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Several Methods for Solving Simultaneous Fermat-Pell Equations

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Abstract

In our previous papers [12] and [13], we have exhibited the structure of certain real bicyclic biquadratic fields and as a byproduct solved the simultaneous Fermat-Pell equations $x^2 - 3y^2 = 1$, $y^2 - 2z^2 = -1$ have only one non-negative integer solution: $(x, y, z) = (2, 1, 1)$. In this paper, we shall investigate similar simultaneous Fermat- Pell equations and solve them by several different methods.

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Introduction

In our papers [12] and [13], we have investigated Hasse's unit indices of certain family of real bicyclic biquadratic fields S_1. Here the real bicyclic biquadratic field $K \in S_1$ is parameterized by the positive integer n and the odd positive integer M. Let K be arbitrary real bicyclic biquadratic field and k_i $(1 \le i \le 3)$ be its three subfields. Let E_K be the unit group of K and ε_i be the fundamental unit of k_i. Put $E =< -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 >$. Then the group index $Q_K = [E_K : E]$ is called Hasse's unit index of $K/\boldsymbol{Q}$ and known to be 1,2, or 4 in general (see e.g. [14], [15] or [26]). In our papers [12] and [13], we have shown $Q_K = 1$ except for finitely many indices n and $Q_K = 2$ for the exceptional n and M. We note that one of the exceptional cases $Q_K = 2$ occurs when the following simultaneous Fermat-Pell equations have positive integer solution (x, y, z)

$$(1) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ y^2 - 2z^2 &= -1. \end{cases}$$

In [12], we have solved the above diophantine equations using practical algorithms developed in Section 4.6 of W. S. Anglin's textbook [1]. On the other hand, in [13], we have solved the same equations with the help of J. H. Rickert's results on the simultaneous Padé approximation to hypergeometric functions [23].

In this paper, we shall also treat the following Fermat-Pell equations similarly

$$(2) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ z^2 - 2y^2 &= 1, \end{cases}$$

$$(3) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ x^2 - 2z^2 &= -1, \end{cases}$$

$$(4) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ y^2 - 2z^2 &= 1, \end{cases}$$

where the diophantine equations (2) are the example treated in [23] and the diophantine equations (3) are the example treated in [3] and the diophantine equations (4) are obtained in the investigations of the Hasse's unit index Q_K of $K \in S_2$. Here $K \in S_2$ is the real bicyclic biquadratic field parameterized by the positive integer n and the "*even*" positive integer M.

In the following, we shall introduce three general methods which are applicable for all the equations (1),(2),(3) and (4). Moreover we shall introduce one more elementary method which is applicable for the cases (3) and (4). In Section 1, we shall introduce the most general method to solve the above simultaneous Fermat-Pell equations based on Baker's theory on linear forms in logarithms which was developed in [1]. In Section 2, we shall show that one can see the above equations as the special cases of J. H. Rickert's results in [23]. In Section [3], we shall reduce the problem of solving the above diophantine equations to the problem to determine all the integer points on certain modular elliptic curves. With the help of the arithmetic of the modular elliptic curves (see e.g. [9], [10] or [24]), we can determine all the integer points on the corresponding elliptic curves. In Section 4, we shall show (3) and (4) can be solved by using the elementary properties of square terms in some non-degenerate binary recurrence sequences.

1. Baker-Davenport's method

Firstly, we will show the simultaneous Fermat-Pell equations

$$(1) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ y^2 - 2z^2 &= -1 \end{cases}$$

have only one positive integer solution $(x, y, z) = (2, 1, 1)$ and no others.
In the following, instead of (1), we will consider the following equivalent equations

$$(5) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ w^2 - 2y^2 &= 2 \end{cases}$$

and show the equations (5) have only one positive integer solution $(x, y, w) = (2, 1, 2)$.

Here we shall quote some lemmas of Waldschmidt and Baker-Davenport.

Lemma 1-1 (**Waldschmidt** [27]).
Let A, A' and E be nonzero, non-negative algebraic numbers, each of degree ≥ 2 and ≤ 4. Let H, H', H'' be their heights. Suppose

$$V \geq \max(1 + \log H, 1 + \log H', 1 + \log H'', |\log A|, |\log A'|, |\log E|).$$

Let m and n be positive integers, and let $W = \max(\log m, \log n)$. Put $L = m \log A - n \log A' + \log E$. Then if $L \neq 0$,

$$|L| > \exp(-2^{101} V^3 (W + \log(64eV)) \log(64eV)).$$

Lemma 1-2 (**Davenport**'s lemma [2]).
Suppose x_1 and x_2 are real numbers. Suppose N is a positive integer with $2+\sqrt{3} > (10^6 N)^{1/N}$. Suppose p and q are integers with $1 \leq q \leq 1000N$ and

$$|x_1 q - p| \leq \frac{2}{1000N}.$$

Suppose m and n are positive integers such that $|mx_1 - n - x_2| \leq (2 + \sqrt{3})^{-N}$.

Then

(1) $m < (\log(10^6 N))/\log(2+\sqrt{3})$,

or

(2) $m > N$,

or

(3) $\|qx_2\| < 0.003$,

where $\|\alpha\|$ *denotes the distance of* α *from the nearest integer.*

Put $A = 3+2\sqrt{2}$, $A' = 2+\sqrt{3}$. Then to find the positive integer solutions of (5) is equivalent to find non-negative integers m, n such that

$$\begin{aligned}
w = p_m &= ((2+\sqrt{2})(3+2\sqrt{2})^m + (2-\sqrt{2})(3-2\sqrt{2})^m)/2, \\
y = q_m &= ((2+\sqrt{2})(3+2\sqrt{2})^m - (2-\sqrt{2})(3-2\sqrt{2})^m)/2\sqrt{2}, \\
x = w_n &= ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2, \\
y = t_n &= ((2+\sqrt{3})^n - (2-\sqrt{3})^n)/2\sqrt{3},
\end{aligned}$$

with $q_m = t_n$. Put

$$E = \frac{\sqrt{3}(2+\sqrt{2})}{\sqrt{2}} = \sqrt{3}+\sqrt{6}.$$

Then elementary calculations show that for $m, n \geq 10$

$$0 < |m \log A - n \log A' + \log E| < (2+\sqrt{3})^{-\max(m,n)}.$$

In our case, we see A, A' are algebraic numbers of degeree 2 with the heights $H = 6$ and $H' = 4$ and E is an algebraic number of degree 4 with the height $H'' = 18$. So in Lemma 1-1, we can take $V = 4$, and we have

$$(2+\sqrt{3})^{-\max(m,n)} > \exp(-11 \cdot 2^{101} 4^3 (W+11))$$

so that

$$11 \cdot 2^{101} 4^3 (W+11) > e^W \log(2+\sqrt{3}).$$

If $W > 55$, this gives

$$2^{107} \times (6/5) \times 11 > (e^W \log(2+\sqrt{3}))/W$$

or

$$74.5 > 107 \log 2 + \log(6/5) + \log 11 - \log\log(2+\sqrt{3}) > W - \log W.$$

Since $g(W) = W - \log(W)$ is a monotone increasing function for $W > 1$ and $g(79) = 74.63\cdots > 74.5$, we have $W < 79$, i.e., $\max(m,n) < e^{79} < 10^{35}$. Now we apply Davenport's lemma. Let $N = 10^{35}$. Then as we have shown, $m < N$, so we are not in case (2) of Davenport's lemma. Let

$$x_1 = \frac{\log A}{\log A'} \text{ and } x_2 = -\frac{\log E}{\log A'}.$$

and put
$p = 56385661377567652409679055189876652197$,
$q = 42126030957897209320409914963098120351$.
Then one can see $|x_1 q - p| < 2 \times 10^{-38}$ and $\|qx_2\| = 0.46217\cdots > 0.004$. Thus we are not in case (3) of Davenport's lemma, also we are not in case (2) of Davenport's lemma. Hence we are in case (1), i.e., $m < \log 10^{41}/\log(2+\sqrt{3}) < 73$. We have checked all the $0 \le m < 73$ and have shown that $q_m = t_n$ occurs only for the case $m = 0$ and $n = 1$, i.e., the simultaneous Fermat-Pell equations (5) have only one positive integer solution $(x, y, w) = (2, 1, 2)$.

Remark 1-1. In [1], W. S. Anglin has shown the above method is applicable for solving any simultaneous Fermat-Pell equations

$$(6) \qquad \begin{cases} x^2 - Ry^2 & = C \\ z^2 - Sy^2 & = D \end{cases}$$

with $0 < R, S, |C|, |D| \le 1000$. Hence in the same way as above, one can show each simultaneous Fermat-Pell equations (2) (resp. (4)) have only one non-negative integer solution $(x, y, z) = (1, 0, 1)$ (resp. $(2, 1, 0)$). Similarly the equations (3) have only two non-negative integer solutions $(x, y, z) = (1, 0, 1)$ and $(7, 4, 5)$.

2. Rickert's method

In this section, we will show the simultaneous Fermat-Pell equations

$$(4) \qquad \begin{cases} x^2 - 3y^2 & = 1 \\ y^2 - 2z^2 & = 1 \end{cases}$$

have only one non-negative integer solution $(x, y, z) = (2, 1, 0)$ and no others.

In the following, instead of (4), we will consider the following equivalent equations

$$(7) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ w^2 - 2y^2 &= -2 \end{cases}$$

where $w = 2z$. With the help of a result of J. H. Rickert (see (1.7) in [21]), we will show that these equations have only one non-negative integer solution: $(x, y, w) = (2, 1, 0)$.

Lemma 2-1 (**Rickert** [21]). *Let u, v be non-zero integers. All integer solutions x, y, z of the following simultaneous Fermat-Pell equations*

$$\begin{cases} x^2 - 3y^2 = u \\ w^2 - 2y^2 = v \end{cases}$$

satisfy

$$\max\{|x|, |y|, |w|\} \leq (10^7 \max\{|u|, |v|\})^{12}.$$

Then to find the non-negative integer solutions of (4) is equivalent to finding all non-negative integers m, n for which

$$\begin{cases} x &= r_n &= ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2, \\ y &= s_n &= ((2+\sqrt{3})^n - (2-\sqrt{3})^n)/(2\sqrt{3}), \\ y &= p_m &= ((1+\sqrt{2})^{2m} + (1-\sqrt{2})^{2m})/2, \\ w &= q_m &= ((1+\sqrt{2})^{2m} - (1-\sqrt{2})^{2m})/\sqrt{2}. \end{cases}$$

From Lemma 2-1, we see that $(1+\sqrt{2})^{2m} < 2p_m \leq 2(10^7 \times 2)^{12}$ for $m \geq 0$ implies $m < (42\log(10) + 6.5\log(2))/\log(1+\sqrt{2}) = 114.836\ldots < 115$. We have checked that for $0 \leq m \leq 114$, $p_m = s_n$ only for $m = 0, n = 1$, i.e., the simultaneous Fermat-Pell equations (7) have only one non-negative integer solution $(x, y, w) = (2, 1, 0)$.

Remark 2-1. In [13], as an application of Lemma 2-1 for the case $u = 1$ and $v = 2$, we have shown the simultaneous Fermat-Pell equations (1) have only one non-negative integer solution $(x, y, z) = (2, 1, 1)$. In [23], as an application of Lemma 2-1 for the case $u = 1$ and $v = 1$, J. H. Rickert has shown the simultaneous Fermat-Pell equations (2) have only one non-negative integer solution

$(x, y, z) = (1, 0, 1)$. Similarly, taking $u = -3$ and $v = 2$, one can easily show that the equations (3) have only two non-negative integer solutions $(x, y, z) = (1, 0, 1)$ and $(7, 4, 5)$.

3. Integer points on the elliptic curves

In the following, we shall reduce the problems of solving the diophantine equations (1), (2), (3) and (4) to the problem of the determination of all the integer points on the following elliptic curves:

$$\begin{aligned} E_1 &: \; Y^2 = X^3 - X^2 - 9X + 9, \\ E_2 &: \; Y^2 = X^3 - X^2 - 2X, \\ E_3 &: \; Y^2 = X^3 - 36X, \\ E_4 &: \; Y^2 = X^3 - X^2 - 17X + 15, \end{aligned}$$

where each curve E_i corresponds to the simultaneous Fermat-Pell equations (i) $(1 \le i \le 4)$.

Firstly we shall treat the simultaneous Fermat-Pell equations

$$(2) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ z^2 - 2y^2 &= 1. \end{cases}$$

Then we have

$$(xz)^2 = (3y^2 + 1)(2y^2 + 1),$$

so

$$(6xyz)^2 = 6y^2(6y^2 + 2)(6y^2 + 3).$$

The substitutions $Y = 6xyz$, $X = 6y^2 + 2$ yields the elliptic curve

$$E_2 : Y^2 = X^3 - X^2 - 2X,$$

which is the curve $96B1(A)$ in Cremona's table [9]. Thus the Mordell-Weil group $E_2(\boldsymbol{Q})$ of E_2 over $\boldsymbol{Q}$ is given by $E_2(\boldsymbol{Q}) = \{O, (0,0), (-1,0), (2,0)\} \cong (\mathbf{Z}/2\mathbf{Z})^2$. Hence the equations (2) have only one non-negative integer solution $(x, y, z) = (1, 0, 1)$.

Similarly the equations (4) imply

$$2(xz)^2 = (y^2 - 1)(3y^2 + 1),$$

so

$$(12xyz)^2 = 6y^2(6y^2 - 6)(6y^2 + 2).$$

The substitutions $Y = 12xyz$, $X = 6y^2 - 1$ yields the elliptic curve

$$E_3 : Y^2 = X^3 - X^2 - 17X - 15,$$

which is the curve $192D2(F)$ in Cremona's table [9]. Thus the Mordell-Weil group $E_3(\boldsymbol{Q})$ of E_3 over $\boldsymbol{Q}$ is given by $E_3(\boldsymbol{Q}) = \{O, (-3,0), (-1,0), (5,0)\} \cong (\mathbf{Z}/2\mathbf{Z})^2$. Hence the equations (4) have only one non-negative integer solution $(x, y, z) = (2, 1, 0)$.

Now we shall treat the simultaneous Fermat-Pell equations

$$(1) \qquad \begin{cases} x^2 - 3y^2 &= 1 \\ y^2 - 2z^2 &= -1, \end{cases}$$

which imply

$$2(xz)^2 = (y^2 + 1)(3y^2 + 1),$$

so

$$(12xyz)^2 = 6y^2(6y^2 + 2)(6y^2 + 6).$$

The substitutions $Y = 12xyz$, $X = 6y^2 + 3$ yields the elliptic curve

$$E_1 : Y^2 = X^3 - X^2 - 9X + 9,$$

which is the curve $192A2(R)$ in Cremona's table [9]. Thus the Mordell-Weil group $E_1(\boldsymbol{Q})$ of E_1 over $\boldsymbol{Q}$ is given by $E_1(\boldsymbol{Q}) \cong \mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^2$. Since E_1 has rank one, we must calculate several arithmetical invariants of E_1. Calculating canonical heights of several integer points on E_1 and using the gap principle of the ordinary height and the canonical height (see e.g. [9], [10]), we can take $P_1 = (0,3)$ as a generator of infinite order of $E_1(\boldsymbol{Q})$. Since the torsion part of $E_1(\boldsymbol{Q})$ is $\{O, (3,0), (1,0), (-3,0)\}$, any integral point on $E_1(\boldsymbol{Q})$ can be written in the form $P = n_1P_1 + n_2(3,0) + n_3(1,0)$ with $0 \leq n_2, n_3 \leq 1$. With the help of the LLL-reduction which was introduced in [10] and [24], we see n_1 is bounded (actually $n_1 \leq 10$). Enumerating all the possible cases, we see $E_1(\mathbf{Z})$ is consisting of the following 13 points:

$$\begin{aligned}&(3,0),(1,0),(-3,0),\\&\pm P_1=(0,\pm 3),\\&\pm(P_1+(3,0))=(-1,\mp 4),\\&\pm(P_1+(1,0))=(9,\pm 24),\\&\pm(P_1+(-3,0))=(-5,\mp 8),\\&\pm(2P_1+(3,0))=(51,\mp 360).\end{aligned}$$

Hence the equations (1) have only one non-negative integer solution $(x,y,z)=(2,1,1)$.

Finally the simultaneous Fermat-Pell equations (3) imply

$$(36xyz)^2=(6x^2)^3-36(6x^2).$$

The substitutions $Y=36xyz$, $X=6x^2$ yields the elliptic curve

$$E_3: Y^2=X^3-36X,$$

which is the curve $576H2$ in Cremona's table [9]. Thus the Mordell-Weil group $E_3(\boldsymbol{Q})$ of E_3 over $\boldsymbol{Q}$ is given by $E_3(\boldsymbol{Q})\cong\mathbf{Z}\times(\mathbf{Z}/2\mathbf{Z})^2$. In the same way as the above E_1, we can take $P_1=(-3,9)$ as a generator of infinite order of $E_3(\boldsymbol{Q})$ and the torsion part of E_3 is $\{O,(0,0),(6,0),(-6,0)\}$. Thus all the integer points on E_3 are the following:

$$\begin{aligned}&(0,0),(6,0),(-6,0),\\&\pm P_1=(-3,\pm 9),\\&\pm(P_1+(0,0))=(12,\pm 36),\\&\pm(P_1+(6,0))=(-2,\mp 8),\\&\pm(P_1+(-6,0))=(18,\mp 72),\\&\pm(2P_1+(6,0))=(294,\pm 5040).\end{aligned}$$

Hence the equations (3) have only two non-negative integer solutions $(x,y,z)=(1,0,1)$ and $(7,4,5)$.

Remark 3-1. In Sections 1 and 2, one can solve the equations (1),(2),(3) and (4) similarly. It is interesting that in the above methods the equations (1), (3) are quite different from (2), (4) because the corresponding elliptic curves have positive rank for the cases (1), (3), while the cases (2), (4) have rank 0.

4. Square terms in binary recurrence sequences

We shall prepare several preliminary lemmas. Let m be a positive integer with $D = m^2 \pm 1 > 0$. Put $\alpha = m + \sqrt{D}$ and $\beta = m - \sqrt{D}$. Then

$$x_n = (\alpha^n + \beta^n)/2 \text{ and } y_n = (\alpha^n - \beta^n)/2\sqrt{D}$$

are the non-degenerate binary recurrence sequences

$$\begin{aligned} x_{2n+2} &= 2mx_{2n+1} \pm x_n \ (n \geq 0) \\ y_{2n+2} &= 2my_{2n+1} \pm y_n \ (n \geq 0), \end{aligned}$$

with the initial terms $x_0 = 1$, $x_1 = m$ and $y_0 = 0$, $y_1 = 1$.

Then we have

$$\begin{aligned} y_{2n+1}^2 - 1 &= \{(\alpha^{2n+1} - \beta^{2n+1})^2 - 4(m^2 \pm 1)\}/4D \\ &= \{\alpha^{4n+2} + \beta^{4n+2} - (4m^2 \pm 2)\}/4D \\ &= (\alpha^{2n+2} - \beta^{2n+2})(\alpha^{2n} - \beta^{2n})/4D \\ &= y_{2n}\ y_{2n+2}. \end{aligned}$$

Moreover inductively we get $(y_{2n}, y_{2n+2}) = 2m$. Hence we have shown the following lemma.

Lemma 4-1.

$$\begin{aligned} &(i) \quad y_{2n+1}^2 - 1 = y_{2n}\ y_{2n+2}, \\ &(ii) \quad (y_{2n}, y_{2n+2}) = 2m. \end{aligned}$$

In his paper [17], W. Ljunggren has shown the following lemma.

Lemma 4-2(Ljunggren [17]**).** *$x^2 - 3y^4 = 1$ has only two positive integer solutions $(x, y) = (2, 1)$ and $(7, 2)$.*

If (x, y, z) is a non-negative integer solution of (4), we see $y^2 - 2z^2 = 1$ implies y is odd. Hence, putting $m = 2$ and $D = 3$ in above, we see $y = y_{2n+1}$ for some n. $2z^2 = y^2 - 1 = y_{2n+1}^2 - 1 = y_{2n}\ y_{2n+2}$ implies one of y_{2n} or y_{2n+2} is a square a^2. Hence, from Lemma 4-2, we see $a = 1$ or 2. Thus the only possible non-negative integer solution of (4) is $(x, y, z) = (2, 1, 0)$.

Finally we shall show the equations (3) can be solved similaly. As was noted in [3], one can solve (4) with the help of the square terms properties of $\{x_n\}$ (see J. H. E. Cohn [8]), but here we shall give another proof based on the square terms properties of $\{y_n\}$. Put $m = 1$ and $D = 2$. Then the solution of (3) satisfies $x^2 - 2z^2 = -1$. Hence $x = x_{2n+1}$ and $z = y_{2n+1}$ for some n. We note that $x^2 - 1 = 2(z^2 - 1) = 2y_{2n}\ y_{2n+2}$. So $x^2 - 1 = 3y^2$ implies the following possibilities;

$$y_{2n} = \square, \quad y_{2n+2} = 6\square,$$

or

$$y_{2n} = 2\square, \quad y_{2n+2} = 3\square,$$

or

$$y_{2n} = 3\square, \quad y_{2n+2} = 2\square,$$

or

$$y_{2n} = 6\square, \quad y_{2n+2} = \square,$$

where $w = \square$ means that the integer w is a square. Here we shall quote a corollary of the more general results of K. Nakamula and A. Pethő [19].

Lemma 4-3 (Nakamula-Pethő). *$y_n = c\square$ for $n > 3$ and $c \in \{1, 2, 3, 6\}$ if and only if $(n, m, c) = (4, 1, 3), (4, 2, 2), (4, 12, 3)$ or $(7, 1, 1)$.*

Hence we see the possible $n = 0$ or 1, i.e., $(x_{2n+1}, y_{2n+1}) = (1, 1)$ or $(7, 5)$. Hence the equations (3) have only two non-negative integer solutions $(x, y, z) = (1, 0, 1)$ and $(7, 4, 5)$.

Remark 4-1. In [3], M. A. Benett has introduced another general method based on a gap principle of two solutions of the given simultaneous diohantine equations and the linear forms in two logarithms and shown the equations (3) have only two non-negative integer solutions as above. So combining the above 5 methods, one can see that the simultaneous diophantine equations (3) can be solved by six different methods.

Remark 4-2. There are several words which express the above simultaneous Fermat-Pell equations, the first one is "simultaneous Pell equations" or "simultaneous pellian equations" as was used in [3], and the second one is "simultaneous

Fermat equations" as was used in [1]. I feel the naming "simultaneous Fermat-Pell equations" is not familiar to many peaple but our coauthor C. Levesque suggested to me to use this naming in our previous papers [12] and [13]. So I decided to use this naming in this paper again.

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