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A Note on Degenerate Kirchhoff Equations with Nonlinear Damping

By

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Abstract

We study the decay property of energy to the initial boundary value problem for nonlinear partial integro-differential equations with nonlinear damping terms.

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1. Introduction and Result

We consider the asymptotic behavior of solutions to the initial boundary value problem for the following nonlinear partial integro-differential equations with nonlinear damping terms :

$$(1.1)\qquad \begin{cases} u_{tt} - \left(\int_\Omega |\nabla u(\cdot,t)|^2 dx\right)^\gamma \Delta u + \delta |u_t|^\beta u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,, \quad \text{and} \quad u(x,t)|_{\partial\Omega} = 0\,, \end{cases}$$

where Ω is a bounded domain in N-dimensional Euclidean space $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, γ a nonnegative constant, δ a positive constant, β a nonnegative constant, $u_t = \partial u/\partial t$, $|\nabla u|^2 = \sum_{j=1}^N |\partial u/\partial x_j|^2$, and $\Delta u = \sum_{j=1}^N \partial^2 u/\partial x_j^2$.

When $N = 1$, Eq.(1.1) describes a small amplitude vibration of an elastic string without the initial axial tension. In the case of $\delta = 0$, Kirchhoff [10] firstly studied such integro-differential equations (with the initial tension), which are called Kirchhoff equations after his name. (Also see [3], [4], [6], [13], [14].)

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The local well-posedness (equivalent to the local existence and uniqueness) in Sobolev space has been already studied by many authors (see [1], [2], [5], [7], [24], [26], and the references cited therein). Indeed, using the fact that if $u_0 \neq 0$, then $\int_\Omega |\nabla u(\cdot,t)|^2 dx > 0$ for some $t > 0$, we can show the existence of a local solution u of Eq.(1.1) (e.g. see [19], [22] and also [1], [8]) :

Propsition 1.1. *Suppose that* $\{u_0, u_1\} \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ *with* $u_0 \neq 0$ *and* $\gamma \geq 1$ *and* $\beta \leq 4/(N-2)$ ($\beta < \infty$ *if* $N \leq 2$). *Then, there exists a unique local solution* u *of Eq.*(1.1) *satisfying* $u \in C_w([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C_w^1([0,T]; H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *and* $u_t \in L^{\beta+2}((0,T)\times\Omega)$ *for some* $T = T(\|u_0\|_{H^2} + \|u_1\|_{H^1}) > 0$.

We note that Eq.(1.1) has not any solutions in the energy class. Moreover, in order to get a global solution of Eq.(1.1), we need to derive the a-priori estimate : $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} < \infty$ for $t \geq 0$, where $\|\cdot\|_{H^j}$ is the norm of $H^j(\Omega)$.

In the case of $\beta = 0$ (i.e. linear damping case) in Eq.(1.1), making the best possible use of an effect of the damping term u_t, Nishihara and Yamada [17] have derived $\|\nabla u_t(t)\|^2/\|\nabla u(t)\|^2 + \|\Delta u(t)\|^2 \leq C$ for $t \geq 0$ ($\|\cdot\|$ is the norm of $L^2(\Omega)$) and shown the existence of a global solution (also see Ono [21] for sharp decay properties). Moreover, in addition to their method, utilizing the sharp energy decay (see (1.4)), we also have proved the global-in-time solvability for Eq.(1.1) with $\beta = 0$ and the perturbation terms $f(u) = \pm|u|^\alpha u, \pm|u|^{\alpha+1}$ in [19, 20]. Indeed, the sharp energy decay estimate has played an important role in the proof. On the other hand, the global-in-time solvability of the degenerate equation (1.1) with $\beta > 0$ has not known at the present. Our goal in this paper is to derive the decay estimate of energy for an assumed solution of Eq.(1.1).

We define the energy $E(t)$ associated with Eq.(1.1) as

$$E(t) \equiv \|u_t(t)\|^2 + (1+\gamma)^{-1}\|\nabla u(t)\|^{2(\gamma+1)}, \tag{1.2}$$

where $\|\cdot\|$ is the norm of $L^2(\Omega)$.

Our main result is as follows.

Theorem 1.2. *Let* $N \geq 1$ *and let* u *be a solution of Eq.*(1.1). *Suppose that* $\beta \leq 4/(N-2)$ *if* $N \geq 3$. *Then, the energy* $E(t)$ *satisfies*

$$E(t) \leq C(1+t)^{-\theta(\gamma,\beta)} \quad \textit{with} \quad \theta(\gamma,\beta) = \frac{2(\gamma+1)}{(2\gamma+1)\beta + 2\gamma} \tag{1.3}$$

for $t \geq 0$.

When $\gamma > 0$ and $\beta = 0$ in Eq.(1.1), the following decay estimate of the energy $E(t)$ is well known (e.g. see Nakao [11]) :

$$\Big((1+\gamma)^{-1}\|\nabla u(t)\|^{2(\gamma+1)} \leq\Big) \quad E(t) \leq C(1+t)^{-(\gamma+1)/\gamma}. \tag{1.4}$$

Then we see $\theta(\gamma, 0) = (\gamma+1)/\gamma$.

Moreover, when the degenerate equation (1.1) has the strong damping term $-\Delta u_t$ instead of $|u_t|^\beta u_t$, in addition to the above decay (1.4), Nishihara [15] has derived the following lower decay estimate :

$$C'(1+t)^{-\theta(\gamma,0)} \le \|\nabla u(t)\|^{2(\gamma+1)} \qquad \text{for} \quad t \ge T_*$$

with some $T_* \ge 0$ (also see [16], [18], [23]), that is, we know that the decay (1.4) is sharp (when $\beta = 0$ in Eq.(1.1)).

On the other hand, when $\gamma = 0$ and $\beta > 0$ in Eq.(1.1), the following decay estimate of the energy $E(t)$ is well known (e.g. see [9], [12], [25], [27]) :

$$E(t) \le C(1+t)^{-2/\beta} \qquad \text{for} \quad t \ge 0\,.$$

Then we see $\theta(0,\beta) = 2/\beta$.

2. Proof

Following Nakao [11, 12], we shall give the proof of Theorem 1.2.

Multiplying (1.1) by $2u_t$ and integrating over Ω, we have the energy identity :

$$\frac{d}{dt}E(t) + 2\|u_t(t)\|_{\beta+2}^{\beta+2} = 0\,, \tag{2.1}$$

where $\|\cdot\|_{\beta+2}$ is the norm of $L^{\beta+2}(\Omega)$, and $E(t)$ is non-increasing, that is, $E(t) \ge E(s)$ for $t \ge s \ge 0$. Integrate (2.1) over $[t, t+1]$ to obtain

$$2\int_t^{t+1} \|u_t(s)\|_{\beta+2}^{\beta+2} ds = E(t) - E(t+1) \qquad \left(\equiv D(t)^{\beta+2}\right). \tag{2.2}$$

Then, it follows that

$$D(t)^{\beta+2} \le E(t) \le E(0)\,, \tag{2.3}$$

and there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t)\|_{\beta+2}^{\beta+2} \le 2D(t)^{\beta+2} \qquad \text{for} \quad j = 1, 2\,. \tag{2.4}$$

Multiplying (1.1) by u and integrating over $\Omega \times [t_1, t_2]$, we have from the Sobolev-Poincaré inequality that

$$\begin{aligned}
&\int_{t_1}^{t_2} \|\nabla u(s)\|^{2(\gamma+1)} ds \\
&\le \int_{t_1}^{t_2} \|u_t(s)\|^2 ds + \sum_{j=1}^{2} \|u_t(t_j)\|\|u(t_j)\| + \int_{t_1}^{t_2} \|u_t(s)\|_{\beta+2}^{\beta+1} \|u(s)\|_{\beta+2} ds \\
&\le C\Big(\int_t^{t+1} \|u_t(s)\|_{\beta+2}^{\beta+2} ds\Big)^{2/(\beta+2)} + C\sum_{j=1}^{2} \|u_t(t_j)\|_{\beta+2} \sup_{t\le s\le t+1} \|\nabla u(s)\| \\
&\quad + C\Big(\int_t^{t+1} \|u_t(s)\|_{\beta+2}^{\beta+2} ds\Big)^{(\beta+1)/(\beta+2)} \sup_{t\le s\le t+1} \|\nabla u(s)\|\,.
\end{aligned} \tag{2.5}$$

Moreover, it follows from (1.2) and (2.2)–(2.5) that

$$\int_{t_1}^{t_2} E(s)\,ds \le \int_{t}^{t+1} \|u_t(s)\|^2 ds + \int_{t_1}^{t_2} \|\nabla u(s)\|^{2(\gamma+1)} ds$$
$$\le CD(t)E(t)^{1/(2(\gamma+1))} + CD(t)^2 .$$

For any $\tau \in [t, t+1]$, integrating (2.1) over $[\tau, t_2]$, we have

$$E(\tau) = E(t_2) + 2\int_{\tau}^{t_2} \|u_t(s)\|_{\beta+2}^{\beta+2} ds$$
$$\le 2\int_{t_1}^{t_2} E(s)\,ds + 2\int_{t}^{t+1} \|u_t(s)\|_{\beta+2}^{\beta+2} ds$$

and from above

$$E(t) \le CD(t)^{2(\gamma+1)/(2\gamma+1)} + CD(t)^2 .$$

Thus, we obtain

$$E(t)^{1+1/\theta(\gamma,\beta)} \le C_1 D(t)^{\beta+2}, \qquad \theta(\gamma,\beta) = \frac{2(\gamma+1)}{(2\gamma+1)\beta+2\gamma}$$
$$= C_1\{E(t) - E(t+1)\} .$$

Setting $\psi(t) = E(t)^{-1/\theta(\gamma,\beta)}$ (see Nakao [12]), we see

$$\psi(t+1) - \psi(t) = \int_0^1 \frac{d}{d\eta}\{\eta E(t+1) + (1-\eta)E(t)\}^{-1/\theta(\gamma,\beta)} d\eta$$
$$= \theta(\gamma,\beta)^{-1} \int_0^1 \{\eta E(t+1) + (1-\eta)E(t)\}^{-1-1/\theta(\gamma,\beta)} d\eta \{E(t) - E(t+1)\}$$
$$\ge \theta(\gamma,\beta)^{-1} E(t)^{-1-1/\theta(\gamma,\beta)} \{E(t) - E(t+1)\} \ge C_1^{-1}\theta(\gamma,\beta)^{-1}$$

and

$$\psi(t+1) \ge \psi(0) + C_1^{-1}\theta(\gamma,\beta)^{-1} t .$$

Hence, we arrive at

$$E(t) \le \{E(0)^{-1/\theta(\gamma,\beta)} + C_1^{-1}\theta(\gamma,\beta)^{-1}[t-1]^+\}^{-\theta(\gamma,\beta)}$$

for $t \ge 0$, where $[a]^+ = \max\{0, a\}$, which implies the desired estimate (1.3). Q.E.D.

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