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Methods of Renormalization and Distributions

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Abstract

In this article, we consider the mathematical meaning of renormalization. We show that divergent integrals can be given finite values in the case of conditional convergence of extended integrals by subtracting an infinite quantity. This can be considered as a kind of renormalization of divergent integrals. Using this method of renormalization, we define distributional extensions of measurable functions with nonsummable singuralities.

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Introduction

In the quantum field theory, there are problems of renormalization of divergent integrals such as Feynman integrals[10]. In this article, we consider the mathematical meaning of renormalization. Using the method of renormalization, we define distributional extensions of measurable functions with nonsummable singuralities. We show that distributional extensions of distributions defined by these functions can be realized as extended Riemann integrals or extended Lebesgue integrals in the case of conditional convergence. Further we show that divergent integrals can be given finite values in the case of conditional convergence of extended integrals by subtracting an infinite quantity. This can be considered as a kind of renormalization of divergent integrals. Further regularizations of divergent integrals in the sense of Gelfand-Shilov[1] can be also considered as distributional extensions of distributions defined by locally integrable functions with nonsummable singularities. Thus we can see that the method of renormalization gives the definition of distributional extensions beyond singularities of distributions defined by locally integrable functions. Further the method of renormalization gives the method of obtaining the finite quantity by subtracting an infinite quantity from a divergent integral which is not necessarily a distribution. This clarifies the method of renormalization used by S. Tomonaga and others in the theory of quantum electrodynamics[7]. At last we give some examples of distributions obtained by the renormalizations of divergent integrals.

1. Extended Riemann integrals and extended Lebesgue integrals

1.1. Extended Riemann integrals. Here we remember the definition of extended Riemann integrals, [3], [5].

Assume $n \geq 1$. Let a set E be a Jordan measurable set in $\mathbf{R}^n$. Let A be a directed set and $\{E_\alpha\}_{\alpha\in A}$ a family of bounded measurable closed sets in E. Then we say that $\{E_\alpha\}_{\alpha\in A}$ converges to E if, for every bounded measurable closed set H in E, there exists some element α_0 in A such that, for every $\alpha \in A$, $\alpha \geq \alpha_0$, $E_\alpha \supset H$ holds. Then we call $\{E_\alpha\}_{\alpha\in A}$ an approximating directed family of E.

Assume that a function $f(x)$ is Jordan measurable on E. A point x_0 in E is defined to be a singular point of $f(x)$ if $f(x)$ is unbounded on every neighborhood of x_0.

Assume that the set S of singular points of $f(x)$ is of Jordan measure zero. Then $f(x)$ is integrable on every bounded measurable closed set included in $E\backslash S$. Then, if, for an approximating directed family $\{E_\alpha\}_{\alpha\in A}$ of $E\backslash S$, there exists the limit $I = \lim_\alpha I(E_\alpha)$ in the sense of Moore-Smith of directed family $\{I(E_\alpha)\}$ defined by the integrals

$$I(E_\alpha) = \int_{E_\alpha} f(x)dx,$$

we call I the extended Riemann integral of $f(x)$. We denote this as

$$I = \int_E f(x)dx.$$

If the integral I does not depend on the choice of an approximating directed family, we say that the integral is absolutely convergent. If the integral I depends on the choice of an approximating directed family, we say the integral is conditionally convergent.

Put

$$f^+(x) = \sup(f(x), 0),\ f^-(x) = -\inf(f(x), 0).$$

Then the situation of convergence and diverdence of an extended integral is as in the following Table 1.1.1.

Table 1.1.1. Convergence and divergence of an extended Riemann integral

conv.=convergent, abs.conv.=absolutely convergent
cond.conv.=conditionally convergent, div.=divergent

$\int_E f(x)dx$	$\int_E \lvert f(x)\rvert dx$	$\int_E f^+(x)dx$	$\int_E f^-(x)dx$
abs.conv.	conv.	conv.	conv.
div.	div.	conv.	div.
div.	div.	div.	conv.
cond.conv. or div.	div.	div.	div.

Example 1.1.1. We consider the integral $\int_{-1}^{1} \frac{1}{x} dx$. Since $\int_{-1}^{0} \frac{1}{x} dx = -\infty$ and $\int_{0}^{1} \frac{1}{x} dx = \infty$ hold, $\int_{-1}^{1} \frac{1}{x} dx$ is conditionally convergent or divergent. In fact, we have the following:

$$\text{v.p.} \int_{-1}^{1} \frac{1}{x} dx = \lim_{\epsilon \to +0} \left(\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \right)$$

$$= \lim_{\epsilon \to +0} ([\log |x|]_{-1}^{-\epsilon} + [\log |x|]_{\epsilon}^{1}) = \lim_{\epsilon \to +0} (\log \epsilon - \log \epsilon) = 0.$$

This is known as the Cauchy principal value(v.p.) of the integral.

Further, we have the following:

$$\text{v.p.}_1 \int_{-1}^{1} \frac{1}{x} dx = \lim_{\epsilon \to +0} \left(\int_{-1}^{-2\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \right)$$

$$= \lim_{\epsilon \to +0} ([\log |x|]_{-1}^{-2\epsilon} + [\log |x|]_{\epsilon}^{1}) = \lim_{\epsilon \to +0} (\log 2\epsilon - \log \epsilon) = \log 2.$$

This is a new principal value.

Example 1.1.2. We consider the integral $\int_{-\infty}^{\infty} \frac{1}{x} dx$. Since $\int_{-\infty}^{0} \frac{1}{x} dx = -\infty$ and $\int_{0}^{\infty} \frac{1}{x} dx = \infty$ hold, $\int_{-\infty}^{\infty} \frac{1}{x} dx$ is conditionally convergent or divergent. In fact, we have the following:

$$\int_{-\infty}^{\infty} \frac{1}{x} dx = \lim_{\epsilon \to +0, a \to +\infty} \left(\int_{-a}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{a} \frac{1}{x} dx \right)$$

$$= \lim_{\epsilon \to +0, a \to +\infty} ([\log |x|]_{-a}^{-\epsilon} + [\log |x|]_{\epsilon}^{a})$$

$$= \lim_{\epsilon \to +0, a \to +\infty} (\log \epsilon - \log a + \log a - \log \epsilon) = 0.$$

Further, we have the following:

for $p > 0$ and $q > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{x} dx = \lim_{\epsilon \to +0, a \to +\infty} (\int_{-pa}^{-q\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{a} \frac{1}{x} dx)$$

$$= \lim_{\epsilon \to +0, a \to +\infty} ([\log |x|]_{-pa}^{-q\epsilon} + [\log |x|]_{\epsilon}^{a})$$

$$= \lim_{\epsilon \to +0, a \to +\infty} (\log q\epsilon - \log pa + \log a - \log \epsilon) = \log \frac{q}{p}.$$

1.2. Extended Lebesgue integral. It is known that the function $\frac{\sin x}{x}$ is extended-Riemann-integrable on $[0, \infty)$ and we have

$$\int_0^{\infty} \frac{\sin x}{x} dx = \lim_{a \to \infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

But, $\frac{\sin x}{x}$ is not Lebesgue-integrable on $[0, \infty)$. Here we consider the extended Lebesgue integral. Then $\frac{\sin x}{x}$ is extended-Lebesgue-integrable.

Here let (R) be the family of Riemann-integrable functions, (ER) the family of extended-Riemann-integrable functions, (L) the family of Lebesgue-integrable functions and (EL) the family of extended-Lebesgue-integrable functions. Then we have the following inclusion relations:

$$(\mathrm{R}) \subset (\mathrm{ER}) \subset (\mathrm{EL}),$$

$$(\mathrm{R}) \subset (\mathrm{L}) \subset (\mathrm{EL}).$$

Here we remember the definition of extended Lebesgue integral, [4], [6].

Assume that $n \geq 1$ and a set E is measurable in $\mathbf{R}^n$. Let A be a directed set and $\{E_\alpha\}_{\alpha \in A}$ the family of bounded measurable subsets of E. Then, we say that $\{E_\alpha\}_{\alpha \in A}$ converges to E, if, for every bounded measurable subset H of E, there exists some element α_0 of A such that, for every $\alpha \in A$, $\alpha \geq \alpha_0$, $E_\alpha \supset H$ holds. Then we say that $\{E_\alpha\}_{\alpha \in A}$ is an approximating directed family of E.

Assume that a function $f(x)$ is measurable on E. We say that a point x_0 in E is a singular point of $f(x)$ if $f(x)$ is not integrable on every measurable neighborhood of x_0.

Assume that the set S of all singular points of $f(x)$ is of measure zero. Then $f(x)$ is integrable on every bounded measurable set included in $E \backslash S$. Then, if, for an approximating directed family $\{E_\alpha\}_{\alpha \in A}$ of $E \backslash S$, there exists the limit

$I = \lim_\alpha I(E_\alpha)$ in the sense of Moore-Smith of the directed family $\{I(E_\alpha)\}$ defined by the integral

$$I(E_\alpha) = \int_{E_\alpha} f(x)dx.$$

We say that I is the extended Lebesgue integral of $f(x)$. We denote this as

$$I = \int_E f(x)dx.$$

If the integral I does not depend on the choice of an approximating directed family, we say that the integral is absolutely convergent. If the integral I depends on the choice of an approximating directed family, we say that the integtal is conditionally convergent.

Put

$$f^+(x) = \sup(f(x), 0), \ f^-(x) = -\inf(f(x), 0).$$

Then the situation of convergence and divergence of an extended Lebesgue integral is as in the following Table 1.2.1.

Table 1.2.1. Convergence and divergence of an extended Lebesgue integral

conv.=convergent, abs.conv.=absolutely convergent
cond.conv.=conditionally convergent, div.=divergent

$\int_E f(x)dx$	$\int_E \lvert f(x)\rvert dx$	$\int_E f^+(x)dx$	$\int_E f^-(x)dx$
abs.conv.	conv.	conv.	conv.
div.	div.	conv.	div.
div.	div.	div.	conv.
cond.conv. or div.	div.	div.	div.

If we define the function $f(x)$ by the condition $f(x) = \dfrac{\sin x}{x}$ (when $x \in [0, \infty)$ is an irrational number) and $f(x) = 0$ (when $x \in [0, \infty)$ is a rational nember), $f(x)$ is discontinuous at every point. $f(x)$ is extended-Lebesgue-integrable, but it is not integrable in any sense of Lebesgue integral and in the extended or narrow sense of Riemann integral.

2. Distributions

2.1. Definition of distributions. Let Ω be an open set in $\mathbf{R}^n$. Let $C_0^\infty(\Omega)$ be the set of all complex-valued C^∞ functions on Ω with compact support. This is a vector space over $\mathbf{C}$. Let K be an arbitrary compact set in Ω. Put

$$\mathcal{D}_K(\Omega) = \{f \in C_0^\infty(\Omega); \mathrm{supp}(f) \subset K\}.$$

We define the seminorms $p_{K,m}$ on $\mathcal{D}_K(\Omega)$ by the relation

$$p_{K,m}(f) = \sup_{|s|\leq m, x\in K} |D^s f(x)|,\ 0 \leq m < \infty.$$

Here we put

$$D^s f(x) = \frac{\partial^{|s|}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}} f(x_1, x_2, \cdots, x_n),$$
$$s = (s_1, \cdots, s_n),\ s_j \geq 0, (j = 1, 2, \cdots, n),$$
$$|s| = \Sigma_{j=1}^n s_j.$$

Then, being endowed with the topology defined by the family of seminorms $\{p_{K,m}; K \subset\subset \Omega, m = 0, 1, 2, \cdots\}$, $\mathcal{D}_K(\Omega)$ becomes a locally convex space. Let K_1 and K_2 be two compact sets in Ω with $K_1 \subset K_2$. Then $\mathcal{D}_{K_1}(\Omega) \subset \mathcal{D}_{K_2}(\Omega)$ holds and the topology of $\mathcal{D}_{K_1}(\Omega)$ coincides with the relative topology induced from that of $\mathcal{D}_{K_2}(\Omega)$. Thus we define $\mathcal{D}(\Omega)$ as the strict inductive limit

$$\mathcal{D}(\Omega) = \lim \mathrm{ind}_{K\subset\Omega} \mathcal{D}_K(\Omega) = \cup_{K\subset\Omega} \mathcal{D}_K(\Omega).$$

Then $\mathcal{D}(\Omega)$ is a locally convex space.

Then we say a continuous linear functional on $\mathcal{D}(\Omega)$ to be a distribution. We denote the set of all distributions on Ω by $\mathcal{D}'(\Omega)$. Then this is a locally convex space. For $T \in \mathcal{D}'(\Omega)$, we denote the value of the distribution T at a test function $\varphi \in \mathcal{D}(\Omega)$ by $T(\varphi) =< T, \varphi >$.

Let $f(x)$ be a complex-valued function on Ω. Then we say $f(x)$ to be a locally integrable function on Ω if, for every compact set K in Ω, it satisfies the condition

$$\int_K |f(x)| dx < \infty.$$

Example 2.1.1. Let $f(x)$ be a locally integrable function on Ω. We define a linear functional T_f on $\mathcal{D}(\Omega)$ by the relation

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x) dx,\ (\varphi \in \mathcal{D}(\Omega)).$$

Then T_f becomes a distribution on Ω.

Example 2.1.2. The function $\dfrac{1}{x}$ is locally intagrable on $\Omega = \{x \in \mathbf{R}; x \neq 0\}$. Put

$$f(x) = \begin{cases} \dfrac{1}{x}, & (x \neq 0), \\ a, & (x = 0). \end{cases}$$

Here a is an arbitrary constant. Then $f(x)$ is an extension of $\dfrac{1}{x}$ over the space $\mathbf{R}$. But $f(x)$ is not locally integrable on $\mathbf{R}$.

$T_{1/x}$ is a distribution on Ω. But we cannot define any distribution on $\mathbf{R}$ such as T_f.

Example 2.1.3. Let m be a completely additive complex-valued measure on Ω. Then we define a linear functional T_m on $\mathcal{D}(\Omega)$ by the relation

$$T_m(\varphi) = \int_\Omega \varphi(x)m(dx), \ (\varphi \in \mathcal{D}(\Omega)).$$

Then T_m is a distribution on Ω.

For example, for a fixed point $a \in \mathbf{R}$, we define T_{δ_a} on $\mathbf{R}$ by the relation

$$T_{\delta_a}(\varphi) = \varphi(a), \ (\varphi \in \mathcal{D}(\Omega)).$$

We denote this by δ_a simply. This is a Dirac measure concentrated at $a \in \mathbf{R}$. For $a = 0 \in \mathbf{R}$, we denote $\delta_0 = \delta$.

2.2. Operations on distributions.

(A) Multiplication by a function. Let Ω be an open set in $\mathbf{R}^n$. For $\alpha \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, we define the product $\alpha T \in \mathcal{D}'(\Omega)$ by the relation

$$(\alpha T)(\varphi) = T(\alpha\varphi), \ (\varphi \in \mathcal{D}(\Omega)).$$

Multiplication by a function αT should be considered as a conjugate operator of an operator of multiplication by the function α on $\mathcal{D}(\Omega)$. The operator of multiplication by a function α should be considered as a differential operator of order 0 mentioned in the next clause.

(B) Differentiation. Let Ω be an open set in $\mathbf{R}^n$. For $T \in \mathcal{D}'(\Omega)$, we define a partial derivative $\partial T/\partial x_j$, $(1 \leq j \leq n)$ by the relation

$$\frac{\partial T}{\partial x_j}(\varphi) = -T(\frac{\partial \varphi}{\partial x_j}), \ (\varphi \in \mathcal{D}(\Omega)).$$

Let $p = (p_1, \cdots, p_n)$ be an n-tuple of nonnegarive integers. Then we define a partial differential operator ∂^p by the relation

$$\partial^p = \partial^{|p|}/\partial x_1^{p_1} \cdots \partial x_n^{p_n},$$

$$|p| = p_1 + \cdots + p_n.$$

Then we have the relation

$$(\partial^p T)(\varphi) = (-1)^{|p|}T(\partial^p\varphi), \ (\varphi \in \mathcal{D}(\Omega)).$$

Thus, for a general differential operator with constant or function coefficients

$$P(\partial/\partial x) = \Sigma_{|p| \leq m} a_p (\partial/\partial x)^p, \ a_p \in \mathbf{C}$$

or

$$P(x, \partial/\partial x) = \Sigma_{|p| \leq m} a_p(x) (\partial/\partial x)^p, \ a_p(x) \in C^\infty(\Omega),$$

we have the relations, for $P(\partial/\partial x)T$ or $P(x, \partial/\partial x)T$,

$$P(\partial/\partial x)T(\varphi) = T(P(-\partial/\partial x)\varphi), \ (\varphi \in \mathcal{D}(\Omega)),$$

or

$$P(x, \partial/\partial x)T(\varphi) = T({}^tP(x, \partial/\partial x)\varphi), \ (\varphi \in \mathcal{D}(\Omega)),$$

Here ${}^tP(x, \partial/\partial x)$ is the formally conjugate operator of $P(x, \partial/\partial x)$, and we define it by the relation

$${}^tP(x, \partial/\partial x)\varphi = \Sigma_{|p|\le m}(-1)^{|p|}(\partial/\partial x)^p(a_p(x)\varphi)$$

for $\varphi \in \mathcal{D}(\Omega)$.

3. Restriction and extensions of distributions

Let Ω_1 and Ω_2 be two open sets in $\mathbf{R}^n$ with $\Omega_1 \subset \Omega_2$. Then, $\mathcal{D}(\Omega_1)$ is a subspace of $\mathcal{D}(\Omega_2)$. Let $T \in \mathcal{D}'(\Omega_2)$. The restriction $T|_{\Omega_1} = S$ of T to Ω_1 is defined by the relation

$$S(\varphi) = T(\varphi), \ (\varphi \in \mathcal{D}(\Omega_1)).$$

Then $S \in \mathcal{D}'(\Omega_1)$.

If, for $S \in \mathcal{D}'(\Omega_1)$, there exists a $T \in \mathcal{D}'(\Omega_2)$ such that $T|_{\Omega_1} = S$ holds, then we call T an extension of S to Ω_2.

Put $\Omega = \mathbf{R}\backslash\{0\}$. Then Ω is an open set in $\mathbf{R}$. Then we can define a distribution $T_{1/x} \in \mathcal{D}'(\Omega)$ by using a locally integrable function $\frac{1}{x}$ on Ω. But, even if we extend $\frac{1}{x}$ to a function $f(x)$ on $\mathbf{R}$ in any way, $f(x)$ does not become a locally integrable function on $\mathbf{R}$. Therefore we cannot define a distribution T on $\mathbf{R}$ such as $T = T_f$. Then, if we use the Cauchy principal value of divergent integral, we can define the distribution v.p.$\frac{1}{x}$ on $\mathbf{R}$ by using the relation, for every $\varphi \in \mathcal{D}(\mathbf{R})$,

$$< \text{v.p.}\frac{1}{x}, \varphi >= \lim_{\epsilon \to +0} \int_{|x|\ge\epsilon} \frac{\varphi(x)}{x} dx = \text{v.p.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx.$$

We call this the Cauchy principal value. Then we have

$$\text{v.p.}\frac{1}{x}|_{\Omega} = T_{1/x} = \frac{1}{x}.$$

Namely, the Cauchy principal value v.p.$\frac{1}{x}$ is a distributional extension of a locally integrable function $\frac{1}{x}$ on Ω onto $\mathbf{R}$. In this sense, we call v.p.$\frac{1}{x}$ a renormalization of a singular function $\frac{1}{x}$.

Here we remember the notion of Cauchy principal value of a divergent integral. For $\varphi \in \mathcal{D}(\mathbf{R})$, we have the following:

$$\text{v.p.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx = \lim_{\epsilon\to+0}\int_{|x|\geq\epsilon}\frac{\varphi(x)}{x}dx$$

$$= \lim_{\epsilon\to+0}\{\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}dx + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx\}$$

$$= \lim_{\epsilon\to+0}\{-\int_{\epsilon}^{\infty}\frac{\varphi(-x)}{x}dx + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx\}$$

$$= \lim_{\epsilon\to+0}\int_{\epsilon}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}dx = \int_{0}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}dx.$$

This is the Cauchy principal value of a divergent integral and a regularization of a divergent integral in the sense of Gelfand and Shilov[1].

The divergent integral in this case is the case of conditional convergence of the extended Riemann integral or the extended Lebesgue integral. Therefore this converges to a different value according to an approximating sequence of convergence.

For example, we have

$$\text{v.p.}_1\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx = \lim_{\epsilon\to+0}[\int_{-\infty}^{-2\epsilon}\frac{\varphi(x)}{x}dx + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx]$$

$$= \lim_{\epsilon\to+0}[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}dx + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx] - \lim_{\epsilon\to+0}\int_{-2\epsilon}^{-\epsilon}\frac{\varphi(x)}{x}dx$$

$$=< \text{v.p.}\frac{1}{x},\varphi > - \lim_{\epsilon\to+0}\int_{-2\epsilon}^{-\epsilon}\frac{\varphi(x)}{x}dx$$

$$=< \text{v.p.}\frac{1}{x},\varphi > - \lim_{\epsilon\to+0}[\varphi(x)\log|x|]_{-2\epsilon}^{-\epsilon} + \lim_{\epsilon\to+0}\int_{-2\epsilon}^{-\epsilon}\varphi'(x)\log|x|dx.$$

Here we have the following:

$$\lim_{\epsilon\to+0}[\varphi(x)\log|x|]_{-2\epsilon}^{-\epsilon} = \lim_{\epsilon\to+0}(\varphi(-\epsilon)\log\epsilon - \varphi(-2\epsilon)\log 2\epsilon)$$

$$= \lim_{\epsilon\to+0}(-\varphi(-2\epsilon)\log 2 + (\varphi(-\epsilon)-\varphi(-2\epsilon))\log\epsilon) = -\varphi(0)\log 2.$$

Further we have the following:

$$|\int_{-2\epsilon}^{-\epsilon}\varphi'(x)\log|x|dx| \leq C\int_{-2\epsilon}^{-\epsilon}|\log|x||dx$$

$$= C\int_{\epsilon}^{2\epsilon}|\log x|dx = C[x - x\log x]_{\epsilon}^{2\epsilon}$$

$$= C(\epsilon - \epsilon \log \epsilon - 2\epsilon \log 2) \longrightarrow 0, \ (\epsilon \longrightarrow +0).$$

Therefore, we have

$$\lim_{\epsilon \to +0} \int_{-2\epsilon}^{-\epsilon} \varphi'(x) \log |x| dx = 0.$$

Therefore, we have

$$< \text{v.p.}_1 \frac{1}{x}, \varphi > = < \text{v.p.} \frac{1}{x}, \varphi > + \log 2\varphi(0) = < \text{v.p.} \frac{1}{x}, \varphi > + < (\log 2)\delta, \varphi > .$$

Therefore we have $\text{v.p.}_1 \frac{1}{x} \in \mathcal{D}'(\mathbf{R})$ and

$$\text{v.p.}_1 \frac{1}{x} = \text{v.p.} \frac{1}{x} + (\log 2)\delta(x).$$

Thus we have

$$\text{v.p.}_1 \frac{1}{x} |_\Omega = T_{1/x} = \frac{1}{x}.$$

$\text{v.p.}_1 \frac{1}{x}$ is a distributional extension of a locally integrable function $\frac{1}{x}$ on Ω onto $\mathbf{R}$. In this sense, $\text{v.p.}_1 \frac{1}{x}$ is also a renormalization of the singular function $\frac{1}{x}$. Thus we cannot extend the singular function $\frac{1}{x}$ onto $\mathbf{R}$ as a locally intagrable function. But we can extend this onto $\mathbf{R}$ as a distribution, and what is more such an extension is not unique. This is a general property of distributional extensions. We call a distributional extension of a locally integrable singular function a renormalization. We wish to study several examples of these phenomena and their aspects.

As for the extendability of distributions, there is Komatsu's work[8]. Here we remember its outline.

Let Ω be an open set in $\mathbf{R}^n$ and $\tilde{\Omega}$ be an open set including Ω.

We consider what is the condition for a distribution on Ω to be extendable to a distribution on $\tilde{\Omega}$. Especially, as a sufficient condition that a locally integrable function $f(x)$ defined on one side of a hypersurface can be extended beyond the hypersurface as a distribution, we may consider the growth condition of the absolute value of $f(x)$ as x approaches to the hypersurface. Let S be the boundary of Ω in $\tilde{\Omega}$. Assume that S has a certain dgree of smoothness.

Proposition 3.1(Komatsu[8], p.90). *We use the notation above. If $T \in \mathcal{D}'(\Omega)$ can be extended to $\tilde{T} \in \mathcal{D}'(\tilde{\Omega})$, T has the extension $\tilde{T} \in \mathcal{D}'(\tilde{\Omega})$ which vanishes on $\tilde{\Omega} \backslash \bar{\Omega}$.*

By the proposition above we assert the existence of such a special extension $\tilde{T}$. But we say nothing about the existence of another extensions.

Proposition 3.2(Komatsu[8], p.90). *We use the notation above. Then $T \in \mathcal{D}'(\Omega)$ can be extended to $\tilde{T} \in \mathcal{D}'(\tilde{\Omega})$ if and only if T is extendable in a certain neighborhood of each x of the boundary S of Ω.*

Namely, the extendability of T depends only on the behavior in a certain neighborhood of S, but it does not depend on the choice of $\tilde{\Omega}$.

Theorem 3.3(Komatsu[8], p.91). *$T \in \mathcal{D}'(\Omega)$ can be extended to $\tilde{T} \in \mathcal{D}'(\tilde{\Omega})$ if and only if, for every point x of the boundary S of Ω, there exists its compact neighborhood K in $\tilde{\Omega}$ such that we have the following :*

There exist some constants m and C, and we have the following estimate:

$$|<T,\varphi>| \leq C \sup_{|\alpha|\leq m,x} |D^\alpha \varphi(x)|,$$

$$(\varphi \in \mathcal{D}(\Omega),\ \mathrm{supp}(\varphi) \subset K).$$

Proposition 3.4(Komatsu[8], p.91). *Let $f(x)$ be a locally integrable function defined on $\mathbf{R}_+ = \{x \in \mathbf{R}; x > 0\}$. Then, if for any $n \in \mathbf{N}$, there exist $c > 0$ and $\delta > 0$ such that we have the estimate*

$$f(x) \geq cx^{-n},\ (0 < x < \delta),$$

T_f cannot be extended as a distribution beyond the origin.

In general, we have the following.

Theorem 3.5(Komatsu[8], p.92). *Let Ω' be an open set in $\mathbb{R}^{n-1}$ and S a hypersurface defined by the relation $x = v(x')$, using a continuous function $v(x')$ on Ω'. Put $\Omega = \{x = (x', x_n); x' \in \Omega',\ x_n > v(x')\}$. Let $f(x)$ be a locally integrable function on Ω. Then, T_f can be extended as a distribution beyond S if the following conditions are satisfied:*

For any compact set $K' \subset \Omega'$, there exist some constants L and C such that we have the estimate

$$\sup_{x'\in K'} |f(x', x_n)| \leq C(x_n - v(x'))^{-L}.$$

Example 3.6. Put $\Omega = \{x \in \mathbf{R}; x > 0\}$ and $\tilde{\Omega} = \mathbf{R}$. Put $f(x) = \dfrac{1}{x}$, $(x > 0)$. Then $f(x)$ is locally integrable on Ω. But, even if we extend $f(x)$ to a measurable function on $\tilde{\Omega}$ in any way, we cannot extend $f(x)$ to a locally integrable function on $\tilde{\Omega}$. But, if, as a renormalization of $\tilde{f} = \dfrac{1}{x}$, $x \neq 0$, we define $\mathrm{v.p.}\dfrac{1}{x} \in \mathcal{D}'(\tilde{\Omega})$, then we have $\mathrm{v.p.}\dfrac{1}{x}|_\Omega = \dfrac{1}{x}$, $(x > 0)$. Namely, $f(x) = \dfrac{1}{x}$, $(x > 0)$ can be extended to a distribution on $\tilde{\Omega}$.

Example 3.7. Put $f(x) = x^{-m}$, $(x > 0)$ for $m \geq 2$. Then $f(x)$ cannot be extended to a locally integrable function on $\mathbf{R}$. But $T_f \in \mathcal{D}'(x > 0)$ can be extended to $\mathrm{Pf}.x^{-m} \in \mathcal{D}'(\mathbf{R})$. namely, $\mathrm{Pf}.x^{-m}$ is a renormalization of x^{-m}, $(x > 0)$.

Example 3.8. Put $f(x) = 1$, $(x > 0)$. Then $f(x)$ is a locally integrable function on $x > 0$. Using Heaviside function $H(x)$:

$$H(x) = \begin{cases} 1, & (x > 0), \\ 0, & (x \leq 0), \end{cases}$$

we extend $f(x)$ to the function $\tilde{f}(x) = H(x)$ on $\mathbf{R}$. Then $T_{\tilde{f}} \in \mathcal{D}'(\mathbf{R})$ is an extension of $T_f \in \mathcal{D}'((0,\infty))$ and we have $\mathrm{supp}(T_{\tilde{f}}) \subset [0,\infty)$ and $T_{\tilde{f}}|_{(0,\infty)} = T_f = 1$, $(x > 0)$.

Example 3.9. Let Ω and $\tilde{\Omega}$ be as in Example 3.6. Assume $f(x) \in L^1_{\mathrm{loc}}(\Omega)$ satifies the condition: For some integer $p \geq 1$ and constants $c > 0$ and $\delta > 0$, $f(x) \sim cx^{-p}$, $(0 < x < \delta)$. Then $f(x)$ cannot be extended to any $\tilde{f}(x) \in L^1_{\mathrm{loc}}(\tilde{\Omega})$. But, using Cauchy's principal value v.p.f or Hadamard's finite parts Pf.f, $(p \geq 2)$, $f(x)$ can be extended as a distribution.

4. Convergence of extended integrals and renormalizations

Here the integral of a function can be considered as the Riemann integral or the Lebesgue integral. We may consider either one of them according to a given function.

Since we have

$$\int_{-1}^{0} \frac{1}{x} dx = -\infty, \quad \int_{0}^{1} \frac{1}{x} dx = \infty,$$

the integral $\int_{-1}^{1} \frac{1}{x} dx$ converges conditionally or diverges according as a choice of an approximating directed family. Here we consider the case of conditional convergence. Until now, we are not especially interested in such a kind of integral because we have considered it as a divergent integral. We found out it is the case of conditional convergence and it has different values according to the choices of approximating families. The well-known Cauchy's principal value is its special case. Namely, we have the following.

Example 4.1 We have:

$$\begin{aligned} \text{v.p.} \int_{-1}^{1} \frac{1}{x} dx &= \lim_{\epsilon \to +0} \int_{\epsilon \leq |x| \leq 1} \frac{1}{x} dx \\ &= \lim_{\epsilon \to +0} \{ \int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \} \\ &= \lim_{\epsilon \to +0} \{ [\log |x|]_{-1}^{-\epsilon} + [\log |x|]_{\epsilon}^{1} \} \\ &= \lim_{\epsilon \to +0} (\log \epsilon - \log \epsilon) = 0. \end{aligned}$$

Now, exchanging the choice of an approximating family, we have thr following.

Example 4.2 We have:

$$\begin{aligned}
\text{v.p.}_1 \int_{-1}^{1} \frac{1}{x} dx &= \lim_{\epsilon \to +0} \Big(\int_{-1}^{-2\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \Big) \\
&= \lim_{\epsilon \to +0} ([\log |x|]_{-1}^{-2\epsilon} + [\log |x|]_{\epsilon}^{1}) \\
&= \lim_{\epsilon \to +0} (\log 2\epsilon - \log \epsilon) = \log 2.
\end{aligned}$$

Example 4.3. Let $a > 0$ and $b > 0$. Then we have:

$$\begin{aligned}
\text{v.p.}_{a,b} \int_{-1}^{1} \frac{1}{x} dx &= \lim_{\epsilon \to +0} \Big(\int_{-1}^{b\epsilon} \frac{1}{x} dx + \int_{a\epsilon}^{1} \frac{1}{x} dx \Big) \\
&= \lim_{\epsilon \to +0} ([\log |x|]_{-1}^{-b\epsilon} + [\log |x|]_{a\epsilon}^{1}) \\
&= \lim_{\epsilon \to +0} (\log b\epsilon - \log a\epsilon) = \log \frac{b}{a}.
\end{aligned}$$

Example 4.4. Let $a > 0,\ b > 0,\ p > 0$ and $q > 0$. Then we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x} dx &= \lim_{\epsilon \to +0, t \to +\infty} \Big(\int_{-pt}^{-b\epsilon} \frac{1}{x} dx + \int_{a\epsilon}^{qt} \frac{1}{x} dx \Big) \\
&= \lim_{\epsilon \to +0, t \to +\infty} ([\log |x|]_{-pt}^{-b\epsilon} + [\log |x|]_{a\epsilon}^{qt}) \\
&= \lim_{\epsilon \to +0, t \to +\infty} (\log b\epsilon - \log pt + \log qt - \log a\epsilon) \\
&= \log \frac{q}{p} + \log \frac{b}{a} = \log \frac{bq}{ap}.
\end{aligned}$$

Therefore, the integrals

$$\int_{-1}^{1} \frac{1}{x} dx, \ \int_{-\infty}^{\infty} \frac{1}{x} dx$$

are the indefinite form such as $\infty - \infty$. But, by subtracting a divergent quantity to $\pm\infty$ from them, we can have some finite quantity of them. It is considered as such an idea that S. Tomonaga and others used in the procedure of renormalization in the quantum electrodynamics[7]. Further the integrals

$$\int_{0}^{1} \frac{1}{x} dx, \ \int_{0}^{\infty} \frac{1}{x} dx$$

are infinite quantities. Then adding to them, respectively, minus infinite quantities

$$\int_{-1}^{0} \frac{1}{x} dx, \ \int_{-\infty}^{0} \frac{1}{x} dx$$

and doing the procedure of renormalizations as above, we complete exactly the scheme $\infty - \infty =$ finite quantity. Using such a method of renormalization of divergent integrals, we study, in section 3, that we can define the renormalizations v.p.$\frac{1}{x}$ or v.p.$_1\frac{1}{x}$ as distributions of a locally integrable function $\frac{1}{x}$, $(x \neq 0)$. Further, the renormalizations v.p.$\frac{1}{x}$ or v.p.$_1\frac{1}{x}$ of a locally integrable function $\frac{1}{x}$, $(x > 0)$ can be understood by the idea of renormalization as above.

Example 4.5 We have:

$$\int_{-1}^{1} \frac{\mathrm{sgn}(x)}{x^2} dx = \lim_{\epsilon \to +0} (\int_{-1}^{-\epsilon} \frac{-1}{x^2} dx + \int_{\epsilon}^{1} \frac{1}{x^2} dx)$$

$$= \lim_{\epsilon \to +0} ([\frac{1}{x}]_{-1}^{-\epsilon} + [-\frac{1}{x}]_{\epsilon}^{1}) = \lim_{\epsilon \to +0} (-\frac{1}{\epsilon} + 1 - 1 + \frac{1}{\epsilon}) = 0.$$

Example 4.6 We have:
For a real number a,

$$\int_{-1}^{1} \frac{\mathrm{sgn}(x)}{x^2} dx = \lim_{\epsilon \to +0} (\int_{-1}^{-\epsilon} \frac{-1}{x^2} dx + \int_{\epsilon/(1+a\epsilon)}^{1} \frac{1}{x^2} dx)$$

$$= \lim_{\epsilon \to +0} ([\frac{1}{x}]_{-1}^{-\epsilon} + [-\frac{1}{x}]_{\epsilon/(1+a\epsilon)}^{1}) = \lim_{\epsilon \to +0} (-\frac{1}{\epsilon} + 1 - 1 + \frac{1+a\epsilon}{\epsilon}) = a.$$

Example 4.7 We have:

$$\int_{-1}^{1} \frac{1}{x^3} dx = \lim_{\epsilon \to +0} (\int_{-1}^{-\epsilon} \frac{1}{x^3} dx + \int_{\epsilon}^{1} \frac{1}{x^3} dx)$$

$$= \lim_{\epsilon \to +0} ([-\frac{1}{2x^2}]_{-1}^{-\epsilon} + [-\frac{1}{2x^2}]_{\epsilon}^{1}) = \lim_{\epsilon \to +0} (-\frac{1}{2\epsilon^2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2\epsilon^2}) = 0.$$

Example 4.8 We have:
For a real number a,

$$\int_{-1}^{1} \frac{1}{x^3} dx = \lim_{\epsilon \to +0} (\int_{-1}^{-\epsilon} \frac{1}{x^3} dx + \int_{\epsilon/(1+2a\epsilon^2)^{1/2}}^{1} \frac{1}{x^3} dx)$$

$$= \lim_{\epsilon \to +0} ([-\frac{1}{2x^2}]_{-1}^{-\epsilon} + [-\frac{1}{2x^2}]_{\epsilon/(1+2a\epsilon^2)^{1/2}}^{1})$$

$$= \lim_{\epsilon \to +0} (-\frac{1}{2\epsilon^2} + \frac{1}{2} - \frac{1}{2} + \frac{1+2a\epsilon^2}{2\epsilon^2}) = a.$$

Example 4.9 We have:
For $n \geq 1$,

$$\int_{-1}^{1} \frac{1}{x^{2n+1}} dx = \lim_{\epsilon \to +0} (\int_{-1}^{-\epsilon} \frac{1}{x^{2n+1}} dx + \int_{\epsilon}^{1} \frac{1}{x^{2n+1}} dx)$$

$$= \lim_{\epsilon\to+0}([-\frac{1}{2nx^{2n}}]_{-1}^{-\epsilon} + [-\frac{1}{2nx^{2n}}]_{\epsilon}^{1})$$

$$= \lim_{\epsilon\to+0}(-\frac{1}{2n\epsilon^{2n}} + \frac{1}{2n} - \frac{1}{2n} + \frac{1}{2n\epsilon^{2n}}) = 0.$$

Example 4.10 We have:
For $n \geq 1$ and a real number a,

$$\int_{-1}^{1}\frac{1}{x^{2n+1}}dx = \lim_{\epsilon\to+0}(\int_{-1}^{-\epsilon}\frac{1}{x^{2n+1}}dx + \int_{\epsilon/(1+2na\epsilon^{2n})^{1/2n}}^{1}\frac{1}{x^{2n+1}}dx)$$

$$= \lim_{\epsilon\to+0}([-\frac{1}{2nx^{2n}}]_{-1}^{-\epsilon} + [-\frac{1}{2nx^{2n}}]_{\epsilon/(1+2na\epsilon^{2n})^{1/2n}}^{1})$$

$$= \lim_{\epsilon\to+0}(-\frac{1}{2n\epsilon^{2n}} + \frac{1}{2n} - \frac{1}{2n} + \frac{1+2na\epsilon^{2n}}{2n\epsilon^{2n}}) = a.$$

Example 4.11 We have:
For $n \geq 1$,

$$\int_{-1}^{1}\frac{\mathrm{sgn}(x)}{x^{2n}}dx = \lim_{\epsilon\to+0}(\int_{-1}^{-\epsilon}\frac{-1}{x^{2n}}dx + \int_{\epsilon}^{1}\frac{1}{x^{2n}}dx)$$

$$= \lim_{\epsilon\to+0}([\frac{1}{(2n-1)x^{2n-1}}]_{-1}^{-\epsilon} + [\frac{-1}{(2n-1)x^{2n-1}}]_{\epsilon}^{1})$$

$$= \lim_{\epsilon\to+0}(\frac{-1}{(2n-1)\epsilon^{2n-1}} + \frac{1}{2n-1} - \frac{1}{2n-1} + \frac{1}{(2n-1)\epsilon^{2n-1}}) = 0.$$

Example 4.12 We have:
For $n \geq 1$ and a real number a,

$$\int_{-1}^{1}\frac{\mathrm{sgn}(x)}{x^{2n}}dx = \lim_{\epsilon\to+0}(\int_{-1}^{-\epsilon}\frac{-1}{x^{2n}}dx + \int_{\epsilon/(1+(2n-1)a\epsilon^{2n-1})^{1/(2n-1)}}^{1}\frac{1}{x^{2n}}dx)$$

$$= \lim_{\epsilon\to+0}([\frac{1}{(2n-1)x^{2n-1}}]_{-1}^{-\epsilon} + [\frac{-1}{(2n-1)x^{2n-1}}]_{\epsilon/(1+(2n-1)a\epsilon^{2n-1})^{1/(2n-1)}}^{1})$$

$$= \lim_{\epsilon\to+0}(\frac{-1}{(2n-1)\epsilon^{2n-1}} + \frac{1}{2n-1} - \frac{1}{2n-1} + \frac{1+(2n-1)a\epsilon^{2n-1}}{(2n-1)\epsilon^{2n-1}}) = a.$$

Example 4.13 We have:
For $n \geq 1$,

$$\int_{-\infty}^{\infty}\frac{1}{x^{2n+1}}dx = \lim_{\epsilon\to+0,p,q\to+\infty}(\int_{-p}^{-\epsilon}\frac{1}{x^{2n+1}}dx + \int_{\epsilon}^{q}\frac{1}{x^{2n+1}}dx)$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}([-\frac{1}{2nx^{2n}}]_{-p}^{-\epsilon} + [-\frac{1}{2nx^{2n}}]_{\epsilon}^{q})$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}(-\frac{1}{2n\epsilon^{2n}}+\frac{1}{2np^{2n}}-\frac{1}{2nq^{2n}}+\frac{1}{2n\epsilon^{2n}})=0.$$

Example 4.14 We have:
For $n \geq 1$ and a real number a,

$$\int_{-\infty}^{\infty}\frac{1}{x^{2n+1}}dx = \lim_{\epsilon\to+0,p,q\to+\infty}(\int_{-p}^{-\epsilon}\frac{1}{x^{2n+1}}dx+\int_{\epsilon/(1+2na\epsilon^{2n})^{1/2n}}^{q}\frac{1}{x^{2n+1}}dx)$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}([-\frac{1}{2nx^{2n}}]_{-p}^{-\epsilon}+[-\frac{1}{2nx^{2n}}]_{\epsilon/(1+2na\epsilon^{2n})^{1/2n}}^{q})$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}(-\frac{1}{2n\epsilon^{2n}}+\frac{1}{2np^{2n}}-\frac{1}{2nq^{2n}}+\frac{1+2na\epsilon^{2n}}{2n\epsilon^{2n}})=a.$$

Example 4.15 We have:
For $n \geq 1$,

$$\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(x)}{x^{2n}}dx = \lim_{\epsilon\to+0,p,q\to+\infty}(\int_{-p}^{-\epsilon}\frac{-1}{x^{2n}}dx+\int_{\epsilon}^{q}\frac{1}{x^{2n}}dx)$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}([\frac{1}{(2n-1)x^{2n-1}}]_{-p}^{-\epsilon}+[\frac{-1}{(2n-1)x^{2n-1}}]_{\epsilon}^{q})$$

$$= \lim_{\epsilon\to+0,p,q\to\infty}(\frac{-1}{(2n-1)\epsilon^{2n-1}}+\frac{1}{(2n-1)p^{2n-1}}$$

$$-\frac{1}{(2n-1)q^{2n-1}}+\frac{1}{(2n-1)\epsilon^{2n-1}})=0.$$

Example 4.16 We have:
For $n \geq 1$ and a real number a,

$$\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(x)}{x^{2n}}dx = \lim_{\epsilon\to+0,p,q\to+\infty}(\int_{-p}^{-\epsilon}\frac{-1}{x^{2n}}dx+\int_{\epsilon/(1+(2n-1)a\epsilon^{2n-1})^{1/(2n-1)}}^{q}\frac{1}{x^{2n}}dx)$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}([\frac{1}{(2n-1)x^{2n-1}}]_{-p}^{-\epsilon}+[\frac{-1}{(2n-1)x^{2n-1}}]_{\epsilon/(1+(2n-1)a\epsilon^{2n-1})^{1/(2n-1)}}^{q})$$

$$= \lim_{\epsilon\to+0,p,q\to+\infty}(\frac{-1}{(2n-1)\epsilon^{2n-1}}+\frac{1}{(2n-1)p^{2n-1}}$$

$$-\frac{1}{(2n-1)q^{2n-1}}+\frac{1+(2n-1)a\epsilon^{2n-1}}{(2n-1)\epsilon^{2n-1}})=a.$$

5. Reguralizations and renormalizations of divergent integrals

Assume that a function $f(x)$ is defined on $\mathbf{R}$ and it is locally integrable everywhere except a point x_0 and it has the nonsummable singularity at x_0. Then we cannot extend the function $f(x)$ to $\mathbf{R}$ as a locally integrable function. But we can extend the distribution T_f defined by $f(x)$ to $\mathbf{R}$ as a distribution $\tilde{T}_f$. We say $\tilde{T}_f$ to be a renormalization of f or T_f. We can consider several methods of such renormalizations. Here we mention the method of regularization of divergent integrals by Gelfand-Shilov[1].

Here we put $\mathcal{D} = \mathcal{D}(\mathbf{R})$. Then, for an arbitrary $\varphi(x) \in \mathcal{D}$, the integral

$$\int f(x)\varphi(x)dx,$$

in general, diverges. But, if $\varphi(x)$ vanishes in a neighborhood of x_0, this integral converges. Using this fact, we can define $\tilde{T}_f \in \mathcal{D}'$ suitably so that, for every $\varphi \in \mathcal{D}$ vanishing in a neighborhood of x_0, we have

$$< \tilde{T}_f, \varphi >= \int f(x)\varphi(x)dx.$$

Gelfand-Shilov say such a $\tilde{T}_f \in \mathcal{D}'$ to be a regularization of the divergent integral or a regularization of $f(x)$. This is an extension to $\mathbf{R}$ of the distribution $T_f \in \mathcal{D}'(\Omega)$ on $\Omega = \mathbf{R}\backslash\{x_0\}$ defined by $f(x)$. In this sense, the method of regularization of $f(x)$ is one method of renormalizaton of $f(x)$ in our definition.

Example 5.1. Let a and b be two arbitrary positive numbers and put $f(x) = \dfrac{1}{x}$, $(x \neq 0)$. Then if we put

$$< \tilde{T}_f, \varphi >= \int_{-\infty}^{-a} \frac{\varphi(x)}{x} dx + \int_{-a}^{b} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{b}^{\infty} \frac{\varphi(x)}{x} dx, \ (\varphi \in \mathcal{D}),$$

$\tilde{T}_f \in \mathcal{D}'$ is a regularization of $f(x) = \dfrac{1}{x}$, $(x \neq 0)$ or a renormalization of $f(x)$. This defines many kinds of distributions according to the choices of a and b. Namely there are an infinite number of regularizations or renormalizations of $f(x) = \dfrac{1}{x}$, $(x \neq 0)$.

Now we show the existence of regularizations. For simplicity we put $x_0 = 0$.

Proposition 5.2(Gelfand-Shilov[1], p.11). *If, for a function $f(x)$ on $\mathbf{R}^n$, there exists some integer $m > 0$ such that $f(x)r^m$ is locally integrable, there exists a regularization of $f(x)$. Here we put $r = |x| = \sqrt{x_1^2 + \cdots x_n^2}$.*

In this case a regularization $\tilde{T}_f$ of $f(x)$ can be defined by the following relation:

$$< \tilde{T}_f, \varphi >= \int f(x)\{\varphi(x) - [\varphi(0) + \frac{\partial \varphi(0)}{\partial x_1} x_1 +$$

$$\cdots + \frac{\partial^m \varphi(0)}{\partial x_n^m}\frac{x_n^m}{m!}]\theta(1-r)\}dx, (\varphi \in \mathcal{D}).$$

Here we put

$$\theta(1-r) = \begin{cases} 1, & (r \leq 1), \\ 0, & (r > 1). \end{cases}$$

Proposition 5.3(Gelfand-Shilov[1], p.11). *If $\tilde{T}_f^0$ is a special solution of the problem of regularization of $f(x)$, the general solution $\tilde{T}_f$ of the problem of regularization of $f(x)$ can be obtained by adding $\tilde{T}_f^0$ a distribution with support in $\{x_0 = 0\}$.*

Example 5.4. Let a regularization of $f(x) = \frac{1}{x}$, $(x \neq 0)$ be given by Example 5.1. Then the difference of two arbitrary regularizations of $f(x)$ is $c\delta(x)$. Here c is a constant.

Proposition 5.5(Gelfand-Shilov[1], p.12). *Assume that a function $f(x)$ satisfies the condition*

$$f(x) \geq F(r)$$

in a certain neighborhood of the origin in a certain solid angle with its vertex at the origin. Here $F(r)$ increases faster than any power of $1/r$ as r approaces to 0. Then there is no regularization of $f(x)$.

After all, if $f(x)$ has an at most countable number of isolated singularities and there are a finite number of singularities in every bounded intervals, there eists a regularization of $f(x)$.

Such a function $f(x)$ can be always represented as

$$f(x) = \Sigma f_k(x)$$

such that each $f_k(x)$ has only one singular point. Therefore the case of an at most countable number of isolated singularities is essentially the same as the mentioned above.

6. Examples

In this section we mention several examples of the method of renormalization by way of regularizations of divergent integrals of Gelfand-Shilov[1].

Example 6.1. We consider the function

$$x_+^\lambda = \begin{cases} 0, & (x \leq 0), \\ x^\lambda, & (x > 0). \end{cases}$$

Here $-1 < \lambda < 0$. Then x_+^λ is locally integrable on $\mathbf{R}$. But $\lambda x_+^{\lambda-1}$ is not locally integrable on $\mathbf{R}$. Then we consider a regularization of the divergent integral

$$\int_0^\infty \lambda x^{\lambda-1}\varphi(x)dx, \ (\varphi \in \mathcal{D}).$$

If we defferentiate x_+^λ in $\mathcal{D}'$, we have, for $\varphi \in \mathcal{D}$,

$$((x_+^\lambda)', \varphi) = -(x_+^\lambda, \varphi') = -\int_0^\infty x^\lambda \varphi'(x)dx$$

$$= -\lim_{\epsilon \to +0} \int_\epsilon^\infty x^\lambda \varphi'(x)dx.$$

Here we have

$$((x_+^\lambda)', \varphi) = -\lim_{\epsilon \to +0} \{[x^\lambda(\varphi(x) + C)]_\epsilon^\infty - \int_\epsilon^\infty \lambda x^{\lambda-1}(\varphi(x) + C)dx\}.$$

Here if we put $C = -\varphi(0)$, the first term $\longrightarrow 0$ as $\epsilon \longrightarrow 0$. Hence we have

$$((x_+^\lambda)', \varphi) = \lim_{\epsilon \to +0} \int_\epsilon^\infty \lambda x^{\lambda-1}(\varphi(x) - \varphi(0))dx$$

$$= \int_0^\infty \lambda x^{\lambda-1}(\varphi(x) - \varphi(0))dx = (\lambda x_+^{\lambda-1}, \varphi).$$

Hence we have

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1}$$

as a distribution. This is a regularization of the function $\lambda x_+^{\lambda-1}$.

Example 6.2. We consider the function

$$\log x_+ = \begin{cases} 0, & (x \leq 0), \\ \log x, & (x > 0). \end{cases}$$

Then we have, for $\varphi \in \mathcal{D}$,

$$((\log x_+)', \varphi) = \lim_{\epsilon \to +0} \int_\epsilon^\infty \frac{\varphi(x) - \varphi(0)\theta(1-x)}{x} dx$$

$$= \int_0^\infty \frac{\varphi(x) - \varphi(0)\theta(1-x)}{x} dx.$$

The restriction of $(\log x_+)'$ to $(0, \infty)$ coincides with $\dfrac{1}{x}$.

Example 6.3. We have

$$\frac{d \log |x|}{dx} = \text{v.p.}\frac{1}{x}.$$

Namely, we have, for $\varphi \in \mathcal{D}$,

$$(\frac{d \log |x|}{dx}, \varphi) = -(\log |x|, \varphi'(x)) = -\int_{-\infty}^\infty \log |x| \varphi'(x)dx$$

$$= -\lim_{\epsilon\to+0}\int_{|x|>\epsilon}\log|x|\varphi'(x)dx = \lim_{\epsilon\to+0}\int_{|x|>\epsilon}\frac{\varphi(x)}{x}dx$$

$$= \text{v.p.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx = (\text{v.p.}\frac{1}{x},\varphi).$$

This can also be regularized by the relation

$$(\frac{d\log|x|}{dx},\varphi) = \int_0^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}dx.$$

Example 6.4. If we put

$$\log(x+i0) = \lim_{y\to+0}\log(x+iy),$$

we have

$$(\log(x+i0))' = \text{v.p.}\frac{1}{x} - i\pi\delta(x).$$

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