J. Math. Univ. Tokushima
Vol. 36 (2002), 7–34

Theory of General Fourier Hyperfunctions

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(Received September 13, 2002)

Abstract

In this article, we construct, by the duality method, the theory of general Fourier hyperfunctions valued in a locally convex topological vector space, which is not necessarily a Fréchet space. We realize, by the duality method, general Fourier analytic-linear mappings and general Fourier hyperfunctions. We prove analogs of Schwartz's Kernel Theorem for them. Further we define several operations on them.

2000 Mathematics Subject Classification. Primary 46F15 ; Secondary 32A45

Introduction

The purpose of this article is the establishment of the theory of general Fourier hyperfunctions valued in a locally convex space which is not necessarily a Fréchet space by the duality method. This is the most general one of the theories of vector-valued Sato-Fourier hyperfunctions. Since, in Sato [25], [26] and Kawai [17], the theory of Sato-Fourier hyperfunctions was established, many authors have tried to extend this theory to the theory of vector-valued Sato-Fourier hyperfunctions (cf. Ion-Kawai [3], Ito [5], [6], [7], [8], [9], [10], [11], [12], Ito-Nagamachi [13], [14], Junker [15], [16], Nagamachi [21], Nagamachi-Mugibayashi [22].) Until now, the case of Fréchet-space-valued Sato-Fourier hyperfunctions is considered as a limiting case, because, except for the case of Fréchet-space-valued ones, we can not prove the Oka-Cartan Theorem B and Problem A after Proposition 3.5 in this article.

As for the Sato-Fourier hyperfunctions, we can realize them as boundary values of holomorphic functions only for Fréchet-space-valued ones. In fact, there are some examples of vector-valued distributions with values in a non-Fréchet space which cannot be realized as boundary values of vector-valued holomorphic functions (see Itano [4] and Vogt [28]). Here we note that distributions are special ones of Sato-Fourier hyperfunctions. By the duality method, we have succeeded in generalizing this theory to the theory of general Fourier hyperfunctions. By the limit of the method of tensor product, this generalization may be considered as the final one.

It is known that the presheaf of Fréchet-space-valued Sato-Fourier hyperfunctions is in fact a flabby sheaf. This fact depends on the pure-codimensionality of the space $\boldsymbol{R}^{*,n}$ with respect to the sheaf ${}^{E}\mathcal{O}^{*}$ of E-valued, slowly increasing, holomorphic functions or on Problem A.

In the case where E is a general locally convex space, we define the space $\mathcal{B}^{*}(\Omega; E)$ of general Fourier hyperfunctions on a relatively compact open set Ω in $\boldsymbol{R}^{*,n}$ by the relation $\mathcal{B}^{*}(\Omega; E) = \mathcal{A}'_{*}(\Omega^{\mathrm{cl}}; E)/\mathcal{A}'_{*}(\partial\Omega; E)$, where $\mathcal{A}'_{*}(K; E)$ denotes the space of general Fourier analytic-linear mappings on a compact set K in $\boldsymbol{R}^{*,n}$. If we put $\mathcal{B}^{*}_{1}(\Omega; E) = \{0\}$ for an open set Ω in $\boldsymbol{R}^{*,n}$ which is not relatively compact and $\mathcal{B}^{*}_{1}(\Omega; E) = \mathcal{B}^{*}(\Omega; E)$ for a relatively compact open set Ω in $\boldsymbol{R}^{*,n}$, we have a presheaf $\{\mathcal{B}^{*}_{1}(\Omega; E)\}$. When we talk about general Fourier hyperfunctions, we only concern with this presheaf. This corresponds to the facts that the spaces of classical functions on open sets in $\boldsymbol{R}^{n}$ such as integrable functions are presheaves but not sheaves.

Generalized functions are extensions of concepts of classical functions. Then it was Sato's idea that the property of the spaces of functions being a sheaf could be taken as a guiding principle. Here we assert that the property of the spaces of functions being a presheaf can be taken as a guiding principle.

At least at this stage of investigations, we have to concern with this presheaf in order to study the most general vector-valued Sato-Fourier hyperfunctions. This presheaf satisfies the condition (S1) of sheaf in Bredon [1] but not the condition (S2) of sheaf in Bredon [1]. But, by virtue of the condition (S1), we can define the concept of support of general Fourier hyperfunctions. This theory contains, as special cases, theories of Sato hyperfunctions, Fourier hyperfunctions, modified Fourier hyperfunctions, mixed Fourier hyperfunctions, partial Fourier hyperfunctions, partial modified Fourier hyperfunctions and partial mixed Fourier hyperfunctions and their vector-valued versions.

In chapter 1, we define the sheaves $\mathcal{O}_*$ and $\mathcal{A}_*$ of partially rapidly decreasing, holomorphic functions or real-analytic functions, and mention their properties.

In chapter 2, we introduce the notion of general Fourier analytic-linear mappings and mention their properties.

In chapter 3, we introduce the notion of general Fourier hyperfunctions valued in a general locally convex space and mention their properties.

In chapter 4, we mention several operations on general Fourier analytic-linear mappings and prove Kernel Theorems.

In chapter 5, we mention several operations on general Fourier hyperfunctions.

In chapter 6, we define the Fourier transformation of general Fourier hyperfunctions. The space of all general Fourier hyperfunctions on the entire ground space is stable for the Fourier transformation. This is the characteristic property for general Fourier hyperfunctions considered in this article.

1. The sheaves $\mathcal{O}_*$ and $\mathcal{A}_*$

In this chapter we recall the sheaves $\mathcal{O}_*$ (resp. $\mathcal{A}_*$) of partially rapidly decreasing, holomorphic functions (resp. real-analytic functions) following Ito [10], section 2.1, p.224ff.

For a natural number n, we denote by $\boldsymbol{D}^n=\boldsymbol{R}^n \sqcup \boldsymbol{S}_\infty^{n-1}$ the directional compactification of $\boldsymbol{R}^n$ in the sense of Kawai [17], Definition 1.1.1, p.468. Here $\boldsymbol{S}_\infty^{n-1}$ denotes the set of points at infinity. We put $\tilde{\boldsymbol{R}}^n=\boldsymbol{D}$ and denote by $\tilde{\boldsymbol{C}}^n$ the space $\tilde{\boldsymbol{C}}^n \times \sqrt{-1}\boldsymbol{R}^n$ endowed with the direct product topology. We also denote by $\tilde{\tilde{\boldsymbol{C}}}^n=\boldsymbol{E}^n$ the directional compactification of $\boldsymbol{C}^n$ considered as $\boldsymbol{R}^{2n}$. That is, $\tilde{\tilde{\boldsymbol{C}}}^n$ is identified with $\boldsymbol{D}^{2n}$. We denote by $\tilde{\tilde{\boldsymbol{R}}}^n=\boldsymbol{D}$ the closure of $\boldsymbol{R}^n$ in $\tilde{\tilde{\boldsymbol{C}}}^n$.

For a pair $n=(n_1, n_2)$ of nonnegative integers with $|n|=n_1+n_2 \neq 0$, we denote by $\boldsymbol{C}^{\#,n}=\boldsymbol{F}^n$ the product space $\tilde{\boldsymbol{C}}^{n_1} \times \tilde{\tilde{\boldsymbol{C}}}^{n_2}$ and by $\boldsymbol{R}^{\#,n}$ the product space $\tilde{\boldsymbol{R}}^{n_1} \times \tilde{\tilde{\boldsymbol{R}}}^{n_2}$. We also denote by $\boldsymbol{C}^{\flat,n}=\boldsymbol{G}^n$ and $\boldsymbol{C}^{\natural,n}=\boldsymbol{H}^n$ the product spaces $\boldsymbol{C}^{n_1} \times \tilde{\boldsymbol{C}}^{n_2}$ or $\boldsymbol{C}^{n_1} \times \tilde{\tilde{\boldsymbol{C}}}^{n_2}$ respectively. Then we denote by $\boldsymbol{R}^{\flat,n}=\boldsymbol{R}^{n_1} \times \tilde{\boldsymbol{R}}^{n_2}$ and by $\boldsymbol{R}^{\natural,n}=\boldsymbol{R}^{n_1} \times \tilde{\tilde{\boldsymbol{R}}}^{n_2}$ the closures of $\boldsymbol{R}^{|n|}=\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$ in the spaces $\boldsymbol{C}^{\#,n}$ and $\boldsymbol{C}^{\natural,n}$ respectively.

For a triplet $n=(n_1, n_2, n_3)=(n_1, n')$ of nonnegative integers with $|n|=n_1+n_2+n_3 \neq 0$, where $n'=(n_2, n_3)$, we denote by $\boldsymbol{C}^{*,n}=\boldsymbol{K}^n$ the product space $\boldsymbol{C}^{n_1} \times \boldsymbol{C}^{\#,n'}=\boldsymbol{C}^{n_1} \times \tilde{\boldsymbol{C}}^{n_2} \times \tilde{\tilde{\boldsymbol{C}}}^{n_3}$ and by $\boldsymbol{R}^{*,n}=\boldsymbol{Y}^n$ the closure of the space $\boldsymbol{R}^{|n|}=\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^{n_3}$

in the space $\boldsymbol{C}^{*,n}$. Then we have $\boldsymbol{R}^{*,n}=\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{\#,n'}=\boldsymbol{R}^{n}\times\tilde{\boldsymbol{R}}^{n_2}\times\tilde{\tilde{\boldsymbol{R}}}^{n_3}$. At last we denote by $\boldsymbol{C}^{|n|}$ the space $\boldsymbol{C}^{n_1+n_2+n_3}=\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2}\times\boldsymbol{C}^{n_3}$. We denote $z=(z', z'', z''')\in\boldsymbol{C}^{|n|}$ so that $z'=(z_1, \cdots, z_{n_1})$, $z''=(z_{n_1+1}, \cdots, z_{n_1+n_2})$ and $z'''=(z_{n_1+n_2+1}, \cdots, z_{|n|})$. Here if $n_3=0$, we put $z=(z', z'')$ and if $n_1=n$, $n_2=n_3=0$, we put $z=z'$.

Let F be a subset of $\boldsymbol{C}^{*,n}$. Then we denote by int (F) the interior of F and by F^{cl} the closure of F with respect to the topology of $\boldsymbol{C}^{*,n}$.

Definition 1. 1 (The sheaf of $\mathcal{O}_*$ of partially rapidly decreasing, holomorphic functions). We define the sheaf $\mathcal{O}_*$ to be the sheaf $\{\mathcal{O}_*(\Omega); \Omega$ is an open set in $\boldsymbol{C}^{*,n}\}$, where the section module $\mathcal{O}_*(\Omega)$ on an open set Ω in $\boldsymbol{C}^{*,n}$ is the space of all holomorphic functions f on $\Omega\cap\boldsymbol{C}^{|n|}$ such that, for every compact set K in Ω, there exists some positive constant δ so that f satisfies the condition

$$\sup\{|f(z)| \exp(\delta(|z''|+|z'''|)); z\in K\cap\boldsymbol{C}^{|n|}\}<\infty.$$

The sheaf $\mathcal{O}_*$ is a nuclear FS-sheaf. Namely every section module $\mathcal{O}_*(\Omega)$ is a nuclear FS-space for an open set Ω in $\boldsymbol{C}^{*,n}$.

Remark 1. By the above definition, it is easy to see that $\mathcal{O}_*|_{\boldsymbol{C}^{|n|}}={}_{|n|}\mathcal{O}$ holds, where ${}_{|n|}\mathcal{O}$ denotes the sheaf of all holomorphic functions over $\boldsymbol{C}^{|n|}$. We put $\mathcal{O}={}_{n_1}\mathcal{O}=\mathcal{O}_*|_{\boldsymbol{C}^{n_1}}$ for $n=(n_1, 0, 0)$, $\underset{\sim}{\mathcal{O}}=\mathcal{O}_*|_{\boldsymbol{C}^{n_2}}$ for $n=(0, n_2, 0)$, $\underset{\approx}{\mathcal{O}}=\mathcal{O}_*|_{\boldsymbol{C}^{n_3}}$ for $n=(0, 0, n_3)$, $\mathcal{O}_\#=\mathcal{O}_*|_{\boldsymbol{C}^{\#,n'}}$ for $n=(0, n')=(0, n_2, n_3)$, $\mathcal{O}_\flat=\mathcal{O}_*|_{\boldsymbol{C}^{\flat,(n_1,n_2)}}$ for $n=(n_1, n_2, 0)$ and $\mathcal{O}_\natural=\mathcal{O}_*|_{\boldsymbol{C}^{\natural,(n_1,n_3)}}$ for $n=(n_1, 0, n_3)$. These sheaves are specializations of the sheaf $\mathcal{O}_*$.

Definition 1. 2 (Topology of $\mathcal{O}_*(K)$). If K is a compact set in $\boldsymbol{C}^{*,n}$, then we endow $\mathcal{O}_*(K)$ with the inductive limit topology lim $\mathrm{ind}_m\mathcal{O}_b^m(U_m)$, where $\{U_m\}$ is a fundamental system of neighborhoods of K satisfying $U_m\supset\supset U_{m+1}$ and $\mathcal{O}_b^m(U_m)$ is the Banach space of all functions $f(z)$ which are holomorphic on $U_m\cap\boldsymbol{C}^{|n|}$ and continuous on $U_m^{\mathrm{cl}}\cap\boldsymbol{C}^{|n|}$ and satisfy the condition $|f(z)|\leqq C\exp(-(|z''|+|z'''|)/m)$, $(z\in U_m\cap\boldsymbol{C}^{|n|})$, for some positive constant C. We define the norm in $\mathcal{O}_b^m(U_m)$ as follows:

$$\|f\|_m=\sup\{|f(z)| \exp((|z''|+|z'''|)/m); z\in U_m\cap\boldsymbol{C}^{|n|}\}$$

The topology of $\mathcal{O}_*(K)$ is well defined and $\mathcal{O}_*(K)$ becomes a nuclear DFS-space.

We define the sheaf $\mathcal{A}_*$ of partially rapidly decreasing, real-analytic functions over $\boldsymbol{R}^{*,n}$ by $\mathcal{A}_*=\mathcal{O}_*|_{\boldsymbol{R}^{*,n}}$. For a compact set K in $\boldsymbol{R}^{*,n}$, we have $\mathcal{A}_*(K)=\mathcal{O}_*(K)$. $\mathcal{A}_*(K)$ is the space of partially rapidly decreasing, real-analytic functions in a neighborhood of K in $\boldsymbol{R}^{*,n}$ and is endowed with the topology of $\mathcal{O}_*(K)$.

If Ω is an open set in $\boldsymbol{R}^{*,n}$, let $\mathcal{A}_*(\Omega)$ be the space of partially rapidly decreasing, real-analytic functions on Ω equipped with the topology

$$\mathcal{A}_*(\Omega)=\lim\mathrm{proj}_{K\subset\Omega}\mathcal{A}_*(K).$$

Here K runs over the family of all compact subsets of Ω. Then $\mathcal{A}_*(\Omega)$ is a nuclear FS-space.

Remark 2. Corresponding to the special cases of O_* in Remark 1, we put $\mathcal{A}=\mathcal{A}_*|_{\boldsymbol{R}^{n_1}}$ for $n=(n_1, 0, 0)$, $\underset{\sim}{\mathcal{A}}=\mathcal{A}_*|_{\tilde{\boldsymbol{R}}^{n_2}}$ for $n=(0, n_2, 0)$, $\underset{\approx}{\mathcal{A}}=\mathcal{A}_*|_{\boldsymbol{R}^{n_3}}$ for $n=(0, 0, n_3)$, $\mathcal{A}_{\#}=\mathcal{A}_*|_{\boldsymbol{R}^{\#,n'}}$ for $n=(0, n')=(0, n_2, n_3)$, $\mathcal{A}_{\flat}=\mathcal{A}_*|_{\boldsymbol{R}^{\flat,(n_1,n_2)}}$ for $n=(n_1, n_2, 0)$, and $\mathcal{A}_{\natural}=\mathcal{A}_*|_{\boldsymbol{R}^{\natural,(n_1,n_3)}}$ for $n=(n_1, 0, n_3)$. If we denote by O_α a representative of sheaves O, $\underset{\sim}{O}$, $\underset{\approx}{O}$, $O_{\#}$, $O_{\flat}$, $O_{\natural}$ and O_*, and $\mathcal{A}_\alpha$ a representative of $\mathcal{A}$, $\underset{\sim}{\mathcal{A}}$, $\underset{\approx}{\mathcal{A}}$, $\mathcal{A}_{\#}$, $\mathcal{A}_{\flat}$, $\mathcal{A}_{\natural}$ and $\mathcal{A}_*$, then $\mathcal{A}_\alpha=O_\alpha|_{M^n}$ holds for a corresponding special space M^n of $\boldsymbol{R}^{*,n}$.

Now we denote by X^n the special case of $\boldsymbol{C}^{*,n}$ corresponding to the sheaf O_α. We denote by $O_{\alpha\beta}$ and $\mathcal{A}_{\alpha\beta}$ the external tensor products of sheaves O_α and O_β, and $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ respectively. As for the notion of external tensor product of sheaves, we refer to Ito [10], section 1.3, p.222ff, where an external tensor product is called a direct tensor product. Then we have the following.

Proposition 1.3. *We use the notation above. Let O_α and O_β be the sheaves of the types as above over X^m and X^n respectively. Then we have the following isomorphisms:*

(1) $O_{\alpha\beta}(\Omega'\times\Omega'')\cong O_\alpha(\Omega')\hat{\otimes} O_\beta(\Omega'')$,

($\Omega'\subset X^m$, $\Omega''\subset X^n$ *are open subsets*).

(2) $O_{\alpha\beta}(K'\times K'')\cong O_\alpha(K')\hat{\otimes} O_\beta(K'')$,

($K'\subset X^m$, $K''\subset X^n$ *are compact subsets*).

(3) $\mathcal{A}_{\alpha\beta}(K'\times K'')\cong \mathcal{A}_\alpha(K')\hat{\otimes} \mathcal{A}_\beta(K'')$,

($K'\subset M^m$, $K''\subset M^n$ *are compact subsets*).

(4) $O_{\alpha\beta}(\Omega'\times\Omega'')\cong O_\alpha(\Omega')\hat{\otimes} O_\beta(\Omega'')$,

($\Omega'\subset M^m$, $\Omega''\subset M^n$ *are open subsets*).

Proof. See Ito [10], Proposition 2.1.10, p.229. Q.E.D.

Then we have the following.

Theorem 1.4. *For every compact set K in $\boldsymbol{R}^{*,n}$, we have $H^1(K, \mathcal{A}_*)=0$.*

Proof. See Ito [10], Theorem 2.1.14, p.231. Q.E.D.

Using the theorem above, we can prove the following.

Theorem 1.5. *Let K_1 and K_2 be two compact sets in $\boldsymbol{R}^{*,n}$. Then we have an exact sequence:*

$$0 \longrightarrow \mathcal{A}_*(K_1\cup K_2) \longrightarrow \mathcal{A}_*(K_1)\oplus\mathcal{A}_*(K_2) \longrightarrow \mathcal{A}_*(K_1\cap K_2) \longrightarrow 0.$$
$$(f_1, f_2) \longrightarrow f_1-f_2$$

Proof. See Ito[10], Theorem 2.1.15, p.232. Q.E.D.

At last we have the following.

Theorem 1.6. *For every compact set K in $\boldsymbol{R}^{*,n}$, $\mathcal{A}_*(\boldsymbol{R}^{*,n})$ is dense in $\mathcal{A}_*(K)$.*

Proof. See Ito [10], Theorem 2.2.1, p.233. Q.E.D.

2. General Fourier analytic-linear mappings

In this chapter we introduce the notion of general Fourier analytic-linear mappings.

In the sequel of this article, E is always assumed to be an arbitrary locally convex, Hausdorff, topological vector space over the complex number field (LCV for short) as far as the contrary is not explicitly mentioned.

Definition 2.1. Let Ω be an open set in $\boldsymbol{C}^{*,n}$ and $\mathcal{O}'_*(\Omega; E)=L(\mathcal{O}_*(\Omega); E)\,(\equiv L_b(\mathcal{O}_*(\Omega); E))$ the space of all continuous linear mappings of $\mathcal{O}_*(\Omega)$ into E equipped with the topology of uniform convergence on every bounded set in $\mathcal{O}_*(\Omega)$. We call an element of $\mathcal{O}'_*(\Omega; E)$ a general Fourier analytic-linear mappings on Ω valued in E or a general Fourier analytic-linear mapping on Ω. We say that $u\in\mathcal{O}'_*(\Omega; E)$ is carried by a compact set K in Ω if u can be extended to $\mathcal{O}_*(K)$. Then we call K a carrier of u. We also say that $u\in\mathcal{O}'_*(\Omega; E)$ is carried by an open set ω in Ω if u is carried by some compact subset of ω. Then ω is said to be a carrier of u. Similarly we define the spaces $\mathcal{O}'_*(K; E)=L(\mathcal{O}_*(K); E)$, $\mathcal{A}'_*(K; E)=L(\mathcal{A}_*(K); E)$ and $\mathcal{A}'_*(\Omega; E)=L(\mathcal{A}_*(\Omega); E)$ for a compact set K in $\boldsymbol{C}^{*,n}$ or in $\boldsymbol{R}^{*,n}$ or for an open set Ω in $\boldsymbol{R}^{*,n}$, respectively. We also say their elements to be general Fourier analytic-linear mappings and define the notion of their carriers in a similar way.

Proposition 2.2. *Let E be complete. Then we have the following isomorphisms:*

(1) $\mathcal{O}'_*(\Omega; E)\cong\mathcal{O}'_*(\Omega)\,\hat{\otimes}\,E$, *($\Omega$ is an open set in $\boldsymbol{C}^{*,n}$).*

(2) $\mathcal{O}'_*(K; E)\cong\mathcal{O}'_*(K)\,\hat{\otimes}\,E$, *($K$ is a compact set in $\boldsymbol{C}^{*,n}$).*

(3) $\mathcal{A}'_*(K; E)\cong\mathcal{A}'_*(K)\,\hat{\otimes}\,E$, *($K$ is a compact set in $\boldsymbol{R}^{*,n}$).*

(4) $\mathcal{A}'_*(\Omega; E)\cong\mathcal{A}'_*(\Omega)\,\hat{\otimes}\,E$, *($\Omega$ is an open set in $\boldsymbol{R}^{*,n}$).*

Proof. See Trèves [27], Proposition 50.5, p.522. Q.E.D.

Definition 2.3. Let Ω be an open set in $\boldsymbol{C}^{*,n}$. A compact set K in Ω is said to have the Runge property if $\mathcal{O}_*(\Omega)$ is dense in $\mathcal{O}_*(K)$.

Proposition 2.4. *Let Ω be an open set in $\boldsymbol{C}^{*,n}$. Suppose that a compact set K in X has the Runge property. Let $u\in\mathcal{O}'_*(\Omega; E)$. Then u is carried by K if and only if u is carried by all open neighborhood of K.*

An element of $\mathcal{A}'_*(\boldsymbol{R}^{*,n}; E)$ is said to be a real, general Fourier analytic-linear mapping.

Theorem 2.5. *For an arbitrary family $\{K_i\}_{i\in I}$ of at most countable compact* *sets in* $\boldsymbol{R}^{*,n}$, *we have* $\cap_{i\in I}\mathcal{A}'_*(K_i; E)=\mathcal{A}'_*(\cap_{i\in I}K_i; E)$.

Proof. (i) At first we prove the case $I=\{1, 2\}$. By virtue of Proposition 1.5, we have the exact sequence

$$0 \longrightarrow \mathcal{A}_*(K_1\cup K_2) \longrightarrow \mathcal{A}_*(K_1)\oplus\mathcal{A}_*(K_2) \longrightarrow \mathcal{A}_*(K_1\cap K_2) \longrightarrow 0.$$

Thus we have the exact sequence

$$0 \longrightarrow \mathcal{A}'_*(K_1\cap K_2; E) \longrightarrow \mathcal{A}'_*(K_1; E)\oplus\mathcal{A}'_*(K_2; E) \xrightarrow{\lambda} \mathcal{A}'_*(K_1\cup K_2; E).$$

Thus we have $\mathrm{Ker}(\lambda)=\mathcal{A}'_*(K_1; E)\cap\mathcal{A}'_*(K_2; E)=\mathcal{A}'_*(K_1\cap K_2; E)$.

(ii) Next we prove the case $I=\{1, 2, \cdots, m\}$. We use the induction. The case $m=2$ holds good by (i). Assuming that the case $m-1$ holds good, we prove the case m. Then we have

$$\begin{aligned}\cap_{i=1}^{m}\mathcal{A}'_*(K_i; E) &= \{\cap_{i=1}^{m-1}\mathcal{A}'_*(K_i; E)\}\cap\mathcal{A}'_*(K_m; E)\\ &= \mathcal{A}'_*(\cap_{i=1}^{m-1}K_i; E)\cap\mathcal{A}'_*(K_m; E)=\mathcal{A}'_*(\{\cap_{i=1}^{m-1}K_i\}\cap K_m; E)\\ &= \mathcal{A}'_*(\cap_{i=1}^{m}K_i; E).\end{aligned}$$

Thus we have the conclusion in this case.

(iii) In the case $I=\{1, 2, 3, \cdots\}$. Put $K=\cap_{i\in I}K_i$ and $L_m=\cap_{i=1}^{m}K_i$. Then we have $L_1\supset L_2\supset\cdots\supset L_m\supset\cdots\supset K$ and $K=\cap_{m=1}^{\infty}L_m$. Thus we have algebraic isomorphisms

$$\begin{aligned}\mathcal{A}'_*(K; E) &\cong L(\lim\operatorname{ind}\mathcal{A}_*(L_m); E)\cong\lim\operatorname{proj}\mathcal{A}'_*(L_m; E)\\ &\cong\lim\operatorname{proj}\cap_{i=1}^{m}\mathcal{A}'_*(K_i; E)\cong\cap_{i=1}^{\infty}\mathcal{A}'_*(K_i; E).\end{aligned}$$

Thus we have the conclusion. Q.E.D.

Theorem 2.6. *Let* $u\in\mathcal{A}'_*(\boldsymbol{R}^{*,n}; E)$ *with* $u\neq 0$. *Then there exists the smallest compact set in* $\boldsymbol{R}^{*,n}$ *which carries* u. *We call it the support of* u *and denote it by* $\mathrm{supp}(u)$.

Proof. Among all carriers of u, we have only to consider compact carriers of u. Let $\{K_\alpha\}$ be the family of all compact carriers of u in M. Then let $\{K_\beta\}$ be an arbitrary totally ordered subfamily of $\{K_\alpha\}$ with $K_\beta\supset K_{\beta'}$ ($\beta\leqq\beta'$). Then we can choose a subsequence $\{K_j\}$ of $\{K_\beta\}$ with $\cap_jK_j=\cap_\beta K_\beta=K$. In fact, let $\{U_j\}$ be a countable family of open neighborhoods of K with $U_1\supset U_2\supset\cdots$ and $\cap_jU_j=K$. Then we choose K_j as the largest compact set among the subfamily of all compact sets $K_\beta\in\{K_\beta\}$ contained in U_j. Then the projective system $\{\mathcal{A}'_*(K_j; E)\}$ is cofinal with the projective system $\{\mathcal{A}'_*(K_\beta; E)\}$. Thus, by virtue of Theorem 2.5, we have the algebraic isomorphisms

$$\begin{aligned}\cap_\beta\mathcal{A}'_*(K_\beta; E) &\cong\lim\operatorname{proj}\mathcal{A}'_*(K_\beta; E)\\ &\cong\lim\operatorname{proj}\mathcal{A}'_*(K_j; E)\cong\mathcal{A}'_*(K; E).\end{aligned}$$

Then $u\in\mathcal{A}'_*(K; E)$. Thus K is a minimal compact carrier of u. Thus, by virtue of Zorn's Lemma, we have the conclusion. Q.E.D.

For $u, u_1, u_2 \in \mathcal{A}'_*(\boldsymbol{R}^{*,n}; E)$, we have

$$\operatorname{supp}(u_1+u_2) \subset \operatorname{supp}(u_1) \cup \operatorname{supp}(u_2),$$
$$\operatorname{supp}(\lambda u) \subset \operatorname{supp}(u), \ (\lambda \in \boldsymbol{C}).$$

Proposition 2.7. *For every two compact sets K_1 and K_2 in $\boldsymbol{R}^{*,n}$ with $K_1 \subset K_2$, there exists a continuous injection $i_{K_1,K_2}: \mathcal{A}'_*(K_1; E) \longrightarrow \mathcal{A}'_*(K_2; E)$.*

Proof. (1) Let ${}^t i_{K_1,K_2}: \mathcal{A}_*(K_2) \longrightarrow \mathcal{A}_*(K_1)$ be a canonical mapping. Then ${}^t i_{K_1,K_2}$ is evidently continuous and of dense image. Let $i_{K_1,K_2}: \mathcal{A}'_*(K_1; E) \longrightarrow \mathcal{A}'_*(K_2; E)$ be its adjoint map. Then i_{K_1,K_2} is evidently a continuous injection. Q.E.D.

Proposition 2.8. *Let K_1 and K_2 be two compact sets in $\boldsymbol{R}^{*,n}$ with $K_1 \subset K_2$. Further, assume that every connected component of K_2 intersects K_1. Then i_{K_1,K_2} has the dense range in $\mathcal{A}'_*(K_2; E)$.*

Proof. By the assumptions, the canonical mapping ${}^t i_{K_1,K_2}$ is injective. Thus $\mathcal{A}'_*(K_1)$ is dense in $\mathcal{A}'_*(K_2)$. Then we have the inclusions

$$\mathcal{A}'_*(K_1) \otimes E \hookrightarrow L(\mathcal{A}_*(K_1); E) \hookrightarrow L(\mathcal{A}_*(K_2); E)$$
$$\hookrightarrow L(\mathcal{A}_*(K_2); \hat{E}) = \mathcal{A}'_*(K_2; \hat{E}).$$

Here $\hat{E}$ denotes the completion of E. Since $\mathcal{A}'_*(K_1) \otimes E$ is dense in $\mathcal{A}'_*(K_2; \hat{E}) \cong \mathcal{A}'_*(K_2) \hat{\otimes} \hat{E}$, we have the conclusion. Q.E.D.

Let now Ω be an open subset of $\boldsymbol{R}^{*,n}$ and K a compact subset of Ω. We call the "compact envelope of K" (in Ω) and denote by $\tilde{K}$ the union of K and the relatively compact connected components (in Ω) of $\Omega \setminus K$. It is again a compact set in Ω.

Corollary 2.9. *Let Ω be a relatively compact open subset of $\boldsymbol{R}^{*,n}$. Let K_1, K_2, ($K_1 \subset K_2$) be two compact subsets of Ω such that $K_i = \tilde{K}_i$ holds ($i=1, 2$). Then $\mathcal{A}'_*((\Omega \setminus K_2)^{\mathrm{cl}}; E)$ is dense in $\mathcal{A}'_*((\Omega \setminus K_1)^{\mathrm{cl}}; E)$.*

3. General Fourier hyperfunctions

Let E be an LCV. First we consider general Fourier hyperfunctions on a relatively compact open set in $\boldsymbol{R}^{*,n}$.

Let Ω be a relatively compact open subset of $\boldsymbol{R}^{*,n}$. We put

$$\mathcal{B}^*(\Omega; E) = \mathcal{A}'_*(\Omega^{\mathrm{cl}}; E) / \mathcal{A}'_*(\partial\Omega; E).$$

Then, since $\mathcal{A}'_*(\partial\Omega; E)$ is dense in $\mathcal{A}'_*(\Omega^{\mathrm{cl}}; E)$, $\mathcal{B}^*(\Omega; E)$ is not endowed with any nontrivial topology.

Definition 3.1. An element of $\mathcal{B}^*(\Omega; E)$ is called a general Fourier hyperfunction on Ω.

Remark 1. We put $\mathcal{B}^*(\Omega; E) = \mathcal{B}(\Omega; E)$ for $n=(n_1, 0, 0)$, $\mathcal{B}^*(\Omega; E) = \tilde{\mathcal{B}}(\Omega; E)$ for $n=(0, n_2, 0)$, $\mathcal{B}^*(\Omega; E) = \tilde{\tilde{\mathcal{B}}}(\Omega; E)$ for $n=(0, 0, n_3)$, $\mathcal{B}^*(\Omega; E) = \mathcal{B}^{\#}(\Omega; E)$ for

$n=(0, n')=(0, n_2, n_3)$, $\mathcal{B}^*(\Omega; E)=\mathcal{B}^\flat(\Omega; E)$ for $n=(n_1, n_2, 0)$, and $\mathcal{B}^*(\Omega; E)=\mathcal{B}^\natural(\Omega; E)$ for $n=(n_1, 0, n_3)$. Here $\mathcal{B}(\Omega; E)$ is the space of all E-valued Sato hyperfunctions on Ω. $\tilde{\mathcal{B}}(\Omega; E)$ is the space of all E-valued Fourier hyperfunctions on Ω. $\tilde{\tilde{\mathcal{B}}}(\Omega; E)$ is the space of all E-valued modified Fourier hyperfunctions on Ω. $\mathcal{B}^\#(\Omega; E)$ is the space of all E-valued mixed Fourier hyperfunctions on Ω. $\mathcal{B}^\flat(\Omega; E)$ is the space of all E-valued partial Fourier hyperfunctions on Ω. $\mathcal{B}^\natural(\Omega; E)$ is the space of all E-valued partial modified Fourier hyperfunctions on Ω. $\mathcal{B}^*(\Omega; E)$ is the space of all E-valued partial mixed Fourier hyperfunctions on Ω which are called general Fourier hyperfunctions on Ω in Definition 3.1.

Let K be a compact set in $\boldsymbol{R}^{*,n}$ containing Ω. Then, by virtue of Proposition 2.7, we have the canonical map

$$\mathcal{A}'_*(\Omega^{\mathrm{cl}}; E) \longrightarrow \mathcal{A}'_*(K; E) \longrightarrow \mathcal{A}'_*(K; E)/\mathcal{A}'_*(K\setminus\Omega; E),$$

whose kernel is the space

$$\mathcal{A}'_*(\Omega^{\mathrm{cl}}; E) \cap \mathcal{A}'_*(K\setminus\Omega; E) = \mathcal{A}'_*(\partial\Omega; E).$$

Thus we have the isomorphism

$$\mathcal{B}'_*(\Omega; E) = \mathcal{A}'_*(\Omega^{\mathrm{cl}}; E)/\mathcal{A}'_*(\partial\Omega; E) \cong \mathcal{A}'_*(K; E)/\mathcal{A}'_*(K\setminus\Omega; E).$$

Let now ω be an open subset of Ω. Then the mapping

$$\mathcal{A}'_*(\Omega^{\mathrm{cl}}; E) \longrightarrow \mathcal{A}'_*(\Omega^{\mathrm{cl}}; E)/\mathcal{A}'_*(\Omega^{\mathrm{cl}}\setminus\Omega; E).$$

defines a mapping

$$\mathcal{B}^*(\Omega; E) \longrightarrow \mathcal{B}^*(\omega; E),$$

which is called the restriction.

If $T\in\mathcal{B}^*(\Omega; E)$, we denote by $T|_\omega$ its image in $\mathcal{B}^*(\omega; E)$. It is clear that, if $\Omega_3 \subset \Omega_2 \subset \Omega_1$ and $T\in\mathcal{B}^*(\Omega_1; E)$, we have

$$(T|_{\Omega_2})|_{\Omega_3} = T|_{\Omega_3}.$$

Thus we have the following.

Proposition 3.2. *Let Ω be a relatively compact open set in $\boldsymbol{R}^{*,n}$. Then the collection $\{\mathcal{B}^*(\omega; E); \omega$ is an open set in $\Omega\}$ becomes a presheaf (of vector spaces) over Ω.*

Proposition 3.3. *We use the notation in Proposition 3.2. Let $\omega=\cap_{i\in I}\omega_i$ be a union of open subsets ω_i of Ω, $(i\in I)$ and $T\in\mathcal{B}^*(\omega; E)$ with $T|_{\omega i}=0$ for all $i\in I$. Then we have $T=0$.*

Proof. By the assumptions, if $u_T\in\mathcal{A}'_*(\omega^{\mathrm{cl}}; E)$ is a representative of T, the image of u_T in $\mathcal{A}'_*(\omega^{\mathrm{cl}}; E)/\mathcal{A}'_*(\omega^{\mathrm{cl}}\setminus\omega_i; E)$ is zero for all $i\in I$. Thus we have

$$u_T\in\mathcal{A}'_*(\omega^{\mathrm{cl}}\setminus\omega_i; E) \text{ for all } i\in I.$$

Hence we have

$$u_T \in \cap_{i \in I} \mathcal{A}'_*(\omega^{\mathrm{cl}} \setminus \omega_i; E) = \mathcal{A}'_*(\cap_{i \in I}(\omega^{\mathrm{cl}} \setminus \omega_i); E)$$
$$= \mathcal{A}'_*(\omega^{\mathrm{cl}} \setminus (\cup_{i \in I} \omega_i); E)$$
$$= \mathcal{A}'_*(\omega^{\mathrm{cl}} \setminus \omega; E) = \mathcal{A}'_*(\partial \omega; E).$$

Hence we have $\mathrm{supp}(u_T) \subset \partial \omega$. Namely $T=0$. Q.E.D.

Thus we have seen that the presheaf $\{\mathcal{B}^*(\omega; E); \omega$ is an open subset of $\Omega\}$ satisfies the condition (S1) of Bredon [1], p.5. But this presheaf does not satisfy the condition (S2) of Bredon [1], p.6. Thus this presheaf does not become a sheaf. Here we remember the conditions (S1) and (S2) of Bredon [1], pp.5-6.

Let X be a topological space and $A=\{A(U)\}$ a presheaf of abelian groups on X. Then we consider two conditions:

(S1) If $U=\cup_\alpha U_\alpha$, with U_α open in X, and $s, t \in A(U)$ are such that $s|_{U\alpha}=t|_{U\alpha}$ for all α, then $s=t$.

(S2) Let $\{U_\alpha\}$ be a collection of open sets in X and let $U=\cup_\alpha U_\alpha$. If $s_\alpha \in A(U_\alpha)$ are given such that $s_\alpha|_{U\alpha \cap U\beta} = s_\beta|_{U\alpha \cap U\beta}$ for all α, β, then there exists an element $s \in A(U)$ with $s|_{U\alpha}=s_\alpha$ for all α.

Then a presheaf $A=\{A(U)\}$ of abelian groups becomes a sheaf if and only if the above two conditions (S1) and (S2) are satisfied.

Proposition 3.4. *We use the notation in Proposition 3.2. If ω is an open subset of Ω and $T \in \mathcal{B}^*(\omega; E)$, then there exists $\tilde{T} \in \mathcal{B}^*(\Omega; E)$ such that $\tilde{T}|_\omega = T$.*

Proof. Let $u_T \in \mathcal{A}'_*(\omega^{\mathrm{cl}}; E)$ be a representative of T. Then we have $u_T \in \mathcal{A}'_*(\Omega^{\mathrm{cl}}; E)$. Thus we define $\tilde{T} \in \mathcal{B}^*(\Omega; E)$ to be the image of u_T in $\mathcal{B}^*(\Omega; E)$. Then evidently we have $\tilde{T}|_\omega = T$. Q.E.D.

In the notation of Proposition 3.2, we define the support of a general Fourier hyperfunction T on Ω to be the smallest closed subset F of Ω such that $T|_{\Omega \setminus F}=0$ holds. We denote it by supp (T). Then we have the following.

Proposition 3.5. *We use the notation in Proposition 3.2. Let K be a compact subset of Ω and put*

$$\mathcal{B}^*_K(\Omega; E) = \{T \in \mathcal{B}^*(\Omega; E); \mathrm{supp}\,(T) \subset K\}.$$

Then we have the inclusion

$$\mathcal{A}'_*(K; E) \subset \mathcal{B}^*_K(\Omega; E).$$

Proof. By the assumptions, we have the inclusion map

$$\mathcal{A}'_*(K; E) \longrightarrow \mathcal{A}'_*(\Omega^{\mathrm{cl}}; E) / \mathcal{A}'_*(\partial \Omega; E) = \mathcal{B}^*(\Omega; E).$$

Let $u \in \mathcal{A}'_*(K; E)$ and $[u]$ its image in $\mathcal{B}^*(\Omega; E)$. We consider the restriction $[u]|_{\Omega \setminus K}$. Since $\mathcal{B}^*(\Omega \setminus K; E) = \mathcal{A}'_*(\Omega^{cl}; E) / \mathcal{A}'_*(\Omega^{\mathrm{cl}} \setminus (\Omega \setminus K); E) = \mathcal{A}'_*(\Omega^{cl}; E) / \mathcal{A}'_*(\partial \Omega \cup K; E)$ holds and $u \in \mathcal{A}'_*(K; E) \subset \mathcal{A}'_*(\partial \Omega \cup K; E)$ holds, we have $[u]|_{\Omega \setminus K}=0$.

Thus we have $\mathcal{A}'_*(K; E) \subset \mathcal{B}^*_K(\Omega; E)$. Q.E.D.

In general, we cannot know the inverse inclusion

$$\mathcal{B}^*_K(\Omega; E) \subset \mathcal{A}'_*(K; E)$$

holds. But, for some special cases, this inverse inclusion holds good. Namely, we have the following.

Corollary 3.6. *Let M be one of the spaces $\tilde{\boldsymbol{R}}^n$, $\tilde{\tilde{\boldsymbol{R}}}^n$ and $\boldsymbol{R}^{\#,n}$ and K be a compact subset of M. Then we have*

(1) $\tilde{\mathcal{B}}_K(M; E) = \underset{\sim}{\mathcal{A}}'(K; E)$.

(2) $\tilde{\tilde{\mathcal{B}}}_K(M; E) = \underset{\approx}{\mathcal{A}}'(K; E)$.

(3) $\mathcal{B}^{\#}_K(M; E) = \mathcal{A}'_{\#}(K; E)$.

Note. I owe the Corollary above to Prof. Nagamachi's suggestion.

In general, in order to prove the inclusion $\mathcal{B}^*_K(\Omega; E) \subset \mathcal{A}'_*(K; E)$, it is sufficient to know the following.

Problem A. For two compact sets K_1 and K_2 in $\boldsymbol{R}^{*,n}$ and put $K = K_1 \cup K_2$. Then is the sequence

$$\mathcal{A}'_*(K_1; E) \oplus \mathcal{A}'_*(K_2; E) \longrightarrow \mathcal{A}'_*(K; E) \longrightarrow 0$$

exact?

If E is a Fréchet space, the answer to the Problem A is affirmative (cf. Ito [10], Theorem 2.3.6, p.240). But in general, we do not know any answer.

Next we will consider general Fourier hyperfunctions on $\boldsymbol{R}^{*,n}$.

Let $\{\mathcal{B}^*_1(\Omega; E); \Omega$ is an open set in $\boldsymbol{R}^{*,n}\}$ be the presheaf over $\boldsymbol{R}^{*,n}$ defined as follows:

$$\text{If } \Omega \text{ is not relatively compact, } \mathcal{B}^*_1(\Omega; E) = \{0\}.$$
$$\text{If } \Omega \text{ is relatively compact, } \mathcal{B}^*_1(\Omega; E) = \mathcal{B}^*(\Omega; E).$$

The restrictions are defined as follows:

$$\mathcal{B}^*_1(\Omega; E) \longrightarrow \mathcal{B}^*_1(\omega; E)$$
$$0 \longrightarrow 0 \quad \text{if } \Omega \text{ is not relatively compact with } \Omega \supset \omega,$$
$$T \longrightarrow T|_\omega \quad \text{if } \Omega \text{ is relatively compact with } \Omega \supset \omega.$$

This presheaf satisfies the condition (S1) of sheaves but not (S2) (cf. Bredon [1], pp.5-6).

We denote by ${}^E\mathcal{B}^*$ the sheaf associated to this presheaf $\{\mathcal{B}^*_1(\Omega; E); \Omega$ is an open set in $\boldsymbol{R}^{*,n}\}$. It is a sheaf of vector spaces over $\boldsymbol{R}^{*,n}$.

Definition 3.7. The sheaf ${}^E\mathcal{B}^*$ is called the sheaf of general Fourier hyperfunctions over $\boldsymbol{R}^{*,n}$.

Then, if Ω is an open set in $\boldsymbol{R}^{*,n}$ and $T \in \Gamma(\Omega, {}^E\mathcal{B}^*) = \mathcal{B}^*(\Omega; E)$. Here we use the notation $\mathcal{B}^*(\Omega; E)$ of the section space on Ω in abuse of language. Then T is a general Fourier hyperfunction on Ω which is defined in the following way:

Let $\{(\Omega_i, T_i)_{i\in I}\}$ be the set of all families $(\Omega_i, T_i)_{i\in I}$ of the covering $\{\Omega_i\}_{i\in I}$ of Ω and sections $T_i \in \mathcal{B}^*(\Omega_i; E)$, $(i\in I)$ such that Ω_i's are relatively compact subsets of Ω and $\Omega = \cup_{i\in I}\Omega_i$ holds, and such that $T_i|_{\Omega_i\cap\Omega_j} = T_j|_{\Omega_i\cap\Omega_j}$ holds for all $i, j\in I$. Two such families $(\Omega_i, T_i)_{i\in I}$ and $(\Omega_{i'}, T_{i'})_{i'\in I'}$ are defined to be equivalent if

$$T_i|_{\Omega_i\cap\Omega_{i'}} = T_{i'}|_{\Omega_i\cap\Omega_{i'}} \text{ for all } i\in I \text{ and all } i'\in I'$$

hold. Then we denote this equivalence as $(\Omega_i, T_i)_{i\in I} \sim (\Omega_{i'}, T_{i'})_{i'\in I'}$. Then we have the quotient space representation

$$\mathcal{B}^*(\Omega; E) = \{(\Omega_i, T_i)_{i\in I}\}/\sim.$$

Thus a general Fourier hyperfunction T on Ω is defined to be an equivalence class $[(\Omega_i, T_i)_{i\in I}]$.

Then we propose the following.

Problem B. Is the sheaf ${}^E\mathcal{B}^*$ flabby?

If E is a Fréchet space, the answer to the Problem B is affirmative (cf. Ito [10], Theorem 2.4.2, p.241). But in general we do not know any answer. If the presheaf $\{\mathcal{B}^*(\omega; E) = \mathcal{A}'_*(\omega^{\mathrm{cl}}; E)/\mathcal{A}'_*(\partial\omega; E);\ \omega$ is an open subset of $\Omega\}$ over a relatively compact open set Ω in $\boldsymbol{R}^{*,n}$ becomes a sheaf, then we can show that the answer to the Problem B is affirmative by a similar way to Ito [10], Lemma 1.2.5, p.221. But this problem is still open for general LCV's E other than Fréchet spaces.

4. Operations on general Fourier analytic-linear mappings and Kernel Theorems

In this chapter we now define several operations on general Fourier analytic-linear mappings.

a) Tensor products and Kernel Theorems. At first we recall the tensor product of general Fourier analytic-linear mappings following Ito [10]. We use the notation similar to proposition 1.3.

Theorem 4.1. *we have the following canonical isomorphisms:*

(1) $O'_\alpha(\Omega')\hat{\otimes} O'_\beta(\Omega'') \cong L(O_\alpha(\Omega'); O'_\beta(\Omega'')) \cong O'_{\alpha\beta}(\Omega'\times\Omega'')$,
(Ω' and Ω'' are open subsets of X^m and X^n respectively.)

(2) $O'_\alpha(K')\hat{\otimes} O'_\beta(K'') \cong L(O_\alpha(K'); O'_\beta(K'')) \cong O'_{\alpha\beta}(K'\times K'')$,
(K' and K'' are compact subsets of X^m and X^n respectively.)

(3) $\mathcal{A}'_\alpha(K')\hat{\otimes}\mathcal{A}'_\beta(K'') \cong L(\mathcal{A}_\alpha(K'); \mathcal{A}'_\beta(K'')) \cong \mathcal{A}'_{\alpha\beta}(K'\times K'')$,
(K' and K'' are compact subsets of M^m and M^n respectively.)

(4) $\mathcal{A}'_\alpha(\Omega') \hat{\otimes} \mathcal{A}'_\beta(\Omega'') \cong L(\mathcal{A}_\alpha(\Omega'); \mathcal{A}'_\beta(\Omega'')) \cong \mathcal{A}'_{\alpha\beta}(\Omega' \times \Omega'')$,

(Ω' and Ω'' are open subsets of M^m and M^n respectively.)

Proof. See Ito [10], Theorem 3.1.1, p.243. Q.E.D.

We note that these statements establish analogs of Schwartz's Kernel Theorem in each case of Sato-Fourier analytic functionals.

Next we consider tensor products of general Fourier analytic-linear mappings.

Theorem 4.2. *We use the notations in Theorem 4.1. Further assume that E_1 and E_2 be two complete* LCV's *and put $E=E_1 \hat{\otimes}_\omega E_2$, where ω stands for the ε- or π- topology in the sense of Trèves [27]. Then we have the following canonical isomorphisms*:

(1) $\mathcal{O}'_\alpha(\Omega'; E_1) \hat{\otimes}_\omega \mathcal{O}'_\beta(\Omega''; E_2) \cong \mathcal{O}'_{\alpha\beta}(\Omega' \times \Omega''; E)$,

(Ω' and Ω'' are open subsets in X^m and X^n respectively.)

(2) $\mathcal{O}'_\alpha(K'; E_1) \hat{\otimes}_\omega \mathcal{O}'_\beta(K''; E_2) \cong \mathcal{O}'_{\alpha\beta}(K' \times K''; E)$,

(K' and K'' are compact subsets in X^m and X^n respectively.)

(3) $\mathcal{A}'_\alpha(K'; E_1) \hat{\otimes}_\omega \mathcal{A}'_\beta(K''; E_2) \cong \mathcal{A}'_{\alpha\beta}(K' \times K''; E)$,

(K' and K'' are compact subsets in X^m and X^n respectively.)

(4) $\mathcal{A}'_\alpha(\Omega'; E_1) \hat{\otimes}_\omega \mathcal{A}'_\beta(\Omega''; E_2) \cong \mathcal{A}'_{\alpha\beta}(\Omega' \times \Omega''; E)$,

(Ω' and Ω'' are open subsets in X^m and X^n respectively.)

Proof. It goes in a similar way to Ito [10], Theorem 3.1.3, p.244. Q.E.D.

We note that the statements in Theorem 4.2 establish analogs of Schwartz's Kernel Theorem in each case of general Fourier analytic-linear mappings.

With the help of Theorem 4.2, we have the following definitions of tensor products of general Fourier analytic-linear mappings of each type.

Definition 4.3. We use the notation in Theorem 4.2. Let $u_1=\varphi_1 \otimes \boldsymbol{e}_1 \in \mathcal{O}'_\alpha(\Omega'; E_1)$ and $u_2=\varphi_2 \otimes \boldsymbol{e}_2 \in \mathcal{O}'_\beta(\Omega''; E_2)$, where $\varphi_1 \in \mathcal{O}'_\alpha(\Omega')$ and $\varphi_2 \in \mathcal{O}'_\beta(\Omega'')$ and $\boldsymbol{e}_i \in E_i$ ($i=1, 2$). Then we define $u_1 \otimes_\omega u_2$ by the following relation:

$$u_1 \otimes_\omega u_2 = (\varphi_1 \otimes \varphi_2) \otimes (\boldsymbol{e}_1 \otimes_\omega \boldsymbol{e}_2),$$

i.e.,

$$(u_1 \otimes_\omega u_2)(f_1 \otimes f_2) = \varphi_1(f_1)\, \varphi_2(f_2)(\boldsymbol{e}_1 \otimes_\omega \boldsymbol{e}_2),$$
$$\text{for } f_1 \in \mathcal{O}_\alpha(\Omega') \text{ and } f_2 \in \mathcal{O}_\beta(\Omega'').$$

In all other cases we define tensor products of general Fourier analytic-linear mappings similarly.

In all real cases, we have

$$\operatorname{supp}(u_1 \otimes_\omega u_2) \subset \operatorname{supp}(u_1) \times \operatorname{supp}(u_2).$$

Here we note that, as far as we are concerned with finite sums of finite tensor products of general Fourier analytic-linear mappings, we need not assume the completeness of LCV's E_1, E_2 and E.

b) Convolution. In this section we define convolutions of general Fourier analytic-linear mappings. For the rest of this chapter, we assume E, E_1 and E_2 be LCV's.

We can show the following Proposition by simple calculations, so that we omit the proof.

Proposition 4.4. *Let $u \in O'(\boldsymbol{C}^{|n|}; E)$, that is, u is assumed to be an analytic-linear mapping with a compact carrier in $\boldsymbol{C}^{|n|}$. Let $f \in O_\alpha(X^n)$, where O_α stands for O, $\underset{\sim}{O}$, $\underset{\approx}{O}$, $O_\#$, $O_\flat$, $O_\natural$ or O_* and X^n stands for a corresponding special space of $\boldsymbol{C}^{*,n}$ respectively. We define $u*f$ by the formula*

$$(u*f)(z) = u_y(f(z-y)).$$

Then $u*f \in O_\alpha(X^n; E)$.

Definition 4.5 (The sheaf of slowly increasing, holomorphic functions). We define O^* to be the sheafification of the presheaf $\{O^*(\Omega);\ \Omega$ is an open set in $\boldsymbol{C}^{*,n}\}$, where the section module $O^*(\Omega)$ on an open set Ω in $\boldsymbol{C}^{*,n}$ is the space of all holomorphic functions f on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any positive number ε and for any compact set K in Ω, the estimate

$$\sup\{|f(z)| \exp(-\varepsilon(|z'|+|z''|));\ z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$$

holds.

As special cases of the sheaf O^* in the above definition, we have the sheaves $O=O^*|_{\boldsymbol{C}^{n_1}}$, $\tilde{O}=O^*|_{\boldsymbol{C}^{n_2}}$, $\tilde{\tilde{O}}=O^*|_{\boldsymbol{C}^{n_3}}$, $O^\#=O^*|_{\boldsymbol{C}^{\#,n'}}$ with $n'=(n_2, n_3)$, $O^\flat=O^*|_{\boldsymbol{C}^{\flat,n''}}$ with $n''=(n_1, n_2)$, and $O^\natural=O^*|_{\boldsymbol{C}^{\natural,n'''}}$ with $n'''=(n_2, n_3)$. We denote by O^α a representative of sheaves O, $\tilde{O}$, $\tilde{\tilde{O}}$, $O^\#$, $O^\flat$, $O^\natural$ and O^* over the corresponding specialization X^n of $\boldsymbol{C}^{*,n}$. Then we have the following.

Proposition 4.6. *Let $u \in O'(\boldsymbol{C}^{|n|}; E)$ and $f \in O^\alpha(X^n)$, we define $u*f$ by the formula*

$$(u*f)(z) = u_y(f(z-y)).$$

Then $u*f \in O^\alpha(X^n; E)$.

We have real analogs of these propositions. We denote by M^n the closure of $\boldsymbol{R}^{|n|}$ in X^n and we put $\mathcal{A}_\alpha = O_\alpha|_{M^n}$ and $\mathcal{A}^\alpha = O^\alpha|_{M^n}$.

Proposition 4.7. *Let $u \in \mathcal{A}'(\boldsymbol{R}^{|n|}; E)$ and $f \in \mathcal{A}_\alpha(M^n)$. If we define $u*f$ by the formula*

$$(u*f)(x) = u_y(f(x-y)),$$

then $u*f \in \mathcal{A}_\alpha(M^n; E)$.

Proposition 4.8. *Let $u \in \mathcal{A}'(\boldsymbol{R}^{|n|}; E)$ and $f \in \mathcal{A}^{\alpha}(M^n)$. If we define $u*f$ by the formula*

$$(u*f)(x) = u_y(f(x-y)),$$

then $u*f \in \mathcal{A}^{\alpha}(M^n; E)$.

Theorem 4.9. *Let $u \in O'_\alpha(X^n; E_1)$ and $v \in O'_\alpha(X^n; E_2)$, one of which has a compact carrier in $\boldsymbol{C}^{|n|}$. Here O_α stands for O, $\underset{\sim}{O}$, $\underset{\approx}{O}$, $O_{\#}$, $O_{\flat}$, $O_{\natural}$ or O_{*}. Then there exists a general Fourier analytic-linear mapping called a convolution product of u and v and denoted by $u *_\omega v$ such that*

$$(u *_\omega v)(f(z)) = (u_\xi \otimes {}_\omega v_\eta)(f(\xi+\eta)),$$
$$\text{for all } f \in O_\alpha(X^n),$$

and $u *_\omega v = v *_\omega u$ holds. Here ω stands for the ε- or π- topology.

Proof. It goes similarly to Ito [10], Theorem 3.2.5, p.246. Q.E.D.

By the definition of the carrier of a general Fourier analytic-linear mapping, we see that, if $u \in O'_\alpha(X^n; E_1)$ and $v \in O'(\boldsymbol{C}^{|n|}; E_2)$ are carried by compact sets K in X^n and L in $\boldsymbol{C}^{|n|}$ respectively, then $u *_\omega v$ is carried by $K+L$. Here we denote by $K+L$ the set $[K \cap \boldsymbol{C}^{|n|}+L]^{\mathrm{cl}}$ for a compact set K in X^n and a compact set L in $\boldsymbol{C}^{|n|}$, where $[\ \ ]^{\mathrm{cl}}$ denotes the closure of the set $[\ \ ]$ in X^n.

Theorem 4.10. *Let $u \in \mathcal{A}'_\alpha(M^n; E_1)$ and $v \in \mathcal{A}'_\alpha(M^n; E_2)$, one of which has a compact support in $\boldsymbol{R}^{|n|}$. Then there exists a general Fourier analytic-linear mapping called the convolution product of u and v and denoted by $u *_\omega v$ such that*

$$(u *_\omega v)(f(x)) = (u_\xi \otimes_\omega v_\eta)(f(\xi+\eta)),$$
$$\text{for } f \in \mathcal{A}_\alpha(M^n).$$

Then we have $u *_\omega v = v *_\omega u$ and

$$\operatorname{supp}(u *_\omega v) \subset \operatorname{supp}(u) + \operatorname{supp}(v).$$

Here the sum in the right hand side has the same meaning as in the complex case above.

Proof. It goes similarly to Ito [10], Theorem 3.2.6, p.246. Q.E.D.

c) Multiplication by a slowly increasing, holomorphic function or a real-analytic function. Let Ω be an open set in X^n. If $f \in O^\alpha(\Omega)$ and $u \in O'_\alpha(\Omega; E)$, we define $fu \in O'_\alpha(\Omega; E)$ by the formula

$$(fu)(g) = u(fg), \text{ for all } g \in O_\alpha(\Omega).$$

Thus $O'_\alpha(\Omega; E)$ is an $O^\alpha(\Omega)$-module.

For a compact set K in X^n (or in M^n) and an open set Ω in M^n, we can define an $O^\alpha(K)$-(resp. $\mathcal{A}^\alpha(K)$-, resp. $\mathcal{A}^\alpha(\Omega)$-) module structure of $O'_\alpha(K; E)$ (resp. $\mathcal{A}'_\alpha(K; E)$, resp. $\mathcal{A}'_\alpha(\Omega; E)$) in a similar way.

For a real general Fourier analytic-linear mapping u and a slowly increasing, real-analytic function f, we have

$$\mathrm{supp}(fu) \subset \mathrm{supp}(u).$$

d) Differentiation. In this section we define the operation of differentiation of general Fourier analytic-linear mappings by a method similar to Ito [10], section 3.4, p.247ff.

Proposition 4.11. *Let Ω be an open set in X^n and $u \in O'_\alpha(\Omega;\, E)$. Then the general Fourier analytic-linear mapping v defined by the formula*

$$v(f) = -u(\partial f/\partial x_j) \ \ or \ \ -u(\partial f/\partial y_j),$$
$$for\, f \in O_\alpha(\Omega),$$

belongs to $O'_\alpha(\Omega;\, E)$.

We define the derivative $\partial u/\partial x_j$ or $\partial u/\partial y_j$ as folows:

$$\partial u/\partial x_j = v \ \text{ or } \ \partial u/\partial y_j = v.$$

By the definition above, we know that a general Fourier analytic-linear mapping is infinitely differentiable in the sense of general Fourier analytic-linear mappings and the partial differentiation of a general Fourier analytic-linear mapping does not depend on the order of differentiation. Namely the equalities

$$\partial^2 u/\partial x_j \partial x_k = \partial^2 u/\partial x_k \partial x_j\, \text{, etc.}$$

are valid. We can define, for a $2|n|$- tuple of nonnegative integers $p=(p_1, p_2, \cdots, p_{2|n|})$,

$$\partial^p u(f) = (-1)^{|p|} u(\partial^p f), \ \text{ for } f \in O_\alpha(\Omega),$$

where ∂^p denotes a partial differential operator

$$\partial^p = \partial^{|p|}/\partial x_1^{p_1} \partial y_2^{p_2} \cdots \partial x_{|n|}^{p_{2|n|-1}} \partial y_{|n|}^{p_{2|n|}}.$$

If we put

$$\partial/\partial z_j = (1/2)(\partial/\partial x_j - i\partial/\partial y_j),$$
$$\partial/\partial \bar{z}_j = (1/2)(\partial/\partial x_j + i\partial/\partial y_j),$$

then we have, for $u \in O'_\alpha(\Omega;\, E)$,

$$\partial u/\partial z_j(f) = -u(\partial f/\partial z_j),$$
$$\partial u/\partial \bar{z}_j(f) = -u(\partial f/\partial \bar{z}_j) = 0,$$
$$\text{for all } f \in O_\alpha(\Omega).$$

Proposition 4.12. *Let K be a compact set in X^n and $u \in O'_\alpha(K;\, E)$. Then the general Fourier analytic-linear mapping v defined by the formula*

$$v(f) = -u(\partial f/\partial x_j) \ \ or \ \ -u(\partial f/\partial y_j),$$
$$for\ f \in O_\alpha(K),$$

belongs to $O'_\alpha(K;\, E)$.

We define the derivative $\partial u/\partial x_j$ or $\partial u/\partial y_j$ as follows:

$$\partial u/\partial x_j = v \text{ or } \partial u/\partial y_j = v.$$

The remarks and the notation following Proposition 4.11 are also applicable for this case.

Proposition 4.13. *Let K be a compact set in M^n and $u \in \mathcal{A}'_\alpha(K; E)$. Then the general Fourier analytic-linear mapping v defined by the formula*

$$v(f) = -u(\partial f/\partial x_j), \text{ for } f \in \mathcal{A}_\alpha(K),$$

belongs to $\mathcal{A}'_\alpha(K; E)$.

We define the derivative $\partial u/\partial x_j$ as follows:

$$\partial u/\partial x_j = v.$$

The similar remarks and the notation following Proposition 4.11 are also applicable for this case.

Proposition 4.14. *Let Ω be an open set in M^n and $u \in \mathcal{A}'_\alpha(\Omega; E)$. Then the general Fourier analytic-linear mapping v defined by the formula*

$$v(f) = -u(\partial f/\partial x_j), \text{ for } f \in \mathcal{A}_\alpha(\Omega),$$

belongs to $\mathcal{A}'_\alpha(\Omega; E)$.

We define the derivative $\partial u/\partial x_j$ as follows:

$$\partial u/\partial x_j = v.$$

The similar remarks and the notation following Proposition 4.11 are also applicable for this case.

By the facts above and those in section c), we can define a general differential operator with constant or variable coefficients:

$$P(\partial/\partial x, \partial/\partial y) = \Sigma_{|p| \leqq m} a_p \partial^p, \ a_p \in \mathbf{C},$$
$$P(\partial/\partial z) = \Sigma_{|p| \leqq m} a_p (\partial/\partial z)^p, \ a_p \in \mathbf{C},$$
$$P(\partial/\partial x) = \Sigma_{|p| \leqq m} a_p (\partial/\partial x)^p, \ a_p \in \mathbf{C},$$
$$P(x, y, \partial/\partial x, \partial/\partial y) = \Sigma_{|p| \leqq m} a_p(z) \partial^p,$$
$$\text{where } a_p(z) \in O^\alpha(\Omega) \text{ or } O^\alpha(K),$$
$$P(z, \partial/\partial z) = \Sigma_{|p| \leqq m} a_p(z) (\partial/\partial z)^p,$$
$$\text{where } a_p(z) \in O^\alpha(\Omega) \text{ or } O^\alpha(K),$$
$$P(x, \partial/\partial x) = \Sigma_{|p| \leqq m} a_p(x) (\partial/\partial x)^p,$$
$$\text{where } a_p(x) \in \mathcal{A}^\alpha(K) \text{ or } \mathcal{A}^\alpha(\Omega).$$

Now we give a topological characterization of differentiation.

For a complex vector $h = h' + ih'' = (h_1, h_3, \cdots, h_{2|n|-1}) + i(h_2, h_4, \cdots, h_{2|n|})$ considered as a real vector $(h_1, h_2, \cdots, h_{2|n|})$, we define the translation operator τ_h by the formula

$$\tau_h f(z) = f(z-h), \text{ for } f \in O_\alpha(\Omega).$$

Then the translation of general Fourier analytic-linear mapping $u \in O'_\alpha(\Omega; E)$ is defined by the formula

$$\tau_h u(f) = u(\tau_{-h} f).$$

Then the linear operator $u \longrightarrow \tau_h u$ is the transpose of the linear operator $f \longrightarrow \tau_{-h} f$ for the two pairs $\{O'_\alpha(\Omega; E), O_\alpha(\Omega)\}$ and $\{O'_\alpha(\tau_{-h}(\Omega); E), O_\alpha(\tau_{-h}(\Omega))\}$.

Let $e_j = (\delta_{jk})_{1 \leqq k \leqq 2|n|}$ be the $2|n|$-dimensional fundamental vector. Put $h = h_j e_{2j-1}$ and $k = k_j e_{2j}$. Then, for a holomorphic function $f \in O_\alpha(\Omega)$, we have, as usual,

$$\frac{\partial f}{\partial x_j} = \lim_{h_j \to 0} \frac{\tau_{-h} f - f}{h_j},$$

$$\frac{\partial f}{\partial y_j} = \lim_{k_j \to 0} \frac{\tau_{-k} f - f}{k_j}.$$

Then we have the following.

Theorem 4.15. *Let Ω be an open set in X^n. For $u \in O'_\alpha(\Omega; E)$, we have*

$$\frac{\partial u}{\partial x_j} = \lim_{h_j \to 0} \frac{\tau_{-h} u - u}{h_j},$$

$$\frac{\partial u}{\partial y_j} = \lim_{k_j \to 0} \frac{\tau_{-k} u - u}{k_j},$$

where $h = h_j e_{2j-1}$ and $k = k_j e_{2j}$ and $e_j = (\delta_{jk})_{1 \leqq k \leqq 2|n|}$.

In a similar way, we have the following.

Theorem 4.16. *Let K be a compact set in X^n. For $u \in O'_\alpha(K; E)$, then we have*

$$\frac{\partial u}{\partial x_j} = \lim_{h_j \to 0} \frac{\tau_{-h} u - u}{h_j},$$

$$\frac{\partial u}{\partial y_j} = \lim_{k_j \to 0} \frac{\tau_{-k} u - u}{k_j},$$

where $h = h_j e_{2j-1}$ and $k = k_j e_{2j}$ and $e_j = (\delta_{jk})_{1 \leqq k \leqq 2|n|}$.

Theorem 4.17. *Let K be a compact set in M^n. Put $e_j = (\delta_{jk})_{1 \leqq k \leqq |n|}$ and $h = h_j e_j$. Then, for $u \in \mathcal{A}'_\alpha(K; E)$, we have*

$$\frac{\partial u}{\partial x_j} = \lim_{h_j \to 0} \frac{\tau_{-h} u - u}{h_j}.$$

Theorem 4.18. *Let Ω be an open set in M^n and h as in Theorem 4.17. Then, for $u \in \mathcal{A}'_\alpha(\Omega; E)$, we have*

$$\frac{\partial u}{\partial x_j} = \lim_{h_j \to 0} \frac{\tau_{-h} u - u}{h_j}.$$

Theorem 4.19. *We use the notation in Theorem 4.2. Let $u \otimes_\omega v \in O'_\alpha(\Omega'; E_1)$*

$\hat{\otimes}_\omega O'_\beta(\Omega''; E_2)$, $O'_\alpha(K'; E_1)\hat{\otimes}_\omega O'_\beta(K''; E_2)$, $\mathcal{A}'_\alpha(K'; E_1)\hat{\otimes}_\omega \mathcal{A}'_\beta(K''; E_2)$ *or* $\mathcal{A}'_\alpha(\Omega'; E_1)\hat{\otimes}_\omega \mathcal{A}'_\beta(\Omega''; E_2)$. *Let* $p=(p', p'')$, $p'=(p_1, p_2, \cdots, p_{2|m|})$ *and* $p''=(p_{2|m|+1}, p_{2|m|+2}, \cdots, p_{2(|m|+|n|)})$ *be a* $2(|m|+|n|)$*-tuple, a* $2|m|$*-tuple and a* $2|n|$*-tuple of nonnegative integers or* $p=(p', p'')$, $p'=(p_1, p_2, \cdots, p_{|m|})$ *and* $p''=(p_{|m|+1}, p_{|m|+2}, \cdots, p_{(|m|+|n|)})$ *a* $(|m|+|n|)$*-tuple, a* $|m|$*-tuple and a* $|n|$*-tuple of nonnegative integers, respectively. Then we have*

$$\partial^p(u\otimes_\omega v)=(\partial^{p'}u)\otimes_\omega(\partial^{p''}v).$$

Theorem 4.20. *In the usual notation, let* $u\in O'_\alpha(X^n; E)$ *or* $\mathcal{A}'_\alpha(M^n; E)$. *Let* $\delta_{(h)}$ *be the Dirac measure at h and* $\delta=\delta_{(0)}$. *Then we have*

$$\delta * u=u,\ \delta_{(h)} * u=\tau_h u,\ \partial^p\delta * u=\partial^p u,$$

where ∂^p *means a differential operator of order* $|p|$.

Thus for a differential polynomial

$$P=\Sigma\, a_p\partial^p,\ \ a_p\in \boldsymbol{C},$$

we have

$$Pu=P\delta * u=(\Sigma\, a_p\delta^{(p)}) * u.$$

Theorem 4.21. *Let* E_1 *and* E_2 *be complete. Let* $u\in O'_\alpha(X^n; E_1)$ *and* $v\in O'_\alpha(X^n; E_2)$ *or* $u\in \mathcal{A}'_\alpha(M^n; E_1)$ *and* $v\in \mathcal{A}'_\alpha(M^n; E_1)$, *one of which has a compact carrier in* $\boldsymbol{C}^{|n|}$ *or* $\boldsymbol{R}^{|n|}$ *respectively. Then we have*

$$\tau_h(u *_\omega v)=(\tau_h u) *_\omega v=u *_\omega(\tau_h v)$$

and

$$\partial^p(u *_\omega v)=(\partial^p u) *_\omega v=u *_\omega(\partial^p v).$$

e) Indefinite integrals of general Fourier analytic-linear mappings. In this section we mention indefinite integrals of general Fourier analytic-linear mappings.

At first we recall the following.

Lemma 4.22. *Let F be the projection* $F{:}\boldsymbol{C}^{|n|}\longrightarrow \boldsymbol{C}^{|n|-1}$ *such as* $F(z_1, \cdots, z_{|n|})=(z_1, \cdots, \hat{z}_j, \cdots, z_{|n|})$, *where* $\hat{z}_j$ *denotes the omission of* z_j. *Let* $V=\mathrm{int}([V_1\times\cdots\times V_{n_1+n_2}\times V''']^{\mathrm{cl}})$ *be a product tubular domain in* X^n *such that* $V_j=\{z_j\in \boldsymbol{C};\ |z_j|<a_j\}$, $(j=1, \cdots, n_1)$, $V_j=\{z_j\in \boldsymbol{C};\ |\mathrm{Im}\, z_j|<a_j\}$, $(j=n_1+1, \cdots, n_1+n_2)$, *and* $V'''=\{z'''\in \boldsymbol{C}^{n_3};\ |\mathrm{Im}\, z'''|<a(1+|\mathrm{Re}\, z'''|)\}$, *for sufficiently small* $a>0$ *and* $a_j>0$, $(j=1, 2, \cdots, n_1+n_2)$. *Put* $V_j=\mathrm{int}([F(V\cap \boldsymbol{C}^{|n|}]^{\mathrm{cl}})$. *Then we have*

(1) *For every* $f\in {}_{|n|}O_\alpha(V)$, *there exists* $g\in {}_{|n|}O_\alpha(V)$ *such that*

(i) $D_j g=f$, *where* $D_j=\partial/\partial z_j$,

(ii) *For any compact subset K of V and for some constant* $\delta>0$, *there exist some compact subset H of V and some constant* $\delta'>0$ *such that*

$$\sup\{|g(z)|e^{\delta|z|};\ z\in K\cap \boldsymbol{C}^{|n|}\}$$

$$\leqq \sup\{|f(z)|e^{\delta'|z|};\, z\in H\cap \boldsymbol{C}^{|n|}\}.$$

Thus the mapping $f \longrightarrow g$ is continuous.

(2) *If $f\in {}_{|n|}O_\alpha(V)$ and $D_j f=0$, then there exists $g\in {}_{|n|-1}O_\alpha(V^j)$ such that $f(z)=g(F(z))$.*

Proof. See Ito[10], Lemma 3.5.1, p.251. Q.E.D.

Theorem 4.23. *Let V be as in Lemma 4.22. Then, for any $v\in {}_{|n|}O'_\alpha(V;\, E)$, there exists $u\in {}_{|n|}O'_\alpha(V;\, E)$ such that $\partial u/\partial z_j=v$. Such two solutions u_1 and u_2 are different one another by an arbitrary general Fourier analytic linear mapping in ${}_{|n|-1}O'_\alpha(F(V);\, E)$, where F is the same projection as defined in Lemma 4.22.*

Proof. It goes similarly to Ito[10], Theorem 3.5.2, p.251. Q.E.D.

Lemma 4.24. *Let K be a compact set in X^n such that* $\mathrm{int}(K\cap \boldsymbol{C}^{|n|})$ *is a convex tubular domain and K has a fundamental system of open neighborhoods of the type of V in Lemma 4.22. Put $K^j=F(K)$, where F is the same as in Lemma 4.22. Then we have*

(1) $D_j({}_{|n|}O_\alpha(K))={}_{|n|}O_\alpha(K)$.

(2) *If $f\in {}_{|n|}O_\alpha(K)$ and $D_j f=0$, then there exists $g\in {}_{|n|-1}O_\alpha(K^j)$ such that $f(z)=g(F(z))$.*

Proof. See Ito [10], Lemma 3.5.3, p.252. Q.E.D.

Theorem 4.25. *Let K be a compact set in X^n such as in Lemma 4.24. Then, for any $v\in {}_{|n|}O'_\alpha(K;\, E)$, there exists $u\in {}_{|n|}O'_\alpha(K;\, E)$ such that $\partial u/\partial z_j=v$. Such two solutions u_1 and u_2 are different one another by an arbitrary general Fourier analytic-linear mapping in ${}_{|n|-1}O'_\alpha(F(K);\, E)$.*

Lemma 4.26. *Let K be a compact set in M^n of the type $K_1\times\cdots\times K_{n_1}\times \tilde{\boldsymbol{R}}^{n_2}\times \tilde{\tilde{\boldsymbol{R}}}^{n_3}$ where $K_j=[-a_j,\, a_j]$, $(a_j>0)$, $(j=1,\, \cdots,\, n_1)$. Put $K_j=\tilde{F}(K)$, where $\tilde{F}$ is the restriction to $\boldsymbol{R}^{|n|}$ of F in Lemma 4.22. Then we have*

(1) $D_j({}_{|n|}\mathcal{A}_\alpha(K))={}_{|n|}\mathcal{A}_\alpha(K)$, *where $D_j=\partial/\partial x_j$.*

(2) *If $f\in {}_{|n|}\mathcal{A}_\alpha(K)$ and $D_j f=0$, then there exists $g\in {}_{|n|-1}\mathcal{A}_\alpha(K^j)$ such that $f(x)=g(\tilde{F}(x))$.*

Theorem 4.27. *Let K be as in Lemma 4.26. Then, for any $v\in {}_{|n|}\mathcal{A}'_\alpha(K;\, E)$, there exists $u\in {}_{|n|}\mathcal{A}'_\alpha(K;\, E)$ such that $\partial u/\partial x_j=v$. Such two solutions u_1 and u_2 are different one another by an arbitrary general Fourier analytic-linear mapping in ${}_{|n|-1}\mathcal{A}'_\alpha(\tilde{F}(K);\, E)$, where $\tilde{F}$ is as in Lemma 4.26.*

f) Analytic diffeomorphism. If X^n is an n-dimensional, complex Euclidean space and if Ω_1 and Ω_2 are two open sets in X^n, $w=\Phi(z)$ denotes a complex-analytic diffeomorphism of Ω_1 onto Ω_2. If X^n is $\tilde{\boldsymbol{C}}^n$ or $\tilde{\tilde{\boldsymbol{C}}}^n$ and if Ω_1 and Ω_2 are two open sets in X^n, $w=\Phi(z)$ denotes a regular, complex, inhomogeneous linear transformation of Ω_1 onto Ω_2. If X^n is one of $\boldsymbol{C}^{\#,n}$, $\boldsymbol{C}^{\flat,n}$, $\boldsymbol{C}^{\natural,n}$ and $\boldsymbol{C}^{*,n}$ and if Ω_1 and Ω_2 are corresponding two open sets in X^n, $w=\Phi(z)$ is a product diffeomorphism of those in the above which maps Ω_1 onto Ω_2.

Then, for $u\in \mathcal{O}_\alpha(\Omega_2; E)$, we define $\Phi^*u\in \mathcal{O}'_\alpha(\Omega_1; E)$ by the formula

$$(\Phi^*u)(f)=u((f\circ\Phi^{-1})\,|J|),\ \text{for } f\in \mathcal{O}_\alpha(\Omega_1),$$

where $|J|$ is the absolute value of the Jacobian J of the mapping Φ^{-1}.

If K_1 and K_2 are two compact sets in X^n and if $w=\Phi(z)$ is a complex-analytic diffeomorphism of the above form which maps a certain neighborhood in X^n of K_1 onto a certain neighborhood in X^n of K_2 such that $\Phi(K_1)=K_2$, then, for $u\in \mathcal{O}'_\alpha(K_2; E)$, we define $\Phi^*u\in \mathcal{O}'_\alpha(K_1; E)$ by the formula

$$(\Phi^*u)(f)=u((f\circ\Phi^{-1})\,|J|),\ \text{for } f\in \mathcal{O}_\alpha(K_1).$$

If K_1 and K_2 are two compact sets in M^n, this is a special case of the above. But, in this case, we have

$$\operatorname{supp}(\Phi^*u)=\Phi^{-1}(\operatorname{supp}(u)),\ \text{for } u\in \mathcal{O}'_\alpha(K_2; E).$$

At last, if M^n is $\boldsymbol{R}^n$, and if Ω_1 and Ω_2 are two open sets in M^n, $y=\Phi(x)$ denotes a real-analytic diffeomorphism of Ω_1 onto Ω_2. If M^n is $\tilde{\boldsymbol{R}}^n$ or $\tilde{\tilde{\boldsymbol{R}}}^n$ and if Ω_1 and Ω_2 are two open sets in M^n, $y=\Phi(x)$ denotes a regular, real, inhomogeneous linear transformation of Ω_1 onto Ω_2. If M^n is $\boldsymbol{R}^{\#,n}$, $\boldsymbol{R}^{\flat,n}$, $\boldsymbol{R}^{\natural,n}$ or $\boldsymbol{R}^{*,n}$ and if Ω_1 and Ω_2 are two open sets in M^n, $y=\Phi(x)$ denotes a product diffeomorphism of those of the two types above which maps Ω_1 onto Ω_2. Then, for $u\in \mathcal{A}'_\alpha(\Omega_2; E)$, we define $\Phi^*u\in \mathcal{A}'_\alpha(\Omega_1; E)$ by the formula

$$(\Phi^*u)(f) = u((f\circ\Phi^{-1})\,|J|),\ \text{for } f\in \mathcal{A}_\alpha(K_1).$$

Then we have

$$\operatorname{supp}(\Phi^*u)=\Phi^{-1}(\operatorname{supp}(u)),\ \text{for } u\in \mathcal{A}'_\alpha(\Omega_2; E).$$

5. Operations on general Fourier hyperfunctions

In this chapter we define several operations on general Fourier hyperfunctions.

a) Tensor products. In this paragraph, we assume that E_1 and E_2 are two complete LCV's and put $E=E_1\hat{\otimes}_\omega E_2$, where ω stands for the ε- or π-topology in the sense of Trèves [27]. Let M^n be one of $\boldsymbol{R}^n$, $\tilde{\boldsymbol{R}}^n$, $\tilde{\tilde{\boldsymbol{R}}}^n$, $\boldsymbol{R}^{\#,n}$, $\boldsymbol{R}^{\flat,n}$, $\boldsymbol{R}^{\natural,n}$ and $\boldsymbol{R}^{*,n}$. Let both $\{\mathcal{F}(\Omega; E_1)$; Ω is an open set in $M^n\}$ and $\{\mathcal{G}(\Omega; E_2)$; Ω is an open set in $M^n\}$ denote one of the presheaves $\{\mathcal{B}(\Omega; E_i)\}$, $\{\tilde{\mathcal{B}}(\Omega; E_i)\}$, $\{\tilde{\tilde{\mathcal{B}}}(\Omega; E_i)\}$, $\{\mathcal{B}^\#(\Omega; E_i)\}$, $\{\mathcal{B}^\flat(\Omega; E_i)\}$, $\{\mathcal{B}^\natural(\Omega; E_i)\}$ and $\{\mathcal{B}^*(\Omega; E_i)\}$, $(i=1, 2)$.

Let Ω_1 and Ω_2 be relatively compact, open sets in M^m and M^n respectively.

Let $T_1 \in \mathcal{F}(\Omega_1; E_1)$ and $T_2 \in \mathcal{G}(\Omega_2; E_2)$. Let $\bar{T}_1 \in \mathcal{A}'_\alpha(\Omega_1^{\mathrm{cl}}; E_1)$ and $\bar{T}_2 \in \mathcal{A}'_\beta(\Omega_2^{\mathrm{cl}}; E_2)$ so that $\bar{T}_1|_{\Omega_1} = T_1$ and $\bar{T}_2|_{\Omega_2} = T_2$, where $\bar{T}_1|_{\Omega_1}$ denotes the image of $\bar{T}_1$ in $\mathcal{F}(\Omega_1; E_1)$ and $\bar{T}_2|_{\Omega_2}$ denotes the image of $\bar{T}_2$ in $\mathcal{G}(\Omega_2; E_2)$. Then we have $\bar{T}_1 \otimes_\omega \bar{T}_2 \in \mathcal{A}'_{\alpha\beta}(\Omega_1^{\mathrm{cl}} \times \Omega_2^{\mathrm{cl}}; E)$. Then we can see that $\bar{T}_1 \otimes_\omega \bar{T}_2|_{\Omega_1 \times \Omega_2}$ does not depend on the choice of representatives $\bar{T}_1$ and $\bar{T}_2$ of T_1 and T_2 respectively. Thus $\bar{T}_1 \otimes_\omega \bar{T}_2|_{\Omega_1 \times \Omega_2}$ is an E-valued general Fourier hyperfunction on $\Omega_1 \times \Omega_2$ which depends only on T_1 and T_2. We denote this by $T_1 \otimes_\omega T_2$ and call it the tensor product of T_1 and T_2. $T_1 \otimes_\omega T_2$ has the properties of tensor products of vectors. Then we have

$$\mathrm{supp}(T_1 \otimes_\omega T_2) \subset \mathrm{supp}(T_1) \times \mathrm{supp}(T_2).$$

Here we note that, as far as we are concerned with finite sums of finite tensor products of general Fourier hyperfunctions, we need not assume the completeness of LCV's E_1, E_2.

b) Convolution. Let M^n be one of $\boldsymbol{R}^n$, $\tilde{\boldsymbol{R}}^n$, $\tilde{\tilde{\boldsymbol{R}}}^n$, $\boldsymbol{R}^{\#,n}$, $\boldsymbol{R}^{\flat,n}$, $\boldsymbol{R}^{\natural,n}$ and $\boldsymbol{R}^{*,n}$. Let $\{\mathcal{F}(\Omega; E_2); \Omega$ is an open set in $M^n\}$ be the presheaf of E_2-valued general Fourier hyperfunctions over M^n.

If M^n is compact, $\mathcal{F}(M^n; E_2) = \mathcal{A}'_\alpha(M^n; E_2)$. Thus, in this case, we have nothing special to do with convolution products of general Fourier hyperfunctions.

If M^n is not compact, let $\tilde{M}^n$ be the radial compactification of M^n. Then $\tilde{M}^n$ is one of $\tilde{\boldsymbol{R}}^n \times \tilde{\boldsymbol{R}}^n$, $\tilde{\boldsymbol{R}}^n \times \tilde{\tilde{\boldsymbol{R}}}^n$, $\tilde{\tilde{\boldsymbol{R}}}^n \times \tilde{\tilde{\boldsymbol{R}}}^n$, $\tilde{\boldsymbol{R}}^n \times \tilde{\boldsymbol{R}}^n \times \tilde{\tilde{\boldsymbol{R}}}^n$ and $\tilde{\tilde{\boldsymbol{R}}}^n \times \tilde{\boldsymbol{R}}^n \times \tilde{\tilde{\boldsymbol{R}}}^n$. Then $\tilde{\mathcal{F}}(\tilde{M}^n; E_2)$ is the space of the type $\mathcal{A}'_{\alpha\beta}(\tilde{M}^n; E_2)$. Then if Ω_2 is relatively compact open set in M^n, then Ω_2 is an open set in $\tilde{M}^n$. Thus, by virtue of Proposition 3.4, we have, for every $T \in \mathcal{F}(\Omega_2; E_2)$, a prolongation $\bar{T} \in \tilde{\mathcal{F}}(\tilde{M}^n; E_2)$ of T so that $\bar{T}|_{\Omega_2} = T$.

Then, for $u \in \mathcal{A}'(\Omega_1; E_1)$, ($\Omega_1$ is an open set in $\boldsymbol{R}^{|n|} \cap M^n$), we can define

$$u * T|_{\Omega'} = u *_\omega \bar{T}|_{\Omega'}$$

for an open set Ω' in M^n such that $(\Omega_1 + (C\Omega_2) \cap \boldsymbol{R}^n) \cap (\Omega' \cap \boldsymbol{R}^{|n|}) = \varnothing$ holds. In particular, if $u \in \mathcal{A}'(\{0\})$, $u*$ defines a morphism of the presheaf $\{\mathcal{F}(\Omega_2; E_2)\}$. We call $u \in \mathcal{A}'(\{0\})$ a local operator of $\mathcal{F}(M^n; E_2)$.

c) Multiplication by a slowly increasing real-analytic function. Let M^n be one of $\boldsymbol{R}^n$, $\tilde{\boldsymbol{R}}^n$, $\tilde{\tilde{\boldsymbol{R}}}^n$, $\boldsymbol{R}^{\#,n}$, $\boldsymbol{R}^{\flat,n}$, $\boldsymbol{R}^{\natural,n}$ and $\boldsymbol{R}^{*,n}$. Let $\{\mathcal{F}(\Omega; E); \Omega$ is an open set in $M^n\}$ be one of the presheaves $\{\mathcal{B}(\Omega; E)\}$, $\{\tilde{\mathcal{B}}(\Omega; E)\}$, $\{\tilde{\tilde{\mathcal{B}}}(\Omega; E)\}$, $\{\mathcal{B}^\#(\Omega; E)\}$, $\{\mathcal{B}^\flat(\Omega; E)\}$, $\{\mathcal{B}^\natural(\Omega; E)\}$, and $\{\mathcal{B}^*(\Omega; E)\}$.

Let Ω be a relatively compact, open set in M^n. If $T \in \mathcal{B}(\Omega; E)$ and $f \in \mathcal{A}^\alpha(\Omega^{\mathrm{cl}})$, then we shall define fT as follows. Let $\bar{T} \in \mathcal{A}'_\alpha(\Omega^{\mathrm{cl}}; E)$ such that $\bar{T}|_\Omega = T$ holds. Then $f\bar{T} \in \mathcal{A}'_\alpha(\Omega^{\mathrm{cl}}; E)$. Then we can see that $f\bar{T}|_\Omega$ does not depend on the choice of the representative $\bar{T}$. Thus $f\bar{T}|_{\Omega}$ is an E-valued general Fourier hyperfunction in $\mathcal{F}(\Omega; E)$ which depend only on f and T and which we denote by fT.

d) Differentiation. Let M^n and Ω and $\mathcal{F}(\Omega; E)$ be as in section c). If $T\in\mathcal{F}(\Omega; E)$, let $\bar{T}\in\mathcal{A}'_{\alpha}(\Omega^{\mathrm{cl}}; E)$ such that $\bar{T}|_{\Omega}=T$ holds. Then, using Proposition 4.12, we define $\partial T/\partial x_j$ as follows:

$$\partial T/\partial x_j=(\partial\bar{T}/\partial x_j)|_{\Omega}.$$

By this definition, we know that an E-valued general Fourier hyperfunction is infinitely differentiable in the sense of E-valued general Fourier hyperfunctions and the partial differentiation of an E-valued general Fourier hyperfunction does not depend on the order of differentiation.

By the above facts and those in section c) of this chapter, we can define a general differential operator with constant or variable coefficients:

$$P(\partial/\partial x)=\Sigma_{|p|\leqq m}\, a_p(\partial/\partial x)^p,\ a_p\in \boldsymbol{C},$$

and

$$P(x, \partial/\partial x)=\Sigma_{|p|\leqq m}\, a_p(x)(\partial/\partial x)^p,$$

where $a_p(x)\in\mathcal{A}^{\alpha}(K)$ or $\mathcal{A}^{\alpha}(\Omega)$, ($\Omega$ is relatively compact) and $(\partial/\partial x)^p$ denotes

$$(\partial/\partial x)^p=\partial^{|p|}/\partial x_1^{p_1}\,\partial x_2^{p_2}\cdots\partial x_{|n|}^{p_{|n|}},$$

$$p=(p_1, p_2, \cdots, p_{|n|}).$$

Theorem 5.1. *Let E_1, E_2 and M^m, M^n and Ω_1, Ω_2 and $\mathcal{F}(\Omega_1; E_1)$, $\mathcal{G}(\Omega_2; E_2)$ be as in section a) of this chapter. Then, for $T_1\in\mathcal{F}(\Omega_1; E_1)$ and $T_2\in\mathcal{G}(\Omega_2; E_2)$, we have*

$$(\partial/\partial x)^p(T_1\otimes_{\omega}T_2)=((\partial/\partial x)^{p'}T_1)\otimes_{\omega}((\partial/\partial x)^{p''}T_2),$$

where $p=(p', p'')$, p', p'' are an $(|m|+|n|)$-tuple, an $|m|$-tuple and an $|n|$-tuple of nonnegative integers, respectively.

Theorem 5.2. *Let $\mathcal{A}'(\Omega_1; E_1)$ and $\mathcal{F}(\Omega_2; E_2)$ be as in section b) of this chapter for an open set $\Omega_1\subset\boldsymbol{R}^{|n|}$ and a relatively compact, open set $\Omega_2\subset M^n$. If Ω' is an open set in M^n such that $(\Omega_1+(C\Omega_2)\cap\boldsymbol{R}^{|n|})\cap(\Omega'\cap\boldsymbol{R}^{|n|})=\varnothing$ holds, then we have, for $u\in\mathcal{A}'(\Omega_1; E_1)$ and $T\in\mathcal{F}(\Omega_2; E_2)$,*

$$(\partial/\partial x)^p(u*T)|_{\Omega'}=((\partial/\partial x)^p u)*_{\omega}T|_{\Omega'}=u*((\partial/\partial x)^p T)|_{\Omega'},$$

and

$$((\partial/\partial x)^p\delta)*T|_{\Omega_2}=(\partial/\partial x)^p\,T|_{\Omega_2},$$

where δ is the Dirac measure.

e) Indefinite integrals. We use the notation in section c) of this chapter. Then we have the following.

Theorem 5.3. *Let M^n, Ω and $\mathcal{F}(\Omega; E)$ be as in section c) of this chapter.*

Then, for every $T \in \mathcal{F}(\Omega; E)$, there exists $S \in \mathcal{F}(\Omega; E)$ such that $\partial S/\partial x_j = T$. Such two solutions S_1 and S_2 are different one another by an arbitrary E-valued general Fourier hyperfunction in $\mathcal{F}(F(\Omega); E)$, where F means a natural extension to M^n of a projection of $\boldsymbol{R}^{|n|}$ on $\boldsymbol{R}^{|n|-1}$ such as $F(x_1, \cdots, x_{|n|}) = (x_1, \cdots, \hat{x}_j, \cdots, x_{|n|})$, denoting by $\hat{x}_j$ the omission of the coordinate x_j.

Proof. Let K be a compact set in M^n of the type in Lemma 4.26 which contains Ω as its subset. Then, by virtue of Proposition 3.4, there exists $\bar{T} \in \mathcal{A}'_\alpha(K; E)$ such that $\bar{T}|_\Omega = T$ holds. Then, by virtue of Theorem 4.27, we have $\bar{S} \in \mathcal{A}'_\alpha(K; E)$ such that $\partial \bar{S}/\partial x_j = \bar{T}$. Then $S = \bar{S}|_\Omega \in \mathcal{F}(\Omega; E)$ satisfies the equation $\partial S/\partial x_j = T$. The latter assertion follows also from Theorem 4.27. Q.E.D.

f) Analytic diffeomorphism. Let M^n, Ω and $\mathcal{F}(\Omega; E)$ be as in section c) of this chapter. If Ω_1 and Ω_2 are relatively compact, open sets, then $y = \Phi(x)$ denotes a complex-analytic diffeomorphism of the form as in section f) of chapter 4 which maps a certain neighborhood in X^n of Ω_1^{cl} onto a certain neighborhood in X^n of Ω_2^{cl} such that $\Phi(\Omega_1^{\mathrm{cl}}) = \Omega_2^{\mathrm{cl}}$. Then, for $T \in \mathcal{F}(\Omega_2; E)$, let $\bar{T} \in \mathcal{A}'_\alpha(\Omega_1^{\mathrm{cl}}; E)$ with $\bar{T}|_{\Omega_2} = T$. Then, using the result in the paragraph f) in chapter 4, we define $\Phi^* \bar{T} \in \mathcal{A}'_\alpha(\Omega_1^{\mathrm{cl}}; E)$. Then we can see that $\Phi^* \bar{T}|_{\Omega_1}$ does not depend on the choice of representatives $\bar{T}$. Thus $\Phi^* \bar{T}|_{\Omega_1}$ is an E-valued general Fourier hyperfunction on Ω_1 which depends only on T. We denote this by $\Phi^* T$. Then we have

$$\mathrm{supp}(\Phi^* T) = \Phi^{-1}(\mathrm{supp}(T)), \text{ for } T \in \mathcal{F}(\Omega_2; E).$$

6. Fourier transformation of general Fourier hyperfunctions

In this chapter we introduce the notion of the Fourier transformation of E-valued mixed Fourier hyperfunctions on $\boldsymbol{R}^{\#,n}$, where $n = (n_1, n_2)$ is a pair of nonnegative integers with $|n| = n_1 + n_2 \neq 0$. Then the Fourier transformation of each type of Fourier hyperfunctions can be treated as a special case of that for E-valued mixed Fourier hyperfunctions.

Proposition 6.1. *If we define $\mathcal{F}_1\varphi$ by the formula*

$$(\mathcal{F}_1\varphi)(\xi) = \int_{\boldsymbol{R}^n} \exp(-i\langle x, \xi\rangle)\, \varphi(x) dx$$

for $\varphi \in \underset{\sim}{\mathcal{A}}(\tilde{\boldsymbol{R}}^n)$, where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$, then $\mathcal{F}_1$ gives a topological isomorphism of $\underset{\sim}{\mathcal{A}}(\tilde{\boldsymbol{R}}^n)$ onto itself.

Proof. See Kawai [17], Proposition 3.2.4, p.483. Q.E.D.

Proposition 6.2. *If we define $\mathcal{F}_2\varphi$ by the formula*

$$(\mathcal{F}_2\varphi)(\xi) = \int_{\boldsymbol{R}^n} \exp(-i\langle x, \xi\rangle)\, \varphi(x) dx$$

for $\varphi \in \underset{\approx}{\mathcal{A}}(\tilde{\tilde{\boldsymbol{R}}}^n)$, *then* $\mathcal{F}_2$ *gives a topological isomorphism of* $\underset{\approx}{\mathcal{A}}(\tilde{\tilde{\boldsymbol{R}}}^n)$ *onto itself.*

Proof. See Saburi [23], Theorem 5.1.2, p.80, [24], Theorem 4.1.1, p.255 and Nagamachi-Mugibayashi [22]. Q.E.D.

Generalizing Propositions 6.1 and 6.2, we obtain

Proposition 6.3. *If we define* $\mathcal{F}f$ *by the formula*

$$(\mathcal{F}\varphi)(\xi)=\int_{\boldsymbol{R}^{|n|}} \exp(-i<x,\ \xi>)\ \varphi(x)dx$$

for $\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n})$, *where* $<x,\ \xi>=x_1\xi_1+\cdots+x_{|n|}\xi_{|n|}$, *then* $\mathcal{F}$ *gives a topological isomorphism of* $\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n})$ *onto itself.*

Proof. This follows from the fact that $\mathcal{F}=\mathcal{F}_1\hat{\otimes}\mathcal{F}_2$ holds. Q.E.D.

Proposition 6.4. *Let* $P(X)$ *be a polynomial of* $|n|$ *indeterminates. Then we have the following:*

(1) $P(\partial/\partial\xi)(\mathcal{F}\varphi)=\mathcal{F}(P(-ix)\varphi(x))$.

(2) $P(\xi)(\mathcal{F}\varphi)=\mathcal{F}(P(\frac{1}{i}\frac{\partial}{\partial x})(\varphi(x))$.

(3) $\mathcal{F}(\varphi*\phi)=(\mathcal{F}\varphi)(\mathcal{F}\phi)$.

(4) $\mathcal{F}(\varphi\phi)=(\mathcal{F}\varphi)*(\mathcal{F}\phi)$.

Here φ and ϕ are in $\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n})$.

Definition 6.5. Let T be an element in $\mathcal{B}^{\#}(\boldsymbol{R}^{\#,n};\ E)=\mathcal{A}'_{\#}(\boldsymbol{R}^{\#,n};\ E)$. Then we define the Fourier transformation $\mathcal{F}^*$ of $\mathcal{B}^{\#}(\boldsymbol{R}^{\#,n};\ E)$ by the formula

$$(\mathcal{F}^*T)(\varphi)=T(\mathcal{F}\varphi),\ \text{for every}\ \varphi\in\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n}).$$

We also define the inverse Fourier transformation $\bar{\mathcal{F}}^*$ of $\mathcal{B}^{\#}(\boldsymbol{R}^{\#,n};\ E)$ by the formula

$$(\bar{\mathcal{F}}^*T)(\varphi)=T(\bar{\mathcal{F}}\varphi),\ \text{for every}\ \varphi\in\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n}).$$

Here we define $\bar{\mathcal{F}}\varphi$ by the formula

$$(\bar{\mathcal{F}}\varphi)(\xi)=(2\pi)^{-|n|}\int_{\boldsymbol{R}^{|n|}} \exp(i<x,\ \xi>)\varphi(x)dx.$$

Then we have the relation.

$$\bar{\mathcal{F}}^*\mathcal{F}^*=\mathcal{F}^*\bar{\mathcal{F}}^*=\text{identity}.$$

In the sequel, we denote $\mathcal{F}=\mathcal{F}^*$.

Definition 6.6. Let $T\in\mathcal{A}'(\boldsymbol{R}^{|n|};\ E)$, whose support is a compact set K in $\boldsymbol{R}^{|n|}$. Then we define the Fourier-Borel transform $\hat{T}(\xi)$ of T by the formula

$$\hat{T}(\xi)=T_x(\exp(i<x,\ \xi>)).$$

Then we have the following.

Proposition 6.7. *For* $T\in\mathcal{A}'(\boldsymbol{R}^{|n|};\ E)$ *whose support is a compact set* K *in* $\boldsymbol{R}^{|n|}$, *the Fourier-Borel transform* $\hat{T}(\xi)$ *of* T *belongs to* $\mathcal{A}^{\#}(\boldsymbol{R}^{\#,n};\ E)$ *and* $\hat{T}(\xi)$ *can be extended to the whole space* $\boldsymbol{C}^{|n|}$ *as an* E*-valued holomorphic function given by* $\hat{T}(\zeta)=T_x(\exp(i<x,\ \zeta>))$ *such that* $\hat{T}(\zeta)\in\mathcal{O}^{\#}(\text{int}[\boldsymbol{R}^{|n|}\times iB^{\circ}]^{\text{cl}};\ E)$, *where* B° *means* *the polar set of a closed convex hull B of K.*

Corollary. *Let T be as in Prposition 6.7. Then we have* $\mathcal{F}T=(\hat{T})^{\vee}$ *considering T as an element in* $\mathcal{A}'_{\#}(\boldsymbol{R}^{\#,n}; E)$, *where* $(\hat{T})^{\vee}$ *is defined by the formula* $(\hat{T})^{\vee}(\xi)=\hat{T}(-\xi)$.

Proof. We have, for every $\varphi\in\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n})$,

$$(\mathcal{F}T)(\varphi)=T(\int_{\boldsymbol{R}^{|n|}}\exp(-i<x,\xi>)\varphi(\xi)\,d\xi)$$
$$=\int_{\boldsymbol{R}^{|n|}}T_x(\exp(-i<x,\xi>)\varphi(\xi)\,d\xi=(\hat{T})^{\vee}(\varphi).$$

This completes the proof. Q.E.D.

Here we study Fourier transforms of convolutions and multiplications in the case of E-valued mixed Fourier hyperfunctions in a similar way to Ito [10], section 5.3, p.562f.

Theorem 6.8. *Let* $S\in\mathcal{A}'(\boldsymbol{R}^{|n|})$ *and* $T\in\mathcal{A}'_{\#}(\boldsymbol{R}^{\#,n}; E)$. *Let* $a(x)\in\mathcal{A}^{\#}(\boldsymbol{R}^{\#,n})$ *be the Fourier-Borel transform of a real analytic functional whose support is a compact set in* $\boldsymbol{R}^{|n|}$. *Then we have*

(1) $\mathcal{F}(S*T)=(\mathcal{F}S)\cdot(\mathcal{F}T)$.

(2) $\mathcal{F}(aT)=(2\pi)^{-|n|}((\mathcal{F}a)*(\mathcal{F}T))$.

Proof. It suffices to prove (1). Indeed, if (1) is proved, the same formula then is true with $\bar{\mathcal{F}}$ replacing $\mathcal{F}$ in (1):

$$\bar{\mathcal{F}}(S*T)=(2\pi)^{|n|}(\bar{\mathcal{F}}S\cdot\bar{\mathcal{F}}T).$$

Thus we have

$$S*T=\bar{\mathcal{F}}((2\pi)^{|n|}\bar{\mathcal{F}}S\cdot\bar{\mathcal{F}}T).$$

But $\bar{\mathcal{F}}S$ may be replaced by $a(x)$ and $\bar{\mathcal{F}}T$ by T. Then S has to be replaced by $\mathcal{F}a$ and T by $\mathcal{F}T$. Thus we obtain (2),

In order to prove (1), we observe that we have, for $\varphi\in\mathcal{A}_{\#}(\boldsymbol{R}^{\#,n})$,

$$\mathcal{F}(S*T)(\varphi)=(S*T)(\mathcal{F}\varphi)=T(\check{S}*\mathcal{F}\varphi).$$

Here $\check{S}$ is defined by the formula

$$<\check{S},f>=<S,\check{f}>$$

where we put $\check{f}(x)=f(-x)$. But we have

$$(\check{S}*\mathcal{F}\varphi)(x)=<S_y,\mathcal{F}\varphi(x+y)>$$
$$=<S_y,\int_{\boldsymbol{R}^{|n|}}\exp(-i<x+y,\xi>)\varphi(\xi)\,d\xi>$$
$$=\int_{\boldsymbol{R}^{|n|}}\exp(-i<x,\xi>)<S_y,\exp(-i<y,\xi>)>\varphi(\xi)\,d\xi$$
$$=\mathcal{F}(\mathcal{F}S\cdot\varphi).$$

Thus we obtain

$$\mathcal{F}(S*T)(\varphi)=T(\mathcal{F}(\mathcal{F}S\cdot\varphi))=(\mathcal{F}T)(\mathcal{F}S\cdot\varphi)=(\mathcal{F}S\cdot\mathcal{F}T)(\varphi).$$

This completes the proof. Q.E.D.

Theorem 6.9. *Let $P(X)$ be a polynomial of $|n|$ indeterminates. Then we have, for every $T\in\mathcal{A}'_{\#}(\boldsymbol{R}^{\#,n}; E)$ and for every vector $p\in\boldsymbol{R}^{|n|}$,*

(1) $P(\partial/\partial\xi)\mathcal{F}T=\mathcal{F}(P(-ix)T)$.

(2) $P(\xi)\mathcal{F}T=\mathcal{F}(P(\frac{1}{i}\frac{\partial}{\partial x})T)$.

(3) $\mathcal{F}(\tau_p T)=\exp(-i<x,\xi>)\mathcal{F}T$.

Proof. It suffices to observe that

$$P(\partial/\partial x)T=[P(\partial/\partial x)\delta]*T,\ \tau_p T=\delta_p*T.\ \text{Q.E.D.}$$

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