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A Note on the Blow-up Pattern for a Parabolic Equation

By
Atsuhito KOHDA and Takashi SUZUKI
Department of Mathematics, Faculty of Engineering, Tokushima University, Minamijosanjima 2-1, Tokushimashi 770-8506 Japan
and
Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyamacho 1-1, Toyonakashi 560-0043 Japan

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Abstract

We consider here some conditions on initial value for parabolic problem which guarantee the blow-up of a solution. Then we study the behaviour of blow-up solution near blow-up time, that is blow-up patterns.

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Introduction

Given a bounded domain $\Omega \subset \mathcal{R}^n$ with smooth boundary $\partial\Omega$, we study the parabolic problem

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0,T), \qquad u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0(x) \tag{1}$$

for $f(u) = \lambda_0 e^u$, λ_0 being a constant. The maximal time for the existence of the classical solution is denoted by $T_{\max}$.

First theorem is stated as follows.

Theorem 1 *Let $v \in C^4(\overline{\Omega})$ satisfy*

$$\Delta^2 v + \lambda_0 e^v |\nabla v|^2 \geq (\Delta v)^2 \quad \textit{in} \quad \Omega, \qquad \Delta v + \lambda_0 = v = 0 \quad \textit{on} \quad \partial\Omega \tag{2}$$

and

$$\Omega_0 \equiv \left\{ x \in \Omega \mid -\Delta v(x) < \lambda_0 e^{v(x)} \right\} \neq \emptyset. \tag{3}$$

Then $T_{\max} < +\infty$, provide that $u_0 \geq v$ in Ω.

Obviously, above theorem is reduced to the case $u_0 = v$. If Ω is the unit ball $B = \{x \in \mathcal{R}^n \mid |x| < 1\}$, and $u_0 = v(x)$ satisfies furthermore that

$$v = v(|x|), \qquad v_r < 0 \quad (0 < r = |x| < 1), \tag{4}$$

we can show that the mapping $t \in (t_0, T_{\max}) \mapsto u(x,t)$ is monotone increasing if $0 < T_{max} - t_0 \ll 1$ and $|x| \ll 1$. Then we have the following theorem. Sharper blow-up profiles are proven under different assumptions on the initial data ([4], [1], [2]).

Theorem 2 *Under those circumstances, for any $K > 0$ there exists some $r \in (0,1]$ satisfying*

$$\lim_{t \uparrow T_{\max}} u(x,t) \geq 2 \log \frac{1}{|x|} + K \qquad (0 < |x| < r). \tag{5}$$

Related to above theorem we have the following.

Theorem 3 *If $n \leq 5$ and $\lambda_0 > \overline{\lambda}$, there exists a funtion $v(x)$ satisfying the assumptions of the previous theorem. Here, $\overline{\lambda}$ denotes the supremum of λ for the existence of a classical solution of*

$$-\Delta v = \lambda e^v \quad in \quad \Omega, \qquad v = 0 \quad on \quad \partial\Omega. \tag{6}$$

Theorems 1, 2, and 3 are proven in sections 2, 3, and 4, respectively.

1 Proof of Theorem 1

First, we note the following.

Lemma 4 *Condition (2) implies that*

$$t \in (0, T_{\max}) \mapsto J(x,t) \qquad \textit{non-decreasing} \tag{7}$$

for any $x \in \Omega$, where $J = e^{-u} u_t$.

Proof: $u(x,t)$ is smooth if $t > 0$ so that

$$u_{tt} - \Delta u_t = \lambda_0 e^u u_t$$

and

$$u_{ttt} - \Delta u_{tt} = \lambda_0 e^u \left(u_t^2 + u_{tt} \right)$$

hold. Therefore, $I = u_{tt} - u_t^2$ satisfies

$$\begin{aligned} I_t - \Delta I &= \lambda_0 e^u I + 2\left|\nabla u_t\right|^2 \\ &\geq \lambda_0 e^u I \qquad \text{in} \quad \Omega \times (0,T), \end{aligned}$$

$I|_{\partial\Omega} = 0$, and

$$\begin{aligned} I &= \Delta u_t + \lambda_0 e^u u_t - \left(\Delta u + \lambda_0 e^u\right)^2 \\ &= \Delta^2 u + \lambda_0 e^u \left|\nabla u\right|^2 - \left(\Delta u\right)^2. \end{aligned}$$

From the standard theory, $u_0 = 0$ on $\partial\Omega$ implies

$$w = u_t \in C\left([0,T), L^p(\Omega)\right)$$

for any $1 < p < \infty$. See [7] or [12].

Let $A_p(t)$ be the realization in $X_p = L^p(\Omega)$ of

$$-\Delta - \lambda_0 e^{u(\cdot,t)}$$

with $\cdot|_{\partial\Omega} = 0$. Then, (2) implies

$$\begin{aligned} w_0 \equiv w|_{t=0} &= \Delta u_0 + \lambda_0 e^{u_0} \\ &\in D_p \equiv D\left(A_p(t)\right) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \end{aligned}$$

Therefore, the regularity theorem for the solution $w(t) \in X_p$ of the evolution

$$\frac{dw}{dt} + A_p(t) w = 0$$

is applicable and $w \in C^1\left([0,T), X_p\right)$ follows. In particular $I \in C\left([0,T), L^p(\Omega)\right)$, and

$$I|_{t=0} = \Delta^2 u_0 + \lambda_0 e^{u_0} \left|\nabla u_0\right|^2 - \left(\Delta u_0\right)^2.$$

Now, the comparison thoerem imply $I \geq 0$ in $\Omega \times (0,T)$. This means

$$j_t = e^{-u}\left(u_{tt} - u_t^2\right) \geq 0$$

and (7) has been proven. □

To prove Theorem 1, suppose that $T_{\max} = +\infty$. Then (17) of our previous paper [9] holds so that

$$\int_\Omega u(x,t)\phi_1(x)dx \leq j_* \qquad (t \geq 0),$$

where $\phi_1(x) > 0$ denoted the L^1-normalized first eigenfuntion of $-\Delta$ and j_* is an absolute constant.

Take $\varepsilon > 0$ and set

$$\Omega_\varepsilon = \{x \in \Omega \mid J(x,0) > \varepsilon\}.$$

Then, above lemma implies

$$u_t(x,t) \geq e^{u(x,t)} J(x,0) > \varepsilon \qquad (x \in \Omega_\varepsilon)$$

and hence

$$j_* \geq \int_{\Omega_\varepsilon} u(x,t)\phi_1(x)dx \geq \varepsilon t \int_{\Omega_\varepsilon} \phi_1(x)dx$$

for $t > 0$. This means $\Omega_\varepsilon = \emptyset$. Therefore,

$$\Omega_0 = \{x \in \Omega \mid J(x,0) > 0\} = \emptyset,$$

a contradiction to assumption (3). □

2 Proof of Theorem 2

The following lemma is due to T. Itoh ([8]). See also [11] for the proof.

Lemma 5 *Let*

$$v = v(|x|, t) \in C^2(B \times (0,T)) \cap C\left(\overline{B} \times [0,T)\right)$$

satisfy

$$v_t > 0, \quad v_r < 0, \quad -\Delta v < e^v \qquad in \quad B_R \times (t_0, T) \tag{8}$$

and

$$\lim_{t \uparrow T} v(0,t) = +\infty, \qquad \int_{B_{r_1}} e^{v(x,t_0)}dx < 4\pi, \tag{9}$$

for $0 < r_1 \leq R \leq 1$ *and* $0 \leq t_0 < T$. *Then,*

$$v(|x|, T) \geq 2\log\frac{1}{|x|} + \log 2 \qquad (|x| < r_2) \tag{10}$$

holds for some $r_2 \in (0, r_1)$

To prove Theorem 2, we put $T = T_{\max}$ for simplicity. The second inequality of (8) holds for $v = u$, $R = 1$, and $t_0 = 0$. Therefore, $T_{\max} < +\infty$ implies the first relation of (9). This meas that

$$J|_{x=0} \leq 0 \qquad (0 \leq t < T)$$

is impossible, because then $u_t(0,t) \leq 0$ for $t \in [0,T)$. Thus, there exists some $t_0 \in [0,T)$ such that $J(0,t_0) > 0$. In use of Lemma 4 we have $J > 0$ on $B_R \times (t_0, T)$ for some $R \in (0,1]$. Namely,

$$u_t > 0, \quad -\Delta u < \lambda_0 e^u \qquad \text{in} \quad B_r \times (t_1, T) \tag{11}$$

holds for $r = R$ and $t_1 = t_0$.

We claim the following.

Lemma 6 $J|_{x=0} \to \lambda_0$ *as* $t \uparrow T$.

Proof: In the case of $n \geq 3$ we can make use of the argument of [3] by (11). The function $w(y,\sigma)$ defined by

$$u(x,t) = w\left(x\,(T-t)^{-1/2}, \log\frac{T}{T-t}\right) - \log(T-t)$$

satisfies

$$w \longrightarrow 0 \qquad \text{locally uniformly in} \quad y \in \mathcal{R}^n$$

as $\sigma \to +\infty$. Furthermore, it satisfies

$$w_\sigma - \Delta w = -\frac{1}{2}y \cdot \nabla w + (\lambda_0 e^w - 1)$$

and hence the parabolic regularity implies that

$$w_\rho,\ w_\sigma \longrightarrow 0 \qquad \text{locally uniformly in} \quad y \in \mathcal{R}^n,$$

as $\sigma \to +\infty$ where $\rho = |y|$. Therefore,

$$u(0,t) + \log(T-t) = w\left(0, \log\frac{T}{T-t}\right) \longrightarrow 0$$

and

$$\begin{aligned}(T-t)u_t(0,t) &= \frac{1}{2}\rho w_\rho\left(0, \log\frac{T}{T-t}\right) \\ &\quad + w_\sigma\left(0, \log\frac{T}{T-t}\right) + \lambda_0 \\ &\quad \longrightarrow \lambda_0\end{aligned}$$

as $t \uparrow T$. Then the desired conclusion follows as

$$\begin{aligned}J(0,t) &= e^{-u}u_t\big|_{t=0} \\ &= e^{-(u(0,T)+\log(T-t))} \cdot (T-t)u_t(0,t) \quad \longrightarrow \lambda_0.\end{aligned}$$

For the case $n = 2$ we make use of [10] instead. □

To prove Theorem 2, take a constant $K > 0$. By Lemma 6, we have some $t_0 \in [0,T)$ satsifying $J(0,t_0) \geq \lambda_0 - 2e^{-K}$, and hence some $r_0 \in (0,R]$ with

$$J(x,t_0) \geq \lambda_0 - e^{-K} \qquad (x \in B_{r_0}).$$

Then Lemma 4 implies

$$J \equiv u_t e^{-u} = \lambda_0 + e^{-u}\Delta u \geq \lambda_0 - e^{-K},$$

or equivalently,

$$-\Delta u \leq e^{u-K} \quad \text{in} \quad B_{r_0} \times (t_0, T).$$

The function $v(|x|, t) \equiv u(|x|, t) - K$ satisfies

$$-\Delta v \leq e^{v} \quad \text{in} \quad B_{r_0} \times (t_0, T).$$

On the other hand if $K > -\log \lambda_0$ we have $e^{u-K} < \lambda_0 e^u$. Thus,

$$v_t = u_t = \Delta u + \lambda_0 e^u > 0 \quad \text{in} \quad B_{r_0} \times (t_0, T).$$

Also $u(0,t) \to +\infty$ implies $v(0,t) \to +\infty$. Finally, we can take some $r_1 \in (0, r_0]$ such that

$$\int_{B_{r_1}} e^{v(x,t_0)} dx < 4\pi.$$

Now Lemma 5 implies (10) for some $r_2 \in (0, r_1]$ and inequality (5) holds with $r = r_2$. □

3 Proof of Theorem 3

By virtue of [5], a priori bound of the solution for

$$-\Delta f = (f + \lambda)^2 \quad \text{in} \quad B, \qquad f = 0 \quad \text{on} \quad \partial B \tag{12}$$

holds for $2 < \frac{n+2}{n-2}$. Namely, when $n \leq 5$, for any $\Lambda > 0$ admits a constant $C > 0$ such that $\|f\|_\infty \leq C$ for any classical solution $f(x)$ of (12) with $0 \leq \lambda \leq \Lambda$. On the other hand, when $0 < \lambda \ll 1$, any classical soltion of (12) is unique and linearized stable. Then, standard argument based on the topological degree gurantees the existence of a solution for any $\lambda > 0$.

Obviously, $f(x) > 0$ in B and [6] assures that this is radial and radially decreasing. Now we take $v(x)$ as the solution of

$$-\Delta v = f + \lambda_0 \quad \text{in} \quad B, \qquad v = 0 \quad \text{on} \quad \partial B. \tag{13}$$

Then, properties (2) and (4) are easy to verify.

Finally, if (6) does not have a solution, then no positive super-solution exists. The function $v(x) > 0$ cannot be a super solution of (6) and hence (3) follows. □

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