup_K u$ if $|\tau| \leq r, z_0 + \tau w + \lambda a e^{-f(\tau)} \in B_r$. Then we have $D_\lambda \cap B_r \subset \hat{K}_\Omega^P$ for every $\lambda \in \Lambda$. Thus Λ is closed, for $\hat{K}_\Omega^P$ is relatively compact in $\Omega \cap \boldsymbol{C}^n$ by (4) because $\hat{K}_\Omega^P \subset \hat{K}_\Omega^{\tilde{P}} \cap \boldsymbol{C}^n \subset \Omega\cap \boldsymbol{C}^n, \hat{K}_\Omega^{\tilde{P}} \subset\subset \Omega$ and $\hat{K}_\Omega^P$ is compact in $\boldsymbol{C}^n$. Here $\hat{K}_\Omega^P$ is defined by the relation

$$\hat{K}_\Omega^P = \{z; z \in \Omega\cap \boldsymbol{C}^n, u(z) \leq \sup_K u \text{ for all } u \in P(\Omega\cap \boldsymbol{C}^n)\}.$$

Thus $D_1 \cap B_r \subset \Omega$. That is, $z_0 + \tau w + ae^{-f(\tau)} \in \Omega \cap \boldsymbol{C}^n$ if $\delta(a) < 1, |\tau| \leq r$ and $z_0 + \tau w + ae^{-f(\tau)} \in B_r$. So that,

$$\delta(z_0 + \tau w, C\Omega \cap \boldsymbol{C}^n) \geq |e^{-f(\tau)}|$$

if $|\tau| \leq r, z_0 + \tau w + ae^{-f(\tau)} \in B_r$, or

$$\sup\{-\log \delta(z_0 + \tau w, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}(z_0 + \tau w)|^2\} \leq \mathrm{Re} f(\tau), (|\tau| \leq r).$$

This proves (2). Q.E.D.

Since the supremum of a family of $\widetilde{\mathcal{O}}$-plurisubharmonic functions is $\widetilde{\mathcal{O}}$-plurisubharmonic if it is continuous, we obtain the following Theorem 5.3 from the condition (2) of Theorem 5.2.

Theorem 5.3. *Let Ω_α be an $\widetilde{\mathcal{O}}$-pseudoconvex open set for every α in an index set A. Then the interior Ω of $\cap_{\alpha \in A} \Omega_\alpha$ is also $\widetilde{\mathcal{O}}$-pseudoconvex.*

Theorem 5.4. *Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$. If, to every point in Ω^{cl}, there is a neighborhood ω such that $\omega \cap \Omega$ is $\widetilde{\mathcal{O}}$-pseudoconvex, then Ω is $\widetilde{\mathcal{O}}$-pseudoconvex.*

Proof. Let $z_0 \in \partial\Omega$. Let ω be a neighborhood of z_0 according to the hypothesis. Then we have $\delta(z, (C\Omega) \cap \boldsymbol{C}^n) = \delta(z, C(\Omega \cap \omega) \cap \boldsymbol{C}^n)$ for all z sufficiently close to z_0. $\sup\{-\log \delta(z, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}$ is $\widetilde{\mathcal{O}}$-plurisubharmonic in a neighborhood of every point on $\partial\Omega$. Then there exists a closed subset F of Ω such that $\sup\{-\log \delta(z, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}$ is $\widetilde{\mathcal{O}}$-plurisubharmonic in $\Omega \backslash F$.

There exists a continuous function $\varphi \in \widetilde{P}(\widetilde{\boldsymbol{C}}^n)$ (for example, a convex increasing function of $|\mathrm{Im}\ z|^2$) such that $\varphi(z) > \sup\{-\log \delta(z, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}$, $(z \in F)$ and $\varphi(z) \to \infty, (|\mathrm{Im}\ z| \to \infty)$. Then we have

$$u(z) = \sup\{\varphi(z), \sup\{-\log(\delta(z, (C\Omega) \cap \boldsymbol{C}^n), |\mathrm{Im}\ z|^2\}\} \in \widetilde{P}(\Omega),$$

for $u = \varphi$ in a neighborhood of F and the supremum of two $\widetilde{\mathcal{O}}$-plurisubharmonic functions is $\widetilde{\mathcal{O}}$-plurisubharmonic. u satisfies the condition (3) of Theorem 5.2, which proves that Ω is $\widetilde{\mathcal{O}}$-pseudoconvex. Q.E.D.

Theorem 5.5. *Let Ω be an $\widetilde{\mathcal{O}}$-pseudoconvex open set in $\widetilde{\boldsymbol{C}}^n$. Let K be a compact subset of Ω and ω an open neighborhood of $\hat{K}_\Omega^{\widetilde{P}} \equiv \hat{K}$. Then we have $\theta(z) \in C^\infty(\Omega \cap \boldsymbol{C}^n)$ so that the following three conditions are satisfied:*

(1) *$\theta(z)$ is strictly $\widetilde{\mathcal{O}}$-plurisubharmonic, i.e. strictly plurisubharmonic in $\Omega \cap \boldsymbol{C}^n$ and bounded on $L \cap \boldsymbol{C}^n$ for every compact subset L of Ω.*

(2) *$\theta < 0$ on $K \cap \boldsymbol{C}^n$, and $\theta > 0$ on $(\Omega \cap C\omega) \cap \boldsymbol{C}^n$.*

(3) *For every $c \in \boldsymbol{R}$, $\{z \in \Omega \cap \boldsymbol{C}^n; \theta(z) < c\} \subset\subset \Omega$.*

Proof. At first we construct a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function v satisfying (1), (2).

Since Ω is an $\widetilde{\mathcal{O}}$-pseudoconvex open set, there exists a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function u_0 on Ω so that (3) is satisfied. If necessary, adding u_0 a certain constant, we may assume that $u_0 < 0$ on $K \cap \boldsymbol{C}^n$.

Put

$$K' = \mathrm{cl}(\{z \in \Omega \cap \boldsymbol{C}^n; u_0(z) \le 2\}).$$
$$L = \mathrm{cl}(\{z \in \Omega \cap \boldsymbol{C}^n; u_0(z) \le 0\}) \cap C\omega.$$

These sets are both compact. For every $z \in L$, we can choose $w \in \widetilde{P}(\Omega)$ such that $w > 0$ at points of $\boldsymbol{C}^n$ in a certain neighborhood of z and $w < 0$ on $K \cap \boldsymbol{C}^n$. By Theorem 3.7 and by using a mollifier, we can obtain a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function w_1 such that $w_1 < 0$ on $K \cap \boldsymbol{C}^n$ and $w_1 > 0$ for points of $\boldsymbol{C}^n$ in a certain neighborhood of z. Since L is compact, by using Borel-Lebesgue's Lemma and the fact that the supremum of a finite number of $\widetilde{\mathcal{O}}$-plurisubharmonic functions is an $\widetilde{\mathcal{O}}$-plurisubharmonic function, we can construct a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function w_2 in a neighborhood of K' so that $w_2 > 0$ for points of $\boldsymbol{C}^n$ in a certain neighborhood of L and $w_2 < 0$ on $K \cap \boldsymbol{C}^n$. Let M be the maximum of w_2 on $K' \cap \boldsymbol{C}^n$. For $z \in \Omega \cap \boldsymbol{C}^n$, we put

$$v(z) = \sup\{w_2(z), Mu_0(z)\}, (\text{if } u_0(z) < 2),$$
$$v(z) = Mu_0(z), (\text{if } u_0(z) > 1).$$

When $1 < u_0(z) < 2$, both definitions are identical. Therefore v is a continuous $\widetilde{\mathcal{O}}$-plurisubharmonic function and evidently satisfies conditions (2), (3). We put

$$\Omega_c = \mathrm{int}(\{z \in \Omega \cap \boldsymbol{C}^n; v(z) < c\}^{\mathrm{cl}}).$$

By using the notation in Theorem 3.7, we put

$$v_j(z) = \int_{\boldsymbol{C}^n \cap \Omega_{j+1}} v(\zeta)\varphi((z-\zeta)/\varepsilon)\varepsilon^{-2|n|}d\lambda(\zeta) + \varepsilon|\mathrm{Im}\ z|^2, (j = 0, 1, 2, \cdots).$$

Then, if we choose ε sufficiently small depending on j, we have $v_j \in C^\infty(\boldsymbol{C}^n)$ such that $v_j > v$ at points of $\boldsymbol{C}^n$ in a certain neighborhood of $\mathrm{cl}(\Omega_j)$ and it is strictly $\widetilde{\mathcal{O}}$-plurisubharmonic. We can choose ε so small that $v_0 < 0, v_1 < 0$ on K and $v_j < v + 1 (j = 1, 2, \cdots)$ on $\boldsymbol{C}^n \cap \Omega_j$. Now, we take a convex function $\chi \in C^\infty(\boldsymbol{R})$ such that $\chi(t) = 0$ for $t < 0$ and $\chi'(t) > 0$ for $t > 0$. Then $\chi(v_j + 1 - j)$ is a strictly $\widetilde{\mathcal{O}}$-plurisubharmonic function in a neighborhood of $\mathrm{cl}(\Omega_j)\backslash\Omega_{j-1}$. Hence, choosing $a_1, a_2, \cdots$ one by one, we have

$$u_m = v_0 + \Sigma_1^m a_j \chi(v_j + 1 - j)$$

so that $u_m > v$ for every point of $\boldsymbol{C}^n$ in a neighborhood of $\mathrm{cl}(\Omega_m)$ and it is a strictly $\widetilde{\mathcal{O}}$-plurisubharmonic function. For $l, m > j$, we have $u_m = u_l$ on $\boldsymbol{C}^n \cap \Omega_j$. Therefore, we have $\theta = \lim_m u_m$ so that it is a strictly $\widetilde{\mathcal{O}}$-plurisubharmonic function. Since $\theta = v_0 < 0$ on $\boldsymbol{C}^n \cap K$ and $\theta > v$ on $\Omega \cap \boldsymbol{C}^n$, we have the properties (1)~(3). Q.E.D.

Theorem 5.6. *Let Ω and Ω' be two open sets in $\widetilde{\boldsymbol{C}}^n$ and $\widetilde{\boldsymbol{C}}^m$ respectively. Let f be an analytic map from $\Omega \cap \boldsymbol{C}^n$ into $\Omega' \cap \boldsymbol{C}^m$ so that, for every compact* *subset L of Ω, $f(L \cap \boldsymbol{C}^n)$ is relatively compact in Ω'. If $u \in \widetilde{P}(\Omega')$, then we have $f^*u \in \widetilde{P}(\Omega)$.*

Theorem 5.7. *Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$ such that $\Omega \cap \boldsymbol{C}^n$ is a domain of holomorphy. Then* $\sup\{-\log \delta(z, C(\Omega \cap \boldsymbol{C}^n)), |\mathrm{Im}\ z|^2\}$ *is $\widetilde{\mathcal{O}}$-plurisubharmonic and continuous.*

Theorem 5.8. *Let Ω be an open set in $\widetilde{\boldsymbol{C}}^n$. Then Ω is an $\widetilde{\mathcal{O}}$-pseudoconvex open set if and only if $\Omega \cap \boldsymbol{C}^n$ is a pseudoconvex open set in $\boldsymbol{C}^n$.*

Theorem 5.9. *If Ω_0 is a pseudoconvex open set in $\boldsymbol{C}^n$, then* $\Omega = \mathrm{int}(\mathrm{cl}(\Omega_0))$ *is an $\widetilde{\mathcal{O}}$-pseudoconvex open set. Here* $\mathrm{int}(\mathrm{cl}(\Omega_0))$ *is defined with respect to the topology of $\widetilde{\boldsymbol{C}}^n$.*

Theorem 5.10. *A domain of $\widetilde{\mathcal{O}}$-holomorphy Ω in $\widetilde{\boldsymbol{C}}^n$ is an $\widetilde{\mathcal{O}}$-pseudoconvex open set.*

Conversely, we propose the following Problem A.

Problem A. Is an $\widetilde{\mathcal{O}}$-pseudoconvex open set in $\widetilde{\boldsymbol{C}}^n$ is a domain of $\widetilde{\mathcal{O}}$-holomorphy?

6. Runge's Theorem

In this chapter we prove Runge's Theorem. This theorem was first proved in Kawai[14], Theorem 2.2, p.474. In this paper, the proof of Theorem 5.5, which is one of the key Lemmas for proving the following Theorem 6.1, is improved by the method of Hörmander[3], Theorem 2.6.1, p.48.

Theorem 6.1. *Let K and $L (K \subset L)$ be two compact subsets of $\widetilde{\boldsymbol{C}}^n$ such that the following two conditions are satisfied:*

(i) *K and L has fundamental systems of $\widetilde{\mathcal{O}}$-pseudoconvex open neighborhoods.*

(ii) *L is contained in the open set* $U = \mathrm{int}(\{x + \sqrt{-1}y \in \boldsymbol{C}^n; |y| < a\}^{\mathrm{cl}})$ *in $\widetilde{\boldsymbol{C}}^n$. Here a denotes a sufficiently small positive number.*

Then $\underset{\sim}{\mathcal{O}}(L)$ is dense in $\underset{\sim}{\mathcal{O}}(K)$.

Corollary 6.2. *Let K and $L (K \subset L)$ be two arbitrary compact sets in $\boldsymbol{D}^n$. Then $\underset{\sim}{\mathcal{A}}(L)$ is dense in $\underset{\sim}{\mathcal{A}}(K)$. Especially $\underset{\sim}{\mathcal{A}}(\boldsymbol{D}^n)$ is dense in $\underset{\sim}{\mathcal{A}}(K)$.*

With some preparations we prove Theorem 6.1 step by step.

Definition 6.3. Let W be an open set in $\widetilde{\boldsymbol{C}}^n$ and $\eta \in \boldsymbol{R}$. We define the space $\underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$ to be the space of all holomorphic functions on $W \cap \boldsymbol{C}^n$ such that, for an arbitrary compact subset K of W,

$$\int_{K \cap \boldsymbol{C}^n} |f|^2 e(-\eta|z|) d\lambda < \infty$$

holds. Here $d\lambda$ denotes the Lebesgue measure in $\boldsymbol{C}^n$.

We define the space $\underset{\sim}{L}^{\eta}_{2,\mathrm{loc}}(W)$ to be the space of all $f \in L_{2,\mathrm{loc}}(W \cap C^n)$ such that, for an arbitrary compact subset K of W,

$$\int_{K\cap C^n} |f|^2 e(-\eta|z|)d\lambda < \infty$$

holds.

$\underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$ becomes an FS-space and $\underset{\sim}{L}^{\eta}_{2,\mathrm{loc}}(W)$ becomes an FS*-space. $\underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$ is a closed subspace of $\underset{\sim}{L}^{\eta}_{2,\mathrm{loc}}(W)$. The dual space of $\underset{\sim}{L}^{\eta}_{2,\mathrm{loc}}(W)$ is realized as $\underset{\sim}{L}^{-\eta}_{2,\mathrm{c}}(W)$. Here we define the space $\underset{\sim}{L}^{-\eta}_{2,\mathrm{c}}(W)$ to be the space of all $f \in L_{2,\mathrm{loc}}(W \cap C^n)$ such that

$$\int_{K\cap C^n} |f|^2 e(\eta|z|)d\lambda < \infty$$

holds and supp(f) is a compact subset of W.

For $\eta' < \eta$ the inclusion relations $\underset{\sim}{\mathcal{O}}^{2,\eta'}_{\mathrm{loc}}(W) \subset \underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$ and $\underset{\sim}{L}^{\eta'}_{2,\mathrm{loc}}(W) \subset \underset{\sim}{L}^{\eta}_{2,\mathrm{loc}}(W)$ hold.

Lemma 6.4. *Let K be a compact set in $\widetilde{C}^n$ and $\{W_j\}$ a fundamental system of neighborhoods of K with $W_j \supset\supset W_{j+1}$. Then we have a topological isomorphism*

$$\underset{\sim}{\mathcal{O}}(K) \simeq \lim_{\substack{\rightarrow \\ j}} \mathrm{ind}\, \underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j).$$

Proof. By the fact following definition 1.4, the topology of $\underset{\sim}{\mathcal{O}}(K)$ is defined by the inductive limit topology $\lim\limits_{\substack{\rightarrow \\ j}} \mathrm{ind}\, \mathcal{O}^{-1/j}_b(W_j)$. But, for $j = 1, 2, 3, \cdots$, we have continuous inclusions

$$\mathcal{O}^{-1/j}_b(W_j) \hookrightarrow \underset{\sim}{\mathcal{O}}^{2,-1/(j+1)}_{\mathrm{loc}}(W_{j+1}),$$

$$\underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j) \hookrightarrow \mathcal{O}^{-1/2j}_b(W_{2j}).$$

Hence we have a topological isomorphism

$$\underset{\sim}{\mathcal{O}}(K) = \lim_{\substack{\rightarrow \\ j}} \mathrm{ind}\, \mathcal{O}^{-1/j}_b(W_j) \simeq \lim_{\substack{\rightarrow \\ j}} \mathrm{ind}\, \underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j). \text{ Q.E.D.}$$

Lemma 6.5. *Let W be an open set in $\widetilde{C}^n$ such that, for a certain positive constant a, $|\mathrm{Im}\, z| < a$ holds for every $z \in W \cap C^n$. Let $\eta' < \eta$. Then $\underset{\sim}{\mathcal{O}}^{2,\eta'}_{\mathrm{loc}}(W)$ is dense in $\underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$.*

Proof. Let $f \in \underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$. By the assumptin on W, we have $f(z)\exp(-z^2/\nu) \in \underset{\sim}{\mathcal{O}}^{2,\eta'}_{\mathrm{loc}}(W), (\nu = 1, 2, 3, \cdots)$. Here we put $z^2 = z_1^2 + \cdots + z_n^2$. On the other hand, for an arbitrary compact set K in W, we have

$$\lim_{\nu\to\infty} \int_{K\cap C^n} |f - fe(-z^2/\nu)|^2 e(-\eta|z|)d\lambda = 0$$

by the Lebesgue convergence theorem. Hence, $\underset{\sim}{\mathcal{O}}^{2,\eta'}_{\mathrm{loc}}(W)$ is dense in $\underset{\sim}{\mathcal{O}}^{2,\eta}_{\mathrm{loc}}(W)$. Q.E.D.

Proof of Theorem 6.1. Let $\{V_j\}$ and $\{W_j\}$ be fundamental systems of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods of K and L respectively. Further assume that $V_j \subset\subset W_j, V_{j+1} \subset\subset V_j, W_{j+1} \subset\subset W_j, (j = 1, 2, \cdots)$. Then $\underset{\sim}{\mathcal{O}}(L)$ and $\underset{\sim}{\mathcal{O}}(K)$ are DFS-spaces and

$$\underset{\sim}{\mathcal{O}}(L) = \lim\operatorname{ind} \underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j),$$

$$\underset{\sim}{\mathcal{O}}(K) = \lim\operatorname{ind} \underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(V_j)$$

hold. Therefore, we here use the following Lemma.

Lemma 6.6. *Assume* $E = \lim\operatorname{ind} E_m$ *and* $F = \lim\operatorname{ind} F_m$ *are* DFS-*spaces and* E *is a subspace of* F. *If, for every* $m(= 1, 2, 3, \cdots)$, E_m *is dense in* F_m, *then* E *is dense in* F.

Therefore we have only to prove the following:

(1) "For sufficiently large j, $\underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j)$ is dense in $\underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(V_j)$."

But, since $\underset{\sim}{\mathcal{O}}^{2,-2/j}_{\mathrm{loc}}(W_j)$ is a subspace of $\underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(W_j)$, we have only to prove the following:

(2) "For sufficiently large j, $\underset{\sim}{\mathcal{O}}^{2,-2/j}_{\mathrm{loc}}(W_j)$ is dense in $\underset{\sim}{\mathcal{O}}^{2,-1/j}_{\mathrm{loc}}(V_j)$".

In the sequel, we assume for j to be sufficiently large so that $W_j \subset U$ holds and put $W_j = W, V_j = V$ and $1/j = \varepsilon$. Then we prove the assertion (2). But, by the Hahn-Banach Theorem, (2) is equivalent to the following assertion:

(3) "If $\mu \in \underset{\sim}{\mathcal{O}}^{2,-\varepsilon}_{\mathrm{loc}}(V)'$ is equal to 0 on $\underset{\sim}{\mathcal{O}}^{2,-2\varepsilon}_{\mathrm{loc}}(W)$, then $\mu = 0$".

On the other hand, since $\underset{\sim}{\mathcal{O}}^{2,-\varepsilon}_{\mathrm{loc}}(V)$ is a subspace of $\underset{\sim}{L}^{-\varepsilon}_{2,\mathrm{loc}}(V)$, there exists $u \in L^{\varepsilon}_{2,\mathrm{c}}(V)$ by the Hahn-Banach Theorem such that μ is represented as

$$< \mu, v >= \int_{V\cap C^n} v\bar{u}d\lambda, (v \in \underset{\sim}{\mathcal{O}}^{2,-\varepsilon}_{\mathrm{loc}}(V)).$$

Here, since $W \subset U$, we consider the function

$$h_\varepsilon(z) = \prod_{j=1}^{n} \cosh(2\varepsilon z_j/n).$$

Then, for sufficiently small $\varepsilon > 0$ there exist positive constants C and C' such that

$$C\exp(-2\varepsilon|z|) \leq |h_\varepsilon(z)|^{-1} \leq C'\exp(-\frac{3\varepsilon}{2}|z|), (z \in W \cap C^n)$$

holds. We assume that we choose a in the assumption of Theorem 6.1 so that this inequality holds. Then we have

$$u/\overline{h_\varepsilon(z)} \in L^{-2\varepsilon}_{2,\mathrm{c}}(V).$$

Now we define the space $\mathcal{O}^2(W; \frac{\varepsilon}{2}|z| + 2\log(1+|z|^2))$ to be the space of all $v \in \mathcal{O}(W \cap \boldsymbol{C}^n)$ such that

$$\int_{W\cap \boldsymbol{C}^n} |v|^2 e(-\frac{\varepsilon}{2}|z| - 2\log(1+|z|^2))d\lambda < \infty$$

holds. Then, for every $v \in \mathcal{O}^2(W; \frac{\varepsilon}{2}|z| + 2\log(1+|z|^2))$, we have $v/h_\varepsilon(z) \in \underset{\sim}{\mathcal{O}}^{2,-2\varepsilon}_{\text{loc}}(W)$. Hence, by the assumption on μ, we have, for $v \in \mathcal{O}^2(W; \frac{\varepsilon}{2}|z| + 2\log(1+|z|^2))$,

$$\int_{V\cap \boldsymbol{C}^n} v\overline{(u/\overline{h_\varepsilon(z)})}d\lambda = \int_{W\cap \boldsymbol{C}^n} (v/h_\varepsilon(z))\overline{u}d\lambda = 0.$$

Here we put $T = \text{cl}\{\text{supp}(u)\}$. Then, by Theorem 5.5, there exist some open neighborhood V' of T which is relatively compact in V and some strictly C^∞-plurisubharmonic function $\theta(z)$ on $W \cap \boldsymbol{C}^n$ such that the following (4) and (5) hold:

(4) $\theta(z) < 0$ on $T \cap \boldsymbol{C}^n$.

(5) $\theta(z) > 0$ in a neighborhood N of $\partial V' \cap \boldsymbol{C}^n$.

Here we remember the Hörmander Theorem.

Theorem 6.7(Hörmander). *Let Ω be an open set in $\boldsymbol{C}^n$ with C^2-pseudoconvex boundary. Let φ, $\psi \in C^2(\Omega^{\text{cl}})$ be two strictly plurisubharmonic functions in Ω. Let $u \in L_2^{p,q-1}(\Omega, -\varphi)$ and $u = 0$ where $\psi > 0$. We assume $< u, v >= 0$ for an arbitrary v such that $\overline{\partial} v = 0$ and $v \in L_2^{p,q-1}(\Omega, \varphi + \lambda\psi^+)$ for a certain $\lambda > 0$. Here we put $\psi^+ = \sup(\psi, 0)$. Then there exists $f \in L^{p,q}_{2,\text{loc}}(\Omega)$ such that we have*

$$\partial' f = (-1)^{p-1}\Sigma'_{I,K}\Sigma_j \frac{\partial f_{I,jK}}{\partial z_j} dz^I \wedge d\overline{z}^K = u, \qquad (*)$$

$$\int_\Omega \Sigma'_{I,K}\Sigma_{j,k} f_{I,jK}\overline{f}_{I,kK} \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k} e(\varphi)d\lambda \le \int_\Omega |u|^2 e(\varphi)d\lambda$$

and $f = 0$ where $\psi > 0$. Here the first equality in $(*)$ is the definition of ∂'.

Proof. See, Hörmander[2], Proposition 2.3.2, p.109. Q.E.D.

Then there exists $f \in L_2^{0,1}(W \cap \boldsymbol{C}^n; -\frac{\varepsilon}{2}|z|)$ by Theorem 6.7 such that we have $u/\overline{h}_\varepsilon = \partial' f$ and $\text{supp}(f) \subset \{z \in W \cap \boldsymbol{C}^n; \theta(z) \le 0)$. Here we choose $\chi \in C^\infty(W \cap \boldsymbol{C}^n)$, so that $0 \le \chi(z) \le 1, \chi(z) = 1$ on $T \cap \boldsymbol{C}^n, \chi(z) = 0$ on $(V' \cap N)^c \cap \boldsymbol{C}^n, \text{supp}(\overline{\partial}\chi) \subset N$ and $\sup|\overline{\partial}\chi| < \infty$ hold. Then, for every $v \in \underset{\sim}{\mathcal{O}}^{2,-4\varepsilon}_{\text{loc}}(V)$, we have $\chi v h_\varepsilon \in L_2(W; \frac{\varepsilon}{2}|z| + 2\log(1+|z|^2)), \overline{\partial}(\chi v h_\varepsilon) \in L_2^{0,1}(W; \frac{\varepsilon}{2}|z|)$ and $\text{supp}(\overline{\partial}(\chi v h_\varepsilon)) \subset N$. Hence, for every $v \in \underset{\sim}{\mathcal{O}}^{2,-4\varepsilon}_{\text{loc}}(V)$, we have

$$< \mu, v >= \int_{V\cap \boldsymbol{C}^n} v\overline{u}d\lambda = \int_{W\cap \boldsymbol{C}^n} (\chi v h_\varepsilon)\overline{(u/\overline{h}_\varepsilon)}d\lambda$$

$$= \int_{W\cap C^n} (\chi v h_\varepsilon)\overline{\partial' f} d\lambda = \int_{W\cap C^n} \overline{\partial}(\chi v h_\varepsilon) \cdot \overline{f} d\lambda = 0.$$

But, since $\underset{\sim}{\mathcal{O}}{}^{2,-4\varepsilon}_{\mathrm{loc}}(V)$ is dense in $\underset{\sim}{\mathcal{O}}{}^{2,-\varepsilon}_{\mathrm{loc}}(V)$ by Lemma 6.5, we have $\mu = 0$. Q.E.D.

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