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Aysmptotic Behavior of Solutions for Semilinear Telegraph Equations

By

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Abstract

For solutions of semilinear telegraph equations in unbounded domain $\mathbb{R}^N$ without the smallness condition on initial data we derive the sharp decay rates in the subcritical case. Even for large data our results can be applied.

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1. Introduction and Results

In this paper we investigate the asymptotic behavior of solutions to the Cauchy problem for the following semilinear wave equation with a dissipative term (i.e. the semilinear telegraph equation) :

$$(F) \qquad \partial_t^2 u - \Delta u + \partial_t u + f(u) = 0 \qquad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+$$

with initial data

$$u(x,0) = u_0(x) \qquad \text{and} \qquad \partial_t u(x,0) = u_1(x)\,,$$

where $\Delta = \sum_{j=1}^N \partial_{x_j}^2$ is the usual Laplace operator and $f(u)$ is the nonlinear function like $f(u) = |u|^\alpha u$ with $\alpha > 0$.

For the related non-dissipative equations, e.g. $\partial_t^2 u - \Delta u + |u|^\alpha u = 0$, the existence, uniqueness, and regularity of global solutions have been already studied by many authors in the subcritical case $0 < \alpha < 4/[N-2]^+$ (see Jörgens [Jö], Strauss [S1, S2], Pecher

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[Pe], Brenner & v.Wahl [BW], Ginibre & Velo [GV], Brenner [Br], etc.). As far as these works are concerned, the dissipative term u_t causes no difficulty and hence the known results for the non-dissipative equations remain valid. (Cf. Grillakis [G1, G2], Struwe [St], Ginibre, Soffer & Velo [GSV], Shatah & Struwe [SS] for the critical case $\alpha = 4/(N-2)$ and $N \geq 3$.)

Under the smallness condition on the initial data $\{u_0, u_1\} \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, Matsumura [Ma] has derived closely related decay estimates of smooth solutions for dissipative wave equations with nonlinear terms $f(u, \nabla u, u_t)$. On the other hand, when $4/N < \alpha < 4/[N-2]^+$, for the solution of Eq.(F) without the small condition on the initial data $\{u_0, u_1\} \in H^{k+l} \cap L^1 \times H^{k+l-1} \cap L^1$, Kawashima, Nakao & Ono [KNO] have shown the following decay property (see (2.1), (3.1), (4.1)) :

$$\|D_x^k D_t^l u(t)\| \leq C(1+t)^{-(k+l)/2-\eta_\alpha}, \qquad \eta_\alpha = (N/4)\min\{1, \alpha\},$$

which is sharp in the case of $l = 0$. This decay property was achieved by combining the energy method and the precise L^p-L^q-type estimates for the linear equation, and hence, the analysis is more delicate. After that, in [O1] we have improved a part of this decay property in the case of $l > 0$,but the results are not enough.

The our goal of this paper is to improve the results in [KNO] and [O1], and to derive the sharp decay rates of the solution for Eq.(F). For simplicity of the proof, we often utilize the known results in [KNO] with respect to the decay estimates.

Let us introduce some notations which will be used through this paper. We denote any partial differential operators of order k with respect to the space variable $x = (x_1, \cdots, x_N)$ by D_x^k i.e. $D_x^k = D_{x_1}^{\alpha_1} \cdots D_{x_N}^{\alpha_N}$ with $k = \alpha_1 + \cdots + \alpha_N$. The partial differential operator with respect to the time t is denoted by D_t, where $D_t^0 u = D_x^0 u = u$. The differential operator D often denotes D_x or D_t. We use only standard function spaces $L^p = L^p(\mathbb{R}^N)$ and $H^s = H^s(\mathbb{R}^N)$ ($H^0 = L^2$) with the norm $\|\cdot\|_p$ and $\|\cdot\|_{H^s}$, respectively. For simplicity, we denote by $\|\cdot\|$ the $L^2(\mathbb{R}^N)$-norm, i.e. $\|\cdot\| = \|\cdot\|_2$. We write $[a]^+ = \max\{0, a\}$, where $1/[a]^+ = \infty$ if $[a]^+ = 0$. Let $k_0, k_1, \cdots$, be nonnegative constants. Various constants which may vary line to line will be denote by C.

To state our results, we shall suppose several conditions below for the nonlinear function $f(u)$.

We suppose that the function $f(u)$ is a continuous function on $\mathbb{R}$ and satisfies

$$\text{(A1)} \qquad f(u)u \geq kF(u) \geq 0 \quad \text{and} \quad |f(u)| \leq k_0|u|^{\alpha+1},$$

where $F(u) = 2\int_0^u f(\eta)\,d\eta$ and k is a positive constant.

Theorem 1. *Let $1 \leq N \leq 3$ and the initial data $\{u_0, u_1\}$ belong to $H^1 \cap L^1 \times L^2 \cap L^1$. Suppose that the function $f(u) \in C^0(\mathbb{R})$ satisfies* (A1) *with*

$$4/N < \alpha \leq 2/(N-2) \qquad (4/N < \alpha < \infty \quad \text{if} \quad N \leq 2).$$

Then the solution $u(t) \in C(\mathbb{R}^+; H^1) \cap C^1(\mathbb{R}^+; L^2)$ of Eq.(F) *has the following : For $0 \leq k, l, k+l \leq 1$,*

$$\text{(1.1)} \qquad \|D_x^k D_t^l u(t)\| \leq C(1+t)^{-k/2-l-N/4}.$$

Moreover, we suppose that $f(u)$ is continuously differentiable and satisfies

(A2) $$|f'(u)| \leq k_1 |u|^{\alpha}.$$

Theorem 2. *Let $1 \leq N \leq 5$ and let the initial data $\{u_0, u_1\}$ belong to $H^2 \cap L^1 \times H^1 \cap L^1$. Suppose that the function $f(u) \in C^1(\mathbb{R})$ satisfies* (A1) *and* (A2) *with*

$$4/N < \alpha < 4/[N-2]^+ .$$

Then the solution $u(t) \in \bigcap_{j=0}^{2} C^j(\mathbb{R}^+; H^{2-j})$ of Eq.(F) *has the following : For $0 \leq k, l, k+l \leq 2$,*

(1.2) $$\|D_x^k D_t^l u(t)\| \leq C(1+t)^{-\theta_{k,l}} \qquad \text{with} \quad \theta_{k,l} = \begin{cases} \omega_{k,l} + \eta_\alpha & \text{if} \quad l \leq 1 \\ \omega_{1,1} + \eta_\alpha & \text{if} \quad l = 2, \end{cases}$$

where we set

(1.3) $$\omega_{k,l} = k/2 + l \qquad \text{and} \qquad \eta_\alpha = (N/4)\min\{1, \alpha\} .$$

Moreover, when $1 \leq N \leq 3$, we can take

(1.4) $$\theta_{0,2} = \max\Big\{\min\Big\{\omega_{0,2} + N/4\, ,\, N\alpha/2\Big\}\, ,\, \omega_{1,1} + N/4\Big\} .$$

Remark. When $N = 6, 7$, the assertion of Theorem 2 holds under a restricted condition (see [KNO])

$$4/N < \alpha \leq 2(N-1)/((N-2)(N-3)) \qquad \Big(< 4/(N-2)\Big) .$$

Further, we suppose that $f(u)$ is a C^2-function and satisfies

(A3) $$|f''(u)| \leq k_2 |u|^{[\alpha-1]^+}$$

Theorem 3. *In addition to the assumptions of Theorem 2, suppose that $\{u_0, u_1\}$ belong to $H^3 \cap L^1 \times H^2 \cap L^1$ and $f(u) \in C^2(\mathbb{R})$ satisfies* (A3). *Then the solution $u(t) \in \bigcap_{j=0}^{3} C^j(\mathbb{R}^+; H^{3-j})$ has the following : For $0 \leq k, l, k+l \leq 3$,*

$$\|D_x^k D_t^l u(t)\| \leq C(1+t)^{-\theta_{k,l}}$$

with

$$(1.5)\qquad \theta_{k,l} = \begin{cases} \omega_{k,l} + N/4 & \text{if } l \le 2 \\ \max\Big\{\min\Big\{\omega_{0,3} + N/4\,,\ N\alpha/2\Big\}\,,\ \omega_{1,2} + N/4\Big\} & \text{if } l = 3\,, \end{cases}$$

where $\omega_{k,l}$ is given by (1.3). Moreover, it holds that for $2 \le q \le \infty$,

$$(1.6)\qquad \|u(t)\|_q \le C(1+t)^{-(N/2)(1-1/q)}\,.$$

By induction, we can obtain the following decay property for the derivatives of higher order.

Theorem 4. *Let $m \ge 3$ be an integer. In addition to the assumptions of Theorem 3, suppose that $\{u_0, u_1\}$ belong to $H^{m+1} \cap L^1 \times H^m \cap L^1$ and $f(u)$ is an m-times continuously differentiable function. Then the solution $u(t) \in \bigcap_{j=0}^{m+1} C^j(\mathbb{R}^+; H^{m+1-j})$ has the following : For $0 \le k, l, k+l \le m+1$,*

$$(1.7)\qquad \|D_x^k D_t^l u(t)\| \le C(1+t)^{-\theta_{k,l}} \qquad \text{with} \quad \theta_{k,l} = \begin{cases} \omega_{k,l} + N/4 & \text{if } l \le m \\ \omega_{1,m} + N/4 & \text{if } l = m+1\,, \end{cases}$$

where $\omega_{k,l}$ is given by (1.3).

Moreover, if $f(u)$ satisfies

$$(\mathrm{A3})\qquad |f^{(j)}(u)| \le k_j |u|^{[\alpha+1-j]^+} \qquad \text{for} \quad j = 3, \cdots, m\,,$$

we can take

$$(1.8)\qquad \theta_{0,m+1} = \max\Big\{\min\Big\{\omega_{0,m+1} + N/4\,,\ N\alpha/2\Big\}\,,\ \omega_{1,m} + N/4\Big\}\,.$$

As a corollary to the above theorem, immediately we have the following decay properties.

Corollary 5. *In addition to the assumptions of Theorem 2, suppose that $\{u_0, u_1\}$ belong to $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ and $f(u) \in C^\infty(\mathbb{R})$ satisfies*

$$|f^{(m)}(u)| \le k_m |u|^{[\alpha+1-m]^+} \qquad \text{for any } m \ge 0\,.$$

Then the solution $u(t)$ of Eq.(F) satisfies

$$\|D_x^k D_t^l u(t)\| \le C(1+t)^{-k/2-l-N/4}$$

and

$$\|D_x^k D_t^l u(t)\|_\infty \leq C(1+t)^{-k/2-l-N/2}$$

for any nonnegative integers k and l.

Finally, for the non-homogeneous telegraph equation :

$$\text{(H)} \qquad \partial_{tt}u - \Delta u + \partial_t u = h(x,t) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+$$

with initial data $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = u_1(x)$, we state the following two propositions, which play an important role through this paper. (See Kawashima, Nakao & Ono [KNO] and Ono [O2] for the proofs.)

Proposition 1.1. *Let the initial data $\{u_0, u_1\}$ belong to $H^{k+l} \cap L^1 \times H^{[k+l-1]^+} \cap L^1$ for nonnegative integers k and l, and let the external force term $h(x,t)$ be an appropriate smooth function. Then, the solution $u(t)$ of Eq.*(H) *satisfies*

$$\begin{aligned}
\|D_x^k D_t^l u(t)\| \leq\ & C(1+t)^{-\omega_{k,l}-N/4}\{\|u_0\|_{H^{k+l}} + \|u_1\|_{H^{[k+l-1]^+}} + \|u_0\|_{L^1} + \|u_1\|_{L^1}\} \\
& + C\int_0^t (1+t-s)^{-\omega_{k,l}-N/4}\|h(s)\|_1 ds \\
& + C\int_0^t e^{-\delta(t-s)}\|D_x^{n_0}h(s)\|\, ds \\
& \left(+ C\sum_{j=1}^{l-1}\|D_x^{n_j}D_t^{l-1-j}h(s)\| \qquad if \quad l \geq 2\right)
\end{aligned}$$

with some $0 < \delta < 1/2$, provided that for nonnegative integers n_0 and n_j such that $n_0 \geq [k+l-1]^+$ and $k+j-1 \leq n_j \leq k+2j$, respectively, where we set $\omega_{k,l} = k/2 + l$.

We define a energy $E(t)$ associated with Eq.(H) as $E(t) \equiv \|D_x u(t)\|^2 + \|D_t u(t)\|^2$.

Proposition 1.2. *Let $u(t)$ be a solution of Eq.*(H). *Suppose that*

$$\|u(t)\| \leq k_0(1+t)^{-\alpha}$$

and for an integer n,

$$\|h(t)\| \leq k_1(1+t)^{-\beta} + k_2(1+t)^{-\gamma}E(t)^{1/2}$$

with certain constants $k_j \geq 0$ $(j = 0,1,2)$, $\alpha \geq 0$, $\beta > 0$, and $\gamma > 0$. Then, the energy $E(t)$ satisfies

$$E(t)^{1/2} \equiv \{\|D_x u(t)\|^2 + \|D_t u(t)\|^2\}^{1/2} \leq C(1+t)^{-\theta}$$

with

$$\theta = \min\left\{1/2 + a\,,\ (\alpha+\beta)/2\,,\ \beta\,,\ \alpha+\gamma\right\}.$$

2. Proof of Theorem 1

We use the following known decay estimates (see Theorem 1 in [KNO]) :

$$\|u(t)\| \leq C(1+t)^{-N/4} \quad \text{and} \quad \|D_x u(t)\| + \|D_t u(t)\| \leq C(1+t)^{-1/2-N/4} \tag{2.1}$$

under the assumptions of Theorem 1.

It is enough to derive the sharp rate of the first derivative with respect to t. Under the condition (A1), we observe from the Gagliardo-Nirenberg inequality and the decay (2.1) that for $1 \leq p \leq 2$,

$$\|f(u(t))\|_p \leq C\|u(t)\|_{p(\alpha+1)}^{\alpha+1} \leq \|u(t)\|^{(\alpha+1)(1-\xi_p)}\|D_x u(t)\|^{(\alpha+1)\xi_p} \leq C(1+t)^{-\theta_p}$$

with $\xi_p = N(1/2-1/(p(\alpha+1))$ and $\theta_p = (\alpha+1)(N/4+\xi_p/2) > 1+N/4$. Then, applying Proposition 1.2 with $(k,l,n_0) = (0,1,0)$, we obtain

$$\begin{aligned} \|D_t u(t)\| &\leq C(1+t)^{-\omega_{0,1}-N/4} + C\int_0^t (1+t-s)^{-\omega_{0,1}-N/4}\|f(u(s))\|_1 ds \\ &\quad + C\int_0^t e^{-\delta(t-s)}\|f(u(s))\|\, ds \\ &\leq C(1+t)^{-1-N/4}\,, \end{aligned}$$

which is the desired estimate. □

3. Proof of Theorem 2

We use the following known decay estimates (see Theorem 3 in [KNO]) : For $0 \leq k, l, k+l \leq 2$,

$$\|D_x^k D_t^l u(t)\| \leq C(1+t)^{-(k+l)/2-\eta_\alpha} \tag{3.1}$$

under the assumptions of Theorem 2, where η_α is given by (1.3).

Immediately we see from the Gagliardo-Nirenberg inequality and the decay (3.1) that

$$\|u(t)\|_q \leq C(1+t)^{-(N/2)(1/2-1/q)-\eta_\alpha} \tag{3.2}$$

for $2 \leq q \leq 2N/[N-4]^+$ ($2 \leq q < \infty$ if $N=4$). Moreover, from the equation (F) and the decay (3.2), we have

$$\|D_t u(t)\| \leq C\{\|D_t^2 u(t)\| + \|D_x^2 u(t)\| + \|u(t)\|_{2(\alpha+1)}^{\alpha+1}\} \leq C(1+t)^{-1-\eta_\alpha}\,, \tag{3.3}$$

and

$$\|D_t f(u(t))\| \leq C\|u(t)\|_{N\alpha}^{\alpha}\|D_x D_t u(t)\| \leq C(1+t)^{-1/2}E_{1+1}(t)\,,$$

where $E_{1+1}(t)$ is as below. Then, applying Proposition 1.2, we obtain

$$E_{1+1}(t)^{1/2} \equiv \{\|D_xD_tu(t)\|^2 + \|D_t^2u(t)\|^2\}^{1/2} \le C(1+t)^{-3/2-\eta_\alpha}. \tag{3.4}$$

Thus the desired decay (1.2) follows from (3.1), (3.3), and (3.4).

Finally, we shall show the decay rate (1.4). When $1 \le N \le 3$, we see $\eta_\alpha = N/4$. It follows from (3.2) that

$$\|f(u(t))\|_1 \le C\|u(t)\|_{\alpha+1}^{\alpha+1} \le C(1+t)^{-N\alpha/2} \tag{3.5}$$

and

$$\|D_xf(u(t))\| \le C\|u(t)\|_\infty^\alpha \|D_xu(t)\| \le C(1+t)^{-N\alpha/2-(N+2)/4}.$$

Then, applying Proposition 1.1 with $(k, l, n_0, n_1) = (0, 2, 1, 1)$, we obtain

$$\begin{aligned}\|D_t^2u(t)\| &\le C(1+t)^{-\omega_{0,2}-N/4} + C\int_0^t (1+t-s)^{-\omega_{0,2}-N/4}\|f(u(s))\|_1 ds \\ &\quad + C\int_0^t e^{-\delta(t-s)}\|D_xf(u(s))\|\,ds + C\|D_xf(u(t))\| \\ &\quad \le C(1+t)^{-\min\{\omega_{0,2}+N/4\,,\,N\alpha/2\}},\end{aligned}$$

which gives the decay rate (1.4). □

4. Proof of Theorem 3

We use the following known decay estimates (see Theorem 4 in [KNO]) : For $k+l=3$,

$$\|D_x^kD_t^lu(t)\| \le C(1+t)^{-3/2-N/4} \tag{4.1}$$

under the assumptions of Theorem 3.

When $1 \le N \le 5$, we see $H^3 \subset L^\infty$ and $\|u(t)\|_\infty \le C < \infty$. Moreover, under the condition $f(u) \in C^2(\mathbb{R})$, we know

$$|f^{(j)}(u(t))| \le c(\|u(t)\|_\infty)|u(t)|^{2-j} \qquad \text{for} \quad j = 0, 1, 2.$$

(Cf. Consider the Taylor expansion of $f(u)$ at $u = 0$.) Hence, we can take $\alpha \ge 1$ and $\eta_\alpha = N/4$ in (1.2), and we find the decay (1.6) follows from the decay (3.1), (4.1), and the Gagliardo-Nirenberg inequality.

From the decay (1.2), (1.6), and (4.1), it is easy to see that

$$\|D_xD_tf(u(t))\| \le C\{\|u\|_\infty^\alpha\|D_xD_tu\| + \|u\|_\infty^{\alpha-1}\|D_xu\|_4\|D_tu\|_4\} \le C(1+t)^{-5/2-N/4}$$

and

$$\|D_t^2f(u(t))\| \le C\{\|u\|_\infty^\alpha\|D_t^2u\| + \|u\|_\infty^{\alpha-1}\|D_tu\|_4^2\} \le C(1+t)^{-5/2-N/4}.$$

Applying Proposition 1.2 together with the estimates $\|D_xD_tu(t)\| \le C(1+t)^{-3/2-N/4}$ and $\|D_t^2u(t)\| \le C(1+t)^{-3/2-N/4}$ (see (3.4)), we obtain

$$E_{2+1}(t)^{1/2} \equiv \{\|D_x^2D_tu(t)\|^2 + \|D_xD_t^2u(t)\|^2\}^{1/2} \le C(1+t)^{-2-N/4} \tag{4.2}$$

and

$$E_{1+2}(t)^{1/2} \equiv \{\|D_xD_t^2u(t)\|^2 + \|D_t^3u(t)\|^2\}^{1/2} \le C(1+t)^{-2-N/4}, \tag{4.3}$$

respectively. Then, from the equation (F) and the decay (1.2), (1.6), and (4.3), we have

$$\|D_t^2u(t)\| \le C\{\|D_t^3u(t)\| + \|D_x^2D_tu(t)\| + \|u(t)\|_\infty^\alpha\|D_tu(t)\|\} \le C(1+t)^{-2-N/4} \tag{4.4}$$

and

$$\|D_t^2f(u(t))\| \le C(1+t)^{-3-N/4}.$$

Again, applying Proposition 1.2 to $E_{1+2}(t)$, we can get

$$E_{1+2}(t)^{1/2} \equiv \{\|D_xD_t^2u(t)\|^2 + \|D_t^3u(t)\|^2\}^{1/2} \le C(1+t)^{-5/2-N/4}. \tag{4.5}$$

Finally, we shall improve the decay rate of the third order derivative with respect to t. Using the fact that

$$\|f(u(t))\|_1\,,\ \|D_x^2f(u(t))\|\,,\ \|D_tf(u(t))\| \le C(1+t)^{-N\alpha/2}$$

and applying Proposition 1.1 with $(k,l,n_0,n_1,n_2) = (0,3,2,0,2)$, we obtain

$$\begin{aligned}\|D_t^3u(t)\| &\le C(1+t)^{-\omega_{0,3}-N/4} + C\int_0^t (1+t-s)^{-\omega_{0,3}-N/4}\|f(u(s))\|_1 ds \\ &\quad + C\int_0^t e^{-\delta(t-s)}\|D_x^2f(u(s))\|\,ds + C\|D_tf(u(t))\| + C\|D_x^2f(u(t))\| \\ &\le C(1+t)^{-\min\{\omega_{0,3}+N/4\,,\,N\alpha/2\}}.\end{aligned} \tag{4.6}$$

Therefore, the desired decay rate (1.5) follows from (1.2), (4.1), (4.2), and (4.4)–(4.6). □

5. Proof of Theorem 4

We shall prove Theorem 4 by induction.

Since $f(u)$ is assumed further to be m-times continuously differentiable with $m \ge 3$, it holds from (A1)–(A2) that

$$|f^{(j)}(u(t))| \le c(\|u(t)\|_\infty)|u(t)|^{3-j} \qquad \text{for} \quad j = 0,1,2,3\,, \tag{5.1}$$

and hence, in this situation we can choose $\alpha \ge 2$.

To get the desired decay property (1.7), it is enough to show the following. Indeed, when $m = 3$ in Proposition 5.1 below, the decay (5.2) is valid by Theorem 3 and hence by induction we conclude the decay (1.7).

Proposition 5.1. *Under the assumptions of Theorem 4, suppose that for* $0 \le k, l, k+l \le m$,

$$\|D_x^k D_t^l u(t)\| \le C(1+t)^{-\theta_{k,l}} \qquad \textit{with} \quad \theta_{k,l} = \begin{cases} \omega_{k,l} + N/4 & \textit{if} \quad l \le m-1 \\ \omega_{1,m-1} + N/4 & \textit{if} \quad l = m. \end{cases} \tag{5.2}$$

Then, it holds that for $0 \le k, l, k+l \le m+1$,

$$\|D_x^k D_t^l u(t)\| \le C(1+t)^{-\theta_{k,l}} \qquad \textit{with} \quad \theta_{k,l} = \begin{cases} \omega_{k,l} + N/4 & \textit{if} \quad l \le m \\ \omega_{1,m} + N/4 & \textit{if} \quad l = m+1. \end{cases} \tag{5.3}$$

Proof of Proposition 5.1. The decay (5.2) implies that for $0 \le n \le m$,

$$\|D_x^{m-n} D_t^n u(t)\| \le C(1+t)^{-a_n}, \qquad a_n = \begin{cases} \omega_{m-n,n} + N/4 & \text{if} \quad n \le m-1 \\ \omega_{1,m-1} + N/4 & \text{if} \quad n = m. \end{cases} \tag{5.4}$$

We have that

$$D^m f(u) = \sum_{j=1}^m f^{(j)}(u) \sum_{\sigma \in \mathcal{S}_j^m} C(D^{\sigma_1} u) \cdots (D^{\sigma_j} u),$$

where we set

$$\mathcal{S}_j^m \equiv \{\sigma = (\sigma_1, \cdots, \sigma_j) \in \mathbb{N}^j \,;\, \sum_{i=1}^j \sigma_i = m,\, 1 \le \sigma_1 \le \cdots \le \sigma_j \le m+1-j\}.$$

Then, it follows from (5.1) that

$$\begin{aligned} \|D^m f(u)\| &\le C \sum_{j=1}^m \|f^{(j)}(u)\|_\infty \sum_{\sigma \in \mathcal{S}_j^m} \|\prod_{i=1}^j |D^{\sigma_i} u|\| \\ &\le C\|u\|_\infty^\alpha \|D^m u\| + C\|u\|_\infty \sum_{\sigma \in \mathcal{S}_2^m} \|\prod_{i=1}^2 |D^{\sigma_i} u|\| + C \sum_{j=3}^m \sum_{\sigma \in \mathcal{S}_j^m} \|\prod_{i=1}^j |D^{\sigma_i} u|\|. \end{aligned}$$

We observe from the Hölder inequality and the Gagliardo-Nirenberg inequality that for $2 \le j \le m$,

$$\|\prod_{i=1}^j |D^{\sigma_i} u|\| \le \prod_{i=1}^j \|D^{\sigma_i} u\|_{2q_i} \le C \prod_{i=1}^j \|D^{\sigma_i} u\|^{1-\xi_i} \|D_x D^{\sigma_i} u\|^{\xi_i},$$

where $\sum_{i=1}^{j} 1/q_i = 1$ $(1 < q_i < \infty)$, $\xi_i = (N/2)(1-1/q_i)$, and $\sum_{i=1}^{j} \xi_i = (j-1)N/2$.

Thus, we obtain from (5.2) that for $0 \le n \le m$,

$$\begin{aligned}\|D_x^{m-n} D_t^n f(u(t))\| &\le C\|u\|_\infty^\alpha \|D_x^{m-n} D_t^n u\| \\ &\quad + C\|u\|_\infty \sum_{\sigma \in S_2^m} \prod_{i=1}^{2} \|D_x^{\sigma_i - n_i} D_t^{n_i} u\|^{1-\xi_i} \|D_x^{\sigma_i - n_i + 1} D_t^{n_i} u\|^{\xi_i} \\ &\quad + C \sum_{j=3}^{m} \sum_{\sigma \in S_j^m} \prod_{i=1}^{j} \|D_x^{\sigma_i - n_i} D_t^{n_i} u\|^{1-\xi_i} \|D_x^{\sigma_i - n_i + 1} D_t^{n_i} u\|^{\xi_i} \\ &\le C(1+t)^{-b_n^{(1)}} + C(1+t)^{-b_n^{(2)}} + C\sum_{j=3}^{m} (1+t)^{-b_n^{(j)}},\end{aligned}$$

where $0 \le n_i \le \sigma_i$, $n = \sum_{i=1}^{j} n_i$ $(2 \le j \le m)$,

$$b_n^{(1)} = N\alpha/2 + a_n \ge 2 + a_n \ge 1 + \omega_{m-n,n} + N/4,$$

and by $n_i \le \sigma_i \le m-1$ $(i = 1, 2)$,

$$\begin{aligned}b_n^{(2)} &= N/2 + \sum_{i=1}^{2} \{(\sigma_i - n_i)/2 + n_i + N/4 + \xi_i/2\} \\ &= N/2 + (m-n)/2 + n + N/2 + N/4 \ge 1 + \omega_{m-n,n} + N/4,\end{aligned}$$

and

$$\begin{aligned}\min_{3 \le j \le m} b_n^{(j)} &= \min_{3 \le j \le m} \sum_{i=1}^{j} \{(\sigma_i - n_i)/2 + n_i + N/4 + \xi_i/2\} \\ &= \min_{3 \le j \le m} \{(m-n)/2 + n + jN/4 + (j-1)N/4\} \ge 1 + \omega_{m-n,n} + N/4.\end{aligned}$$

Therefore, it follows that

$$\|D_x^{m-n} D_t^n f(u(t))\| \le C(1+t)^{-b_n}, \qquad b_n = 1 + \omega_{m-n,n} + N/4. \tag{5.5}$$

Applying Proposition 1.2 together with (5.4) and (5.5), we have that for $0 \le n \le m$,

$$\|D_x^{m+1-n} D_t^n u(t)\| + \|D_x^{m-n} D_t^{n+1} u(t)\| \le C(1+t)^{-\theta_{m+1-n,n}} \tag{5.6}$$

with

$$\theta_{m+1-n,n} = 1/2 + a_n = \begin{cases} \omega_{m+1-n,n} + N/4 & \text{if} \quad n \le m-1 \\ \omega_{2,m-1} + N/4 & \text{if} \quad n = m, \end{cases}$$

which gives the desired decay rate (5.3) expect for the case $n = m$.

Since it follows from the equation (F) and the decay (5.6) and (5.5) that

$$\|D_t^m u(t)\| \le C\{\|D_t^{m+1} u(t)\| + \|D_x^2 D_t^{m-1} u(t)\| + \|D_t^{m-1} f(u(t))\|\} \le C(1+t)^{-\omega_{0,m} - N/4},$$

applying Proposition 1.2 together with (5.5), again, we obtain

$$\|D_x D_t^m u(t)\| + \|D_t^{m+1} u(t)\| \le C(1+t)^{-\omega_{1,m}-N/4}. \tag{5.7}$$

Thus the desired decay estimate (5.3) follows from (5.6) and (5.7). □

Finally, under the assumption (A3), we improve the decay rate of the L^2-norm of $D_t^{m+1}u(t)$. We observe from the decay (1.6), (5.2), and (5.3) that

$$\begin{aligned}\|D_x^m f(u(t))\| &\le C\|u\|_\infty^\alpha \|D_x^m u\| + C\sum_{j=2}^m \|u\|_\infty^{[\alpha+1-j]^+} \sum_{\sigma\in S_j^m}\prod_{i=1}^j \|D_x^{\sigma_i}u\|^{1-\xi_i}\|D_x^{\sigma_i+1}u\|^{\xi_i} \\ &\le C(1+t)^{-b_1} + C\sum_{j=2}^m (1+t)^{-b_j},\end{aligned}$$

where $b_1 = N\alpha/2 + m/2 + N/4$ and

$$\begin{aligned}b_j &= [\alpha+1-j]^+ N/2 + \sum_{i=1}^j \{\sigma_i/2 + N/4 + \xi_i/2\} \\ &= [\alpha+1-j]^+N/2 + m/2 + jN/4 + (j-1)N/4 \ge N\alpha/2 + m/2 + N/4,\end{aligned}$$

and hence, we know

$$\|f(u(t))\|_1,\ \|D_x^m f(u(t))\|,\ \|D_x^j D_t^{m-j} f(u(t))\| \le C(1+t)^{-N\alpha/2} \tag{5.8}$$

for $2 \le j \le m$ (see (3.5)). Applying Proposition 1.1 with $(k, l, n_0, n_j) = (0, m+1, m, j)$ together with (5.8), we have

$$\begin{aligned}\|D_t^{m+1}u(t)\| &\le C(1+t)^{-\omega_{0,m+1}-N/4} + C\int_0^t (1+t-s)^{-\omega_{0,m+1}-N/4}\|f(u(s))\|_1 ds \\ &\quad + C\int_0^t e^{-\delta(t-s)}\|D_x^m f(u(s))\|\, ds + C\sum_{j=2}^m \|D_x^j D_t^{m-j} f(u(s))\| \\ &\le C(1+t)^{-\min\{\omega_{0,m+1}+N/4\,,\,N\alpha/2\}},\end{aligned}$$

which gives the desired decay rate (1.8). □

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